THE ROLE OF ZEROS IN THE PERFORMANCE OF MULTI-INPUT, MULTI-OUTPUT FEEDBACK SYSTEMS*

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ABSTRACT

The concepts of pole and zero have been both intuitively appealing and practically useful in the development of feedback control engineering. Moreover, the evolution of notions of zero for systems with multiple inputs and outputs provides a selection of interesting interplays between applied engineering, control theory, and mathematics. In this paper, we provide a review of the early history attached to the idea of zeros for these systems, a history attributed essentially to H. H. Rosenbrock and phrased, with permission, in his words. Also provided is a discussion of the consensus definitions for such zeros, together with the ways in which they have become involved in control thinking and control applications during recent years. A control zero principle (CZP) and control pole principle (CPP) explain the intuitive way in which zeros and poles influence the performance of feedback systems. Moreover, these are presented also in the multivariable case (MCZP, MCPP) by means of a pair of new results; and this provides a framework to unify the overall presentation. Finally, zeros at infinity are introduced; the causality of controllers is fitted into place; and recent developments in time-varying zeros, nonlinear zeros, and generic zeros are described.

I. INTRODUCTION

Even the design of single-input, single-output (SISO) control systems leads inexorably to the issues of multi-input, multi-output (MIMO) feedback. It is clear why this should be the case. In the linearization of a nonlinear plant, say in differential equation form, there are both a nominal control and a nominal state. Accordingly, in any physical implementation predicated upon such a linearization, the loop must be entered at two points, one for the control and one for the state. If the controller is to function at more than one nominal (control, state) pair, as in the servomechanism problem, then dynamically changing signals access the loop at both of these points. Moreover, as is now well known to researchers, an adequate monitoring of the loop involves a consideration of the outputs of in-loop compensators, in addition to the plant itself. Thus, even in the elementary situation of unity negative feedback, there are at least two outputs to be examined. We can advance, therefore, the hypothesis that realistic control

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problems usually have at least two inputs and two outputs. In such cases, the zeros of the system, in particular the transmission zeros, represent one of the most difficult obstacles to an achievement of high performance specifications.

General systems satisfying the state equations

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) \\ y(i) &= Cx(i) + Du(i) \end{aligned} \tag{1}$$

with $x(i) \epsilon k^n$, $u(i) \epsilon k^m$, and $y(i) \epsilon k^p$ are of interest. Usually the field k is considered to be the field of real numbers, R, in control problems. An external description of the system can be formed in the customary manner, assuming initial conditions are zero, by applying the z-transform to both members of (1) to obtain

$$\hat{y}(z) = [C(zI - A)^{-1}B + D]\hat{u}(z) = G(z)\hat{u}(z). \tag{2}$$

The matrix $G(z)$ in (2) is termed the transfer function matrix corresponding to the system in (1).

Similarly, continuous systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3}$$

with $x(t) \epsilon k^n$, $u(t) \epsilon k^m$, $y(t) \epsilon k^p$ can be associated with state-space matrices (A, B, C, D) and a transfer function matrix $G(s)$ using Laplace transforms, assuming zero initial conditions. In Sections II and V, s is written as the transformed variable, which should be interpreted in context.

The study of zeros in the SISO case has been extensive. Consider a SISO transfer function

$$g(z) = \frac{n(z)}{d(z)} \tag{4}$$

with $(n(z), d(z))$ coprime and both elements of $k[z]$, the ring of polynomials in z over k. Then the zeros of the polynomials $n(z)$ and $d(z)$ in (4) are called the zeros and poles of the transfer function, respectively. In the case of MIMO systems, many zero generalizations are based upon a matrix representation such as the Smith or Smith-McMillan form. For the purpose of notation, we briefly introduce these forms here, although detailed descriptions can be found, for example, in [1].

Given any $p \times m$ polynomial matrix $P(z)$ of rank r, there exist appropriate polynomial matrices $U_1(z)$ and $U_2(z)$ both with unit determinant in $k[z]$, or alternatively, unimodular over $k[z]$, such that

$$\Lambda(z) = U_1(z)P(z)U_2(z) = \begin{bmatrix} \Lambda^*(z)_{r,r} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix}. \tag{5}$$

The subscripts in (5) denote submatrix dimensions and

$$\Lambda^*(z) = diag(\lambda_1(z), \ldots, \lambda_r(z)) \tag{6}$$

where each $\lambda_i(z)$ in (6) is a unique monic polynomial satisfying the division property; i.e.,

$$\lambda_i(z) | \lambda_{i+1}(z), \quad 1 \leq i \leq r-1. \tag{7}$$

Moreover, if we define $\Delta_i(z)$ to be the monic greatest common divisor of all the $i \times i$ nonzero minors of $P(z)$, then

$$\lambda_i(z) = \Delta_i(z)/\Delta_{i-1}(z); \;\; \Delta_0(z) = 1. \tag{8}$$

The matrix $\Lambda(z)$ is identified as the Smith form of $P(z)$ over $k[z]$, the $\{\Delta_i(z)\}$ are referred to as the determinantal divisors of $P(z)$, and the $\{\lambda_i(z)\}$ are the invariant polynomials of $P(z)$.

For matrices whose elements are rational functions, the Smith-McMillan form over $k[z]$, $M(z)$, is introduced. Given a $p \times m$ rational matrix $G(z)$ of rank r, it can be rewritten as $N(z)/d(z)$, a polynomial matrix of rank r divided by the least common monic denominator of the elements of $G(z)$. Then the Smith form of $N(z)$ over $k[z]$, say $\Lambda_N(z)$, can be found and $M(z)$ can be characterized by $\Lambda_N(z)/d(z)$ with reduced elements; that is,

$$M(z) = \begin{bmatrix} M^*(z)_{r,r} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix} \tag{9}$$

with

$$M^*(z) = diag(\varepsilon_i(z)/\psi_i(z)),\ 1 \le i \le r, \tag{10}$$

and $(\varepsilon_i(z), \psi_i(z))$ coprime. Furthermore, we find the following three properties: (i) $\varepsilon_i(z)|\varepsilon_{i+1}(z), \quad 1 \le i \le r-1$, (ii) $\psi_{i+1}(z)|\psi_i(z), \quad 1 \le i \le r-1$, (iii) $d(z) = \psi_1(z)$.

The next section presents a brief treatise on the origins of the system matrix as recalled by Rosenbrock. Section III, following, introduces and illustrates the consensus definitions of zeros in the finite plane. In Section IV, we summarize the developments in the theory of zeros, since Rosenbrock's pioneering work. Material is displayed concisely using a digraph. The focus of the paper, zeros and control limitations, will be examined in Section V. We then explain in the following section how the ideas of finite zeros have been extended to the point at infinity, and how this relates to the concept of controller causality. Finally, new directions in current zeros research are mentioned.

II. HISTORY OF THE SYSTEM MATRIX [2]

The fifteen years after 1945 were pre-eminently the period of frequency-response methods, and of their application to industrial problems. By the late 1950's, most of the theoretical developments which could be implemented with the available equipment had already been made. There had also been a very satisfactory application of these methods in industry.

In 1958, the Rand Report by Bellman, Glicksberg, and Gross [3] appeared, and from 1960 the work of Kalman and others, which laid the foundations of state-space methods. These were adaptable not only to linear constant systems, but also to nonlinear and time-varying problems, and were especially suited to aerospace, and to off-line batch computing methods of design. In their linear form, they provided definitive answers to certain problems of redundancy in systems – uncontrollability and unobservability – which had been difficult to handle by frequency-response methods and were poorly understood.

Those who attempted to apply these newer methods to industrial problems, on the other hand, met a number of new difficulties, including the following:

(i) Frequency-response methods had a certain inherent robustness, expressed by the describing function and by circle theorems. It was more difficult to achieve robustness by state-space methods.

(ii) The order of many industrial systems, especially in the process industries, was excessively high – hundreds or perhaps thousands – while their frequency-response behavior was relatively simple. Methods for order-reduction of theoretical models posed difficulties, while if a state-space description was obtained from process measurements the result was obviously artificial.

(iii) Much of the modelling process was excluded from the state-space framework. A plant, for example, would be described by a mixed set of nonlinear algebraic and differential equations. These would be linearized around some operating condition, and then had to be reduced to state-space form. In one investigation, as Rosenbrock recollects, linearization gave 8 algebraic equations and 13 differential equations, some of second order, while the state-space representation had order 11. No answer could be obtained within the state-space framework at that time to such questions as: did the process of reduction to state-space form change the order?; or did this process change the controllability and observability properties?

(iv) Most plants consist of interconnected sub-units. Each of these can be described by state-space equations, and their interconnections are generally described by algebraic equations. Taken together, these are not in state-space form, and if they are brought to this form the identity of the separate sub-units is lost. With it is lost the opportunity to make use of engineering knowledge about their separate behavior.

For these and similar reasons there was a strong incentive to achieve a common framework within which both frequency-response and state-space descriptions could find a place, and both could be generalized and extended. A starting point in this program was to notice that the linearized equations for a system will generally have the form, after Laplace transformation,

$$\begin{aligned} T(s)\zeta &= U(s)u \\ y &= V(s)\zeta + W(s)u \end{aligned} \tag{11}$$

where ζ is a vector of system variables and $T,\ U,\ V,\ W$ are polynomial matrices. The number of system variables will generally not be equal to the order n of the system, but by a trivial expansion if necessary we can ensure that the dimension of ζ is at least equal to n.

An obvious second step is to write (11) in the form

$$\begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \begin{bmatrix} \zeta \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}. \tag{12}$$

Here all the information about the system is contained in the first matrix, which is called the system matrix and for state-space systems has the form

$$\begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}. \tag{13}$$

A change of basis in the state space is expressed by the transformation

$$\begin{bmatrix} H^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - A_1 & B_1 \\ -C_1 & 0 \end{bmatrix} \tag{14}$$

which we call system similarity (SS).

Now Weierstrass showed that two matrices $A,\ A_1$ are similar if and only if the two pencils $sI - A,\ sI - A_1$ are equivalent, that is if

$$M(s)(sI - A)N(s) = sI - A_1 \tag{15}$$

where M and N are unimodular. A complete set of invariants of the transformation (15) is easy to find: it is the set of invariant polynomials, and these can be characterized by their zeros. Thus, the awkward problem of finding a complete set of invariants under similarity has been replaced by a simpler problem with a simpler solution. This is achieved by replacing the similarity transformation of A by the more powerful transformation of equivalence applied to $sI - A$.

By analogy, we therefore look for a more powerful transformation which will replace (14). One such transformation is strict system equivalence (SSE),

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - A_1 & B_1 \\ -C_1 & 0 \end{bmatrix}, \tag{16}$$

where $M,\ N$ are again unimodular. We can show that two system matrices in state-space form are SS if and only if they are SSE. An alternative transformation has been given by Fuhrmann [4].

The transformation of SSE solves many of the problems raised at the beginning of this section. The system matrix in (12), for a strictly proper system, can always be brought to state-space form by SSE. We can easily relate state-space and frequency-response descriptions. The system matrix can represent interconnected subsystems while preserving their separate identity.

We can also define a large number of zeros of the system matrix which remain invariant under SSE. These are briefly mentioned here and are examined in detail in the following section.

(a) The zeros of the invariant polynomials of $T(s)$, which are the system poles.

(b) The input decoupling zeros, which lie over the poles of the uncontrollable part of the system.

(c) The output decoupling zeros, which lie over the poles of the unobservable part of the system.

(d) The input output decoupling zeros, which lie over the poles of the uncontrollable and unobservable part of the system.

(e) The transmission zeros, which are the zeros of the numerator polynomials in the Smith-McMillan form of the transfer function matrix.

Taking account of the intersections of these sets, we have:

(f) The poles of the transfer function matrix are given by $\{f\} = \{a\} - \{b\} - \{c\} + \{d\}$.

(g) The zeros of the system can be defined by $\{g\} = \{b\} + \{c\} - \{d\} + \{e\}$.

Unfortunately, in contradistinction to the result obtained by Weierstrass, we do not obtain a complete invariant of the transformation from these zeros. There is a residual structure in the system matrix which is difficult to describe uniquely and which accounts for most of the difficulty in transferring frequency-response methods from single-input single-output systems to multivariable systems. In the transfer function matrix, for example, this residual structure gives a frequency-dependent cross-coupling between inputs and outputs.

III. THE MEANING OF ZEROS

Although many authors before 1970 alluded to the concept of zeros of multivariable systems (see, for example, [5,6]), Rosenbrock is credited with the first definition of zeros of a multivariable system or, loosely termed, multivariable zeros. The original definitions of Rosenbrock will be examined in great detail as they provide a means of classification of multivariable zeros.

Rosenbrock's first multivariable zeros are distinctly related to the standard state-space tests for controllability and observability [7]. These zeros are termed decoupling zeros. The complex numbers $\{z_0\}$ which satisfy

$$rank[z_0 I - A \qquad B] < n \tag{17}$$

are called input decoupling zeros (i.d. zeros). Similarly, those complex numbers $\{z_0\}$ which satisfy

$$rank \begin{bmatrix} z_0 I - A \\ -C \end{bmatrix} < n \tag{18}$$

are called output decoupling zeros (o.d. zeros). The number of i.d. zeros is equal to the row rank defect of the controllability matrix

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}; \tag{19}$$

and the number of o.d. zeros is equal to the column rank defect of the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{20}$$

Consider the system matrix Σ_1 obtained by removing i.d. zeros from a given system matrix Σ by the order-reduction process described in [7]. Then the set difference of o.d. zeros of Σ and the o.d. zeros of Σ_1 is called the set of input output decoupling zeros (i.o.d. zeros) of Σ. The set of i.d. zeros of Σ defined by (17) contains the set of i.o.d. zeros of Σ. Trivially, the set of o.d. zeros of Σ defined by (18) contains the set of i.o.d. zeros of Σ.

At the same time, Rosenbrock introduced zeros of a transfer function matrix, $G(z)$ [7]. For a matrix $G(z)$ put into Smith-McMillan form over $k[z]$, the zeros of the numerator polynomials, $\varepsilon_i(z)$, are called the zeros of $G(z)$ and the zeros of the denominator polynomials, $\psi_i(z)$, are called the poles of $G(z)$. Because $G(z)$ is the transmittance matrix and its zeros are physically associated with the transmission-blocking properties of the system it describes, the zeros of $G(z)$ are commonly called transmission zeros. Since the transfer function matrix describes that part of a system which is both reachable and observable, the sets of transmission zeros (t.z.) and decoupling zeros (d.z.) are conceptually distinct.

In addition to the previous definitions, Rosenbrock later defined system zeros [8,9]. System zeros, loosely the combined set of decoupling and transmission zeros, are defined using the Rosenbrock system matrix,

$$\Sigma = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}. \tag{21}$$

In a manner similar to his definition of transmission zeros, Rosenbrock first defined the zeros of a system by putting Σ into Smith form over $k[z]$. The zeros of a system were then defined to be the zeros of the polynomials $\lambda_i(z)$, $1 \le i \le r$, taken all together [8]. However, this definition was later revised in order to establish the exact set equality (g) of the previous section [9]. The first definition of system zeros, although not an appropriate definition to establish this set equality, was in equivalent definitions designated as yielding invariant zeros (i.z.), the roots of the invariant polynomials, $\lambda_i(z)$. Invariant zeros contain the complete set of transmission zeros and some, but not necessarily all, of the decoupling zeros.

The revised definition of zeros of a system is based also upon the system matrix Σ. If Σ has rank $n + r$ where $0 \le r \le min(m, p)$, then consider the $(n + r)$-order nonzero minors of Σ of the form

$$\Sigma^{1,2,\ldots,n,n+i_1,n+i_2,\ldots,n+i_r}_{1,2,\ldots,n,n+j_1,n+j_2,\ldots,n+j_r}. \tag{22}$$

Alternatively, consider that subset of minors of Σ of maximum order containing the first n rows and n columns of Σ. Let $\phi(z)$ be the monic greatest common divisor of all of these minors of Σ. The system zeros (s.z.) are then the complex roots including multiplicities of $\phi(z)$. As noted in [9], the set of invariant zeros and the set of system zeros coincide, for example, when $m = p$ and $det(\Sigma) \neq 0$, that is, Σ is nondegenerate. The set of invariant zeros is a subset of the set of system zeros. The following set equality was shown to be true by Rosenbrock:

$$\{system\ zeros\} = \{transmission\ zeros\} + \{i.d.\ zeros\} + \{o.d.\ zeros\} - \{i.o.d.\ zeros\}. \tag{23}$$

Example 1: Consider a system $S(A, B, C, D)$ satisfying (1) with

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \ B = \begin{bmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{24}$$

Using (17), it can be easily seen that -4 is the only i.d. zero of S. Likewise, (18) indicates that there are no o.d. zeros of S and consequently no i.o.d. zeros. These results can be confirmed using (19) and (20). The Smith-McMillan form of $G(z)$ associated with S over the ring $R[z]$ is

$$\begin{bmatrix} \dfrac{z}{(z+1)(z+2)(z+3)} & 0 \end{bmatrix}; \tag{25}$$

therefore there is one transmission zero at 0. Since the Smith form of Σ associated with S over the ring $R[z]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & z(z+4) & 0 \end{bmatrix}, \tag{26}$$

there are two invariant zeros; one at 0 and one at -4. Notice that the transmission zero is also an invariant zero. Now examine the two 5th-order nonzero minors of Σ containing the first four rows and the first four columns. Then

$$\phi(z) = z(z+4) \tag{27}$$

and the system zeros are at 0 and -4. The set equation (23) is also satisfied.

IV. ZERO THEORIES AND APPLICATIONS

Within the topic of zeros of multivariable systems, the largest class of papers revolves around the use of the transfer function matrix. Among these is a subset which defines finite transmission zeros merely as specific complex numbers satisfying a given property. Structural properties such as multiplicity or invariant factors are not evident within this group.

A common approach involving transfer function matrices is the use of coprime factorization. Desoer and Schulman presented a definition of transmission zeros in terms of a drop in the local rank below normal rank of the "numerator" matrix [10]. Here, the order or multiplicity of such a zero cannot be determined; instead, it is noted that the Smith-McMillan form of $G(z)$ over $k[z]$ must be used (see also [11] where transmission zeros are determined using the Smith form over $k[z]$ of the "numerator" matrix). The definition of transmission zeros given in [10] was generalized by

MacFarlane as those complex numbers which cause a drop in the local rank below normal rank of $G(z)$ [12]. Patel combined coprime factorization with minimal order inverses [13].

Such structural properties as order of a transmission zero prompted other investigations. In much the same way as Rosenbrock's definition of transmission zeros (zeros of a transfer function matrix), many authors choose not only to use the transfer function matrix as a means of investigating multivariable zeros, but also to utilize the Smith-McMillan form of the transfer function matrix over $k[z]$; for example, see [14,15]. In addition, the transmission zeros were shown to be the poles, including multiplicity, of a reflexive generalized inverse [16], a natural extension of the SISO case [10].

MacFarlane was also instrumental in the development of the geometric approach to multivariable zeros. Kouvaritakis and MacFarlane presented geometric interpretations of the locations of invariant zeros in a two-part series of papers [17,18]. All state-space system models involved were assumed to be completely controllable and observable; therefore, the terms invariant zero, transmission zero, and zero were used interchangeably.

The concentration on the state-space description matrices (A, B, C, D) as well as the system matrix Σ in defining multivariable zeros led directly to the invariant zero. Nearly coincident with Rosenbrock's definition of invariant zero [8], Wolovich [19] proposed that zeros of a linear, time-invariant, multivariable system be defined via a system matrix rank test as did Davison and Wang [20]. Other invariant zero definitions have addressed structural questions by using minors [15].

Wonham's geometric approach to linear multivariable control related invariant zeros to the supremal controllability subspace contained in the kernel of C, R^*, and the supremal (A, B)-invariant subspace contained in the kernel of C, V^* [21]. Note that the maximal unobservable subspace defined in [22] is, in fact, V^*. Using minimal system inverses, Bengtsson adopted these geometric concepts and established invariant transmission zeros as a subset of the poles of a left (right) inverse system, if one exists [23]. Under certain conditions, Corfmat and Morse identified transmission zeros in terms of these subspaces [24].

The concept of invariant zero directions, initially developed in [15], was established in conjunction with MacFarlane and Karcanias's invariant zero to characterize the output-zeroing problem. Shaked and Karcanias utilized this theory in model reduction [25], while Kouvaritakis examined the duality between poles and pole directions and zeros and zero directions using inverse systems [26]. Owens established the geometric multiplicity of an invariant zero as the dimension of the subspace generated by its corresponding state zero direction [27].

System zeros have been defined in set equality form using invariant and decoupling zeros [15,27-29]. Structural questions of system zeros are addressed in [9,27].

When studying zeros of multivariable systems, one must specify which definition of multivariable zero pertains to the problem at hand. Individual element zeros in a transfer function matrix are related to transmission zeros only in special cases such as $G(z)$ diagonal. When a system is controllable and observable, the set of system zeros, the set of invariant zeros, and the set of transmission zeros all coincide.

Some investigations into multivariable zeros consider the system or transfer function matrix as a means of obtaining zero information. Zeros have also been approached through the use of modules [30-35]. Those definitions which consider zeros merely as complex numbers, although deceivingly simple, create difficulties and produce little structural information. The use of invariant factors includes multiplicity information while zero directions introduce further insight into the output-zeroing problem. The module theory setting virtually provides all zero information in one concise package.

In order to summarize the developments in the theory of zeros since Rosenbrock's pioneering work, we use the graphical method of Figure 1. Here, solid arrows indicate cause and affect while dashed lines designate relationships. This digraph, established chronologically by publication date from top to bottom, demonstrates pictorially key zero developments relating finite zeros addressed thus far, module zeros mentioned previously and discussed briefly in Section VII [30-35], and infinite zeros examined in Section VI [36-38]. For a more detailed discussion of these ideas, see [39].

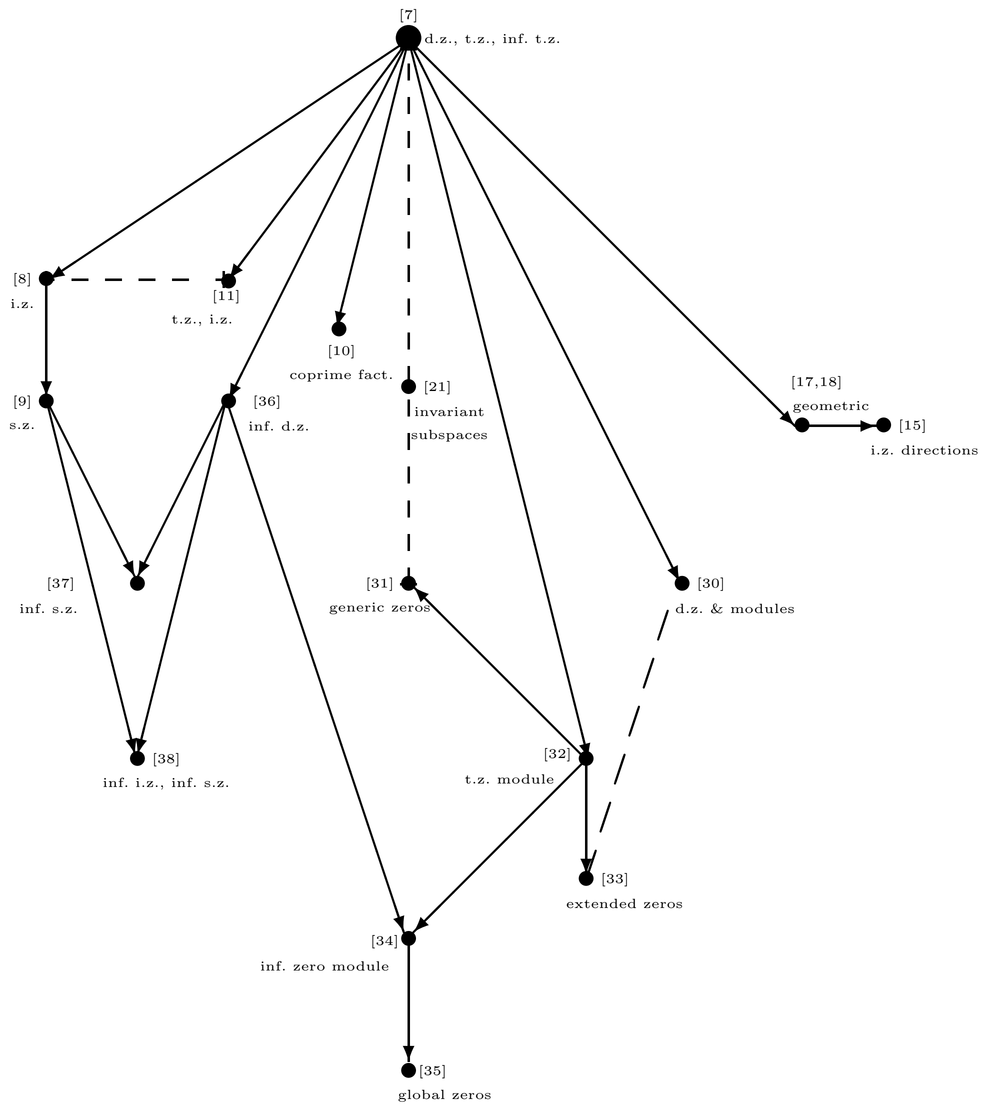

Figure 1. Key Zero Developments

The study of zeros of linear multivariable systems has led to the solution of many control problems. For example, properties of zeros have led to the design of stable high gain feedback systems [40], distributed systems [41], and nonminimum phase systems [42], and conditions for controllability [43] or conditional reachability and observability [44]. The physical output-zeroing problem [45-47] as well as characterization of fixed modes [48-50] have been approached using zeros. Moreover, zeros have been used to resolve stability questions with respect to block decoupling [51]. The effect of the sampling rate in the sampling of a continuous system on the resulting zeros of the sampled system has been examined extensively [52-54]. Also, zeros of discrete systems are extremely important in adaptive control [55,56]. A concentration on assignment or cancellation of zeros is also in evidence in the literature [57-67]. Zeros are used in model reduction [25] and in the selection of weights for LQ regulator design [68]. Additionally, zeros have been defined in systems with delays [69], determined in spacecraft control [70], and investigated in large space structures [71,72]. Zeros are intimately related to the study of linear singular systems [73]. Time-domain behavior due to zero location has been extended to the multivariable case [74]. Theoretically, zeros are used in the hierarchical theory of systems [75] and in greatest common divisor extraction of polynomial matrices [76]. Furthermore, zeros are central in the solution of model matching and factorization problems [77-80] and in the theory of inverse systems [81].

V. ZEROS AND CONTROL LIMITATIONS

In this section, we wish to examine a number of the relationships between zeros and the performance of feedback systems. Of particular interest will be any control or other performance limitations which arise from, and can be stated in terms of, zeros. A number of such relationships have already been mentioned, in quite general terms, in the section preceding; and the reader can refer to the references cited there for more detail. Our purpose in this section is to be more explicit, and to convey certain very basic ideas concerning the role of the zero concept in multivariable systems.

Consider the general feedback configuration of Figure 2. In order to emphasize the fact that the foregoing

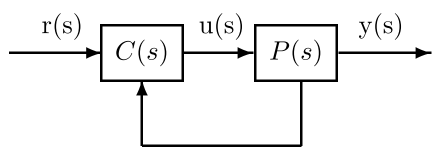

Figure 2

discussions apply equally as well to a Laplace transform scenario as to that of a z-transform, we represent the transform variable as s in place of z. Moreover, in order to simplify the notation somewhat, we shall write a transform $\hat{u}(s)$ without its hat, in the manner $u(s)$. The matrices $P(s)$ and $C(s)$ contain transfer functions of the type discussed in Sections I and III. Clearly, the action of the plant $P(s)$ can be understood in the usual manner by $y(s) = P(s)u(s)$, while that of the controller $C(s)$ should be understood by the calculation

$$u(s) = C(s) \begin{bmatrix} r(s) \\ y(s) \end{bmatrix}, \tag{28}$$

in which the vectors $r(s)$ and $y(s)$ are written one above the other. It is customary to say that the feedback loop of Figure 2 is well posed if $u(s)$ is uniquely determined by $r(s)$. Let this be the case, and denote by $M(s)$ the transfer function matrix which achieves this construction, namely $u(s) = M(s)r(s)$. It is then clear that $y(s)$ is also determined uniquely by $r(s)$ in the manner $y(s) = T(s)r(s)$, and furthermore that

$$T(s) = P(s)M(s). \tag{29}$$

Equation (29) is called a model matching equation. Given $P(s)$, together with a desired relationship $T(s)$ between $y(s)$ and $r(s)$, it follows that there is a fundamental connection between solving (29) for $M(s)$ and designing a controller $C(s)$ in Figure 2.

As the reader may be aware, there are a number of ways in which a model of the plant $P(s)$ may in some way be used as part of the system which produces the controller transfer function matrix $C(s)$. For a survey of ideas of this type, the paper by Garcia, Prett, and Morari [82] has appeared recently. Although (29) is referred to as model matching, it is not necessarily true that controllers based upon it are going to make use of a model of the plant $P(s)$. Rather, (29) is somewhat more basic than that, holding for every feedback control loop which is well posed, no matter how it is designed or what its architecture.

As a matter of fact, whenever (29) is satisfied, there is an architecture for Figure 2 which corresponds directly to it; and this is depicted in Figure 3. Suppose, for instance, that $k = R$, the field of real numbers, that $M(s)$ and $T(s)$

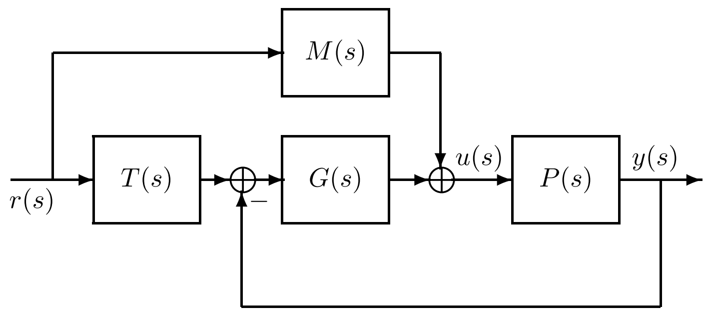

Figure 3

are stable matrices in the sense that each of their elements has a strictly Hurwitz polynomial for its denominator, and that $G(s)$ is designed so that the loop formed by itself with $P(s)$ is also stable. Notice that this last condition does not depend directly on either $M(s)$ or $T(s)$. Then there is clearly a natural association possible between Figures 2-3 and the model matching equation (29). We leave it to the reader to establish that the transfer function matrix between $r(s)$ and $y(s)$ in Figure 3 is $T(s)$, and that the transfer function matrix between $r(s)$ and $u(s)$ is $M(s)$.

Accordingly, we shall examine control limitations in Figure 2 by phrasing them in terms of (29). Let us begin with the single-input, single-output case. Write the transfer functions $m(s)$, $p(s)$, and $t(s)$ in coprime polynomial ratios in the manner

$$m(s) = \frac{n_m(s)}{d_m(s)},\ p(s) = \frac{n_p(s)}{d_p(s)},\ t(s) = \frac{n_t(s)}{d_t(s)}. \tag{30}$$

If $p(s)$ and $t(s)$ are given, with $p(s)$ the plant model and $t(s)$ an output performance specification, then

$$m(s) = \frac{d_p(s)n_t(s)}{n_p(s)d_t(s)}. \tag{31}$$

Note carefully that $n_t(s)$ can have no factor in common with $d_t(s)$, other than a nonzero constant. A similar statement holds for $d_p(s)$ and $n_p(s)$. Thus, if a nontrivial cancellation is to occur between numerator and denominator in (31), it must occur between $n_t(s)$ and $n_p(s)$, or between $d_p(s)$ and $d_t(s)$. Because of these observations, we can write down the following two principles:

Control Zero Principle

The zeros of the controller transfer function $m(s)$ consist of those zeros of $t(s)$ which are not zeros of $p(s)$ and those poles of $p(s)$ which are not poles of $t(s)$.

Control Pole Principle

The poles of the controller transfer function $m(s)$ consist of those zeros of $p(s)$ which are not zeros of $t(s)$ and those poles of $t(s)$ which are not poles of $p(s)$.

Some interesting conclusions follow from these two principles. Observe, once again, that $u(s) = m(s)r(s)$, even in Figure 3. Except for rare instances where poles or zeros of $r(s)$ may cancel with those of $m(s)$, the poles and zeros of $u(s)$ will be made up of those of $m(s)$ and those of $r(s)$. Moreover, if we consider impulse responses, those of $r(s)$ do not come into consideration. In other words, the frequency domain character of $u(s)$ is intrinsically tied to that of $m(s)$.

Consequently, if we desire that the control action transfer function $u(s)$ should have no right-half plane (RHP) zeros whenever, say, $r(s)$ is stable, then the only RHP zeros to appear in $t(s)$ must be those of $p(s)$, in the case when $p(s)$ is stable. In the event that $p(s)$ is not stable, then $u(s)$ will in general display RHP zero behavior even if $p(s)$ does not. With regard to the poles of $u(s)$, it is of course required that they display no RHP character whatsoever as a necessary condition for the feedback system to be stable. This means that every RHP zero of $p(s)$ must be a RHP zero of $t(s)$, where we understand that $t(s)$ is to be designed stable as well.

Example 1: Consider a plant described by

$$p(s) = \frac{s-1}{s-2}. \tag{32}$$

If we select

$$t(s) = \frac{s-1}{s+1}, \tag{33}$$

then by (29) we obtain

$$m(s) = \frac{s-2}{s+1}. \tag{34}$$

Because $p(s)$ is unstable, $m(s)$ displays a RHP zero at the same point. Observe that the factor $(s-2)$, which cancels between $m(s)$ and $p(s)$ in the model matching equation, need not lead to an unstable, uncontrollable or unobservable mode in the architecture of Figure 3. Indeed, suppose that $G(s)$ is chosen to be -1.5. Then the complete set of closed loop poles for the system is given by $-1, -1$, and -1. The first two of these come from $m(s)$ and $t(s)$, respectively, while the third arises from the loop. If desired, $m(s)$ and $t(s)$ may be realized with a single common pole, in the manner

$$\begin{bmatrix} \frac{s-2}{s+1} \\ \\ \frac{s-1}{s+1} \end{bmatrix}, \tag{35}$$

so as to simplify the controller.

We turn now to the case of multiple plant inputs and outputs, when $P(s)$ may have more than one row or column. Is there a way in this more general situation to develop a Control Zero Principle (CZP) and a Control Pole Principle (CPP)? The answer is yes; but a perusal of Section III indicates that the definition of zero is more elaborate and makes use of a Smith-McMillan form. Accordingly, it will be necessary to replace (31) by an appropriate matrix means.

The basic idea which we propose to employ is as follows. We shall say that a matrix $A_z(M(s))$, with elements in the field k, *characterizes the transmission zeros* of a matrix $M(s)$ if the non-unit polynomials in the Smith form of $(sI - A_z(M(s)))$ are identical to the non-unit polynomials in the list $\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)$ which appear in the Smith-McMillan form of $M(s)$ and if the number of rows and columns in $A_z(M(s))$ is equal to the degree of the product $\varepsilon_1(s)\varepsilon_2(s)\ldots\varepsilon_r(s)$.

Example 2: Suppose that

$$M(s) = \begin{bmatrix} \frac{s+1}{s} & 2 \\ 0 & \frac{1}{s+1} \end{bmatrix}. \tag{36}$$

Then $\varepsilon_1(s) = 1,\ \varepsilon_2(s) = s + 1$, and $A_z(M(s)) = [-1]$ may be chosen.

In like fashion, we will also say that a matrix $A_p(M(s))$, with elements in the field k, *characterizes the transmission poles* of a matrix $M(s)$ if the non-unit polynomials in the Smith form of $(sI - A_p(M(s)))$ are identical to the non-unit polynomials in the list $\psi_r(s), \psi_{r-1}(s), \ldots, \psi_1(s)$ which appear in the Smith-McMillan form of $M(s)$ and if the number of rows and columns in $A_p(M(s))$ is equal to the degree of the product $\psi_r(s)\psi_{r-1}(s)\ldots\psi_1(s)$.

Example 3: For the $M(s)$ of (36), we have $\psi_2(s) = 1,\ \psi_1(s) = s(s + 1)$, and $A_p(M(z))$ may be taken as

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{37}$$

Example 4: For Example 1 in Section III, we can choose

$$A_z(G(s)) = [0], \quad A_p(G(s)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}. \tag{38}$$

Notice that the number of rows and columns in A_z or A_p may be zero, when there are no zeros or no poles, respectively.

With these definitions, we can state a CPP and a CZP for the matrix case. It is assumed that the image of $T(s)$, considered as a linear map over the field $k(s)$ of transfer functions, is contained in the image of $P(s)$, also so considered. In this way, we know that there is at least one solution $M(s)$ to the model matching equation (29). We begin with CPP.

Multivariable CPP

There exists a solution $M(s)$ to the model matching equation (29) with the property that its transmission poles are characterized by the matrix

$$A_p(M(s)) = \begin{bmatrix} A_p^z & X_p \\ 0 & A_p^p \end{bmatrix}, \tag{39}$$

in which A_p^z can be found from the relation

$$A_z(P(s)) = \begin{bmatrix} A_p^z & Y_p \\ 0 & A_z([T(s)\ P(s)]) \end{bmatrix}, \tag{40}$$

and A_p^p from the corresponding equation

$$A_p([T(s)\ P(s)]) = \begin{bmatrix} A_p(P(s)) & Z_p \\ 0 & A_p^p \end{bmatrix}. \tag{41}$$

Furthermore, $A_p(M(s))$ has the minimum number of rows and columns of any matrix characterizing the transmission poles of a solution to (29); and any other matrix A_p having this size and characterizing the transmission poles of a solution to (29) has the same non-unit polynomials in the Smith form of $(sI - A_p)$.

Remark: All the matrices denoted by A in this CPP are square. Matrices X_p, Y_p, and Z_p exist, but are specified only in a concrete example. Moreover, if we form the characteristic polynomial of (39), then

$$|sI - A_p(M(s))| = |sI - A_p^z||sI - A_p^p|, \tag{42}$$

which tells us that the minimal set of poles in a solution $M(s)$ to (29) consists of those zeros associated with A_p^z and those poles associated with A_p^p. Moreover, from (40), we see that the zeros arising from A_p^z are those zeros from $A_z(P(s))$ which are not zeros from $A_z([T(s)\ P(s)])$. Finally, from (41), we have that the poles arising from A_p^p are those poles from $A_p([T(s)\ P(s)])$ which are not poles from $A_p(P(s))$. A comparison with the single-input, single-output version of CPP shows great similarity, except that $[T(s)\ P(s)]$ replaces $t(s)$, which is one of the novelties of the multi-input, multi-output control problem. The Multivariable CPP (MCPP) can be deduced from the researches of Conte, Perdon, and Wyman [83], who stated their results in an entirely different way, for other purposes.

Example 5: Suppose that $T(s) = I$, so that the feedback control system is solving an inverse problem. In this case,

$$A_p([T(s)\ P(s)]) = A_p(P(s)), \tag{43}$$

so that A_p^p has zero rows and zero columns. Then (39) is determined by A_p^z. Moreover, in this case, the Smith-McMillan form for $[T(s)\ P(s)]$ has all unit numerators, so that $A_z([T(s)\ P(s)])$ also has zero rows and zero columns. Then

$$A_p(M(s)) = A_z(P(s)), \tag{44}$$

where $M(s)$ is one of the solutions described in MCPP. This indicates, of course, that "the poles of $M(s)$ must cancel the zeros of P(s)." Notice also that neither $P(s)$ nor $M(s)$ is required to be square.

The situation with regard to CZP is more involved. In fact, when we mention recent developments in Section VII, we will dwell briefly on the nature of this added complication. The foundation for a multi-input, multi-output CZP has been laid by Sain, Wyman, and Peczkowski [80]. There are three new issues that have to be addressed in this situation. First, there *is not* always a solution $M(s)$ with "smallest" zero structure. Note that this is in marked contrast to MCPP. Second, there *is* a "smallest" zero structure, nonetheless. At first, this may sound like a contradiction; but it is not. It is just this: one can deduce a zero structure which must be contained in the zero structure of all solutions $M(s)$; but one cannot always find a solution $M(s)$ which has precisely this zero structure. Third, the "smallest" zero structure involves a generalized notion of zero, which we will describe in Section VII. The most direct way to simplify this discussion is to require that $T(s)$ be onto, as a linear mapping over the transfer function field $k(s)$. Then, for a solution to exist, $P(s)$ must be onto as well. Finally, we shall examine only solutions $M(s)$ which are onto also. This means, for example, that there must be no more controls than there are reference commands. Under these agreements, we get a CZP.

Multivariable CZP

Under the conditions described in the paragraph preceding, there exists a solution $M(s)$ to the model matching equation (29) with the property that its transmission zeros are characterized by the matrix

$$A_z(M(s)) = \begin{bmatrix} A_z^z & X_z \\ 0 & A_z^p \end{bmatrix}, \tag{45}$$

in which A_z^z can be found from the relation

$$A_z(T(s)) = \begin{bmatrix} A_z^z & Y_z \\ 0 & A_z([T(s)\ P(s)]) \end{bmatrix}, \tag{46}$$

and A_z^p from the corresponding equation

$$A_p([T(s)\ P(s)]) = \begin{bmatrix} A_p(T(s)) & Z_z \\ 0 & A_z^p \end{bmatrix}. \tag{47}$$

Furthermore, $A_z(M(s))$ has the minimum number of rows and columns of any matrix characterizing the transmission zeros of a solution to (29), under the conditions outlined in the preceding paragraph; and any other matrix A_z having this size and characterizing the transmission zeros of a solution to (29) under these same conditions has the same non-unit polynomials in the Smith form of $(sI - A_z)$.

Remark: As in the interpretation of MCPP, this Multivariable CZP (MCZP) has the possibility to be understood as follows. The minimal set of zeros in a solution $M(s)$ to (29), under the conditions preceding the statement of MCZP, consists of those zeros associated with A_z^z and those poles associated with A_z^p. In turn, from (46), we have that the zeros arising from A_z^z are those zeros from $A_z(T(s))$ which are not zeros from $A_z([T(s)\ P(s)])$. Moreover, from (47), we see that the poles arising from A_z^p are those poles from $A_p([T(s)\ P(s)])$ which are not poles from $A_p(T(s))$. In this instance, $[T(s)\ P(s)]$ has replaced $P(s)$ in the single-input, single-output CZP.

Example 6: Suppose again, as in Example 5, that $T(s) = I$. Then $A_z(T(s))$ has zero rows and zero columns, as do $A_z([T(s)\ P(s)])$ and A_z^z. Thus $A_z(M(s))$ derives its character from A_z^p. But (43) and (47) tell us that

$$|sI - A_p(P(s))| = |sI - A_p(T(s))|\ |sI - A_z^p|, \tag{48}$$

so that $A_z(M(s))$ is supplying those poles from $A_p(P(s))$ which are not poles from $A_p(T(s))$. However, $A_p(T(s))$ has zero rows and zero columns, so that A_z^p is essentially just $A_p(P(s))$, which means that "the zeros of $M(s)$ must cancel the poles of $P(s)$."

In this section, we have developed a control zero principle CZP and a control pole principle CPP for the single-input, single-output case; and we have generalized them to principles MCZP and MCPP for the case of multiple inputs and outputs. These principles show the fundamental role that poles and zeros and their multivariable structure play in feedback design. We wish to conclude the section with a series of remarks.

Remark: The matrix $A_p(M(s))$ can be understood as the state matrix in a minimal realization of $M(s)$. It is unique up to a similarity transformation. If $M(s)$ has an inverse, then $A_z(M(s))$ is equal to $A_p(M^{-1}(s))$. This is helpful in developing pedagogical examples.

Remark: If the spectra of A_p^z and A_p^p are disjoint, then X_p may be chosen to be zero. Similarly, if the spectra of A_z^z and A_z^p are disjoint, then X_z may be chosen to be zero. Similar conclusions can be made in (40) and (41), or in (46) and (47), if the corresponding assumptions are made. For example, in the case of (41) and (47), let $T(s)$ and $P(s)$ have no poles in common.

Remark: All these discussions are directed toward zeros in the finite plane. The next section touches upon zeros at infinity.

VI. ZEROS AT INFINITY

The zeros of multivariable systems discussed up to this point have been considered to be finite: zeros at finite points in the complex z plane. This is due to the fact that the unimodular matrices $U_1(z)$ and $U_2(z)$ used to generate the Smith or Smith-McMillan form over the polynomial ring $k[z]$ do not retain information regarding behavior at $z = \infty$ as they can modify poles and zeros at infinity by both introduction and or deletion [1].

Alternatively, consider the ring O_∞, the ring of proper rational functions corresponding to the valuation

$$ord_\infty(n(z)/d(z)) = deg\ d(z) - deg\ n(z) \tag{49}$$

where $n(z)$ and $d(z)$ lie in $k[z]$. Now the Smith-McMillan form at infinity [34,84] is obtained using unimodular matrices over the ring O_∞, not $k[z]$. For a $p \times m$ rational matrix $G(z)$ of rank r, there exist unimodular matrices over O_∞, $U_{\infty_1}(z)$ and $U_{\infty_2}(z)$, such that

$$G(z) = U_{\infty_1}(z)\Lambda_\infty(z)U_{\infty_2}(z) \tag{50}$$

where

$$\Lambda_\infty(z) = \begin{bmatrix} \Lambda^*_\infty(z)_{r,r} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix} \tag{51}$$

and

$$\Lambda^*_\infty(z) = diag(z^{-\nu_1}, \ldots, z^{-\nu_r}); \quad \nu_1 \leq \ldots \leq \nu_r. \tag{52}$$

Then $G(z)$ has a zero at ∞ of order ν_j for any positive ν_j.

Example 1: Consider the system $S(A, B, C, D)$ of Example 1 in Section III. Then for

$$U_{\infty_1}(z) = \frac{z^2}{(z+1)(z+2)} \tag{53}$$

and

$$U_{\infty_2}(z) = \begin{bmatrix} 1 & \frac{2(z+2)}{z+3} \\ 0 & 1 \end{bmatrix} \tag{54}$$

in (50),

$$\Lambda_\infty(z) = [z^{-1} \quad 0] \tag{55}$$

and $G(z)$ has a zero at ∞ of order 1. To check that $U_{\infty_1}(z)$ and $U_{\infty_2}(z)$ are indeed unimodular over O_∞, verify that the reciprocals of their determinants lie in O_∞. Alternatively, note that the numerator degree of each determinant must equal its denominator degree.

Transmission zeros at infinity have received considerable attention in the literature [7,84-86]. Infinite transmission zeros are important in determining the asymptotic behavior of multivariable root loci [87-90]. They have also been investigated in the light of the problems of decoupling [91], realization [38], and model matching [92].

At this point, it is appropriate to mention some investigations into the infinite counterparts of the other finite zero definitions studied in Section III. Decoupling zeros at infinity have been defined [36,93] and the topic of infinite invariant zeros has been addressed [38]. In addition, [38] contains an infinite system zero definition and [37] extends Rosenbrock's system zero definition to consider the point at infinity.

We turn now to a brief discussion of the way in which the use of a ring such as O_∞, in place of $k[z]$, impacts the model matching equation (29), where we interpret this expression in terms of z rather than s. Notice that although the transfer functions in $T(z)$, $P(z)$, and $M(z)$ form a field, which is denoted by $k(z)$, it is not necessarily of interest to solve (29) with regard to that field. For example, if $P(z)$ is unstable as in Example 1 of Section V, we nonetheless are interested in stable solutions for $T(z)$ and $M(z)$; and one such pair is indicated in (33) and (34). In the field of transfer functions, the subset of stable transfer functions is not a field in its own right; rather, it is a ring. This ring differs from a field in just one operation: one cannot always divide. Thus, the transfer function $(z-1)$ is stable (i.e., has denominator 1 which is strictly Hurwitz); but its reciprocal $1/(z-1)$ is not. This means that the coefficients in our equation, namely elements of the matrix $P(z)$, lie in the field $k(z)$, whereas our desired matrices $T(z)$ and $M(z)$ have elements which are more restricted. Thus, the "standard" matrix theory, based upon vector spaces, falls just a bit short of resolving the issues. This is one reason for making so much use of the ring $k[z]$ in Section III, in place of the field $k(z)$. Indeed, if we attempted to form "Smith-McMillan" forms for a transfer function over $k(z)$, all the "$\lambda_i(z)$" would be 1; and no zero or pole information would make itself evident. Thus, the use of the ring $k[z]$, in place of the field $k(z)$, can be viewed as essential to being able to "see" the poles and zeros of the transfer function matrix.

A similar situation occurs in (29) if we look for causal pairs $(T(z),\ M(z))$ which satisfy the equation for a given $P(z)$. Then we seek solutions $T(z)$ and $M(z)$ whose elements are proper transfer functions, while $P(z)$ itself may be proper or not. Intuitively, a matrix of transfer functions is proper if it has no poles at infinity; and so the question of zeros and poles, this time at infinity, can arise once again in this new context.

Let us briefly sketch the situation in a single-input single-output case. Here, in place of (30), we shall write

$$p(z) = \frac{n_p(z)/a(z)}{d_p(z)/a(z)},\ t(z) = \frac{n_t(z)/b(z)}{d_t(z)/b(z)}, \tag{56}$$

where $a(z)$ is a nonzero polynomial of degree equal to the maximum of the degrees of $n_p(z)$ and $d_p(z)$, and where $b(z)$ is chosen in like manner. Equation (56) can be understood as a factorization of $p(z)$ and $t(z)$ over the ring O_∞ of proper transfer functions, instead of over the ring $k[z]$ of polynomials. Notice that $a(z)$, say, can have no nontrivial factor in common with both $n_p(z)$ and $d_p(z)$, because these last two ploynomials were chosen to be relatively prime. Just as the factorization (30) is unique up to nonzero constants, so also is the factorization (56) unique up to transfer functions which have equal degree in both numerator and denominator. If $f(z)$ is such a transfer function, then

$$\frac{[n_p(z)/a(z)]f(z)}{[d_p(z)/a(z)]f(z)} \tag{57}$$

is another such factorization, and so forth.

Now refer once again to (31). In our present set of considerations, we can write

$$m(z) = \frac{[d_p(z)/a(z)][n_t(z)/b(z)]}{[n_p(z)/a(z)][d_t(z)/b(z)]}. \tag{58}$$

Without loss of generality, assume that $d_p(z)$ and $a(z)$ have equal degrees, and that $d_t(z)$ and $b(z)$ have equal degrees. The reader should have no difficulty making the appropriate adjustments in case they do not. Then, as z gets large, we may as well examine

$$\frac{z^{-(deg\ b(z) - deg\ n_t(z))}}{z^{-(deg\ a(z) - deg\ n_p(z))}}. \tag{59}$$

Therefore, it is possible to think of zeros at infinity of $t(z)$ which are not zeros at infinity of $p(z)$, toward a control zero principle. Or, toward a control pole principle, one can look at zeros at infinity of $p(z)$ which are not zeros at infinity of $t(z)$. The other situations of CZP and CPP in Section V arise in case $d_p(z)$ and $a(z)$ do not have equal degrees, and so forth.

In like manner, an MCZP and an MCPP can be developed. The procedure is also based upon [80], where the result follows merely by changing the ring from $k[z]$ to O_∞, and the "A" matrices are nilpotent. The details of such methods are omitted, because of space limitations. We shall, however, mention them briefly in the next section. It is possible, moreover, to employ a ring O_{ps} of proper and stable transfer functions. Here one can appeal either to the method of [80] or to the "global" approach of [94].

VII. RECENT DEVELOPMENTS

The current research on zeros has recently taken a number of new directions. Time-varying zeros, nonlinear zeros, and generalized or extended zeros are mentioned and briefly explicated.

Linear time-varying systems have been examined from a zero viewpoint. Zeros of SISO linear time-varying systems have been defined with emphasis on the discrete-time case in terms of noncommutative factorizations of operator polynomials [95,96]. Consider the commutative ring A (A_C), the set of all real-valued (complex-valued) functions defined on the set of integers. Then the skew ring $A[z]$ ($A_C[z]$) is the set of all operator polynomials in the left shift operator z with coefficients in A (A_C) with the usual polynomial addition and the skew polynomial multiplication, $\circ$, defined by

$$\begin{aligned} z^i \ \circ \ z^j &= z^{i+j} \\ z^i \ \circ \ a(j) &= a(j+i)z^i, \ a(j) \ \epsilon \ A \ (A_C). \end{aligned} \tag{60}$$

The input-output equation for a SISO linear discrete-time system can be described in terms of

$$a(z,j)y(j) = b(z,j)u(j) \tag{61}$$

where

$$a(z,j) = z^n + \sum_{i=0}^{n-1} a_i(j)z^i \ \epsilon \ A[z] \tag{62}$$

and

$$b(z,j) = \sum_{i=0}^{n} b_i(j)z^i \ \epsilon \ A[z]. \tag{63}$$

If there exists a $h(z,j) \ \epsilon \ A_C[z]$ such that

$$b(z,j) = h(z,j) \ \circ \ [z - q(j)], \tag{64}$$

then $q(j) \ \epsilon \ A_C$ is a right zero of the system. Moreover, this zero notion can be interpreted as a transmission-blocking zero. Left, inner, and outer zeros can be defined in like manner.

Recently, a special class of such systems, discrete-time periodic systems, has been examined in the multivariable case in regard to the structure at infinity, and, thus, the zeros at infinity [97]. In addition, the notions of transmission zero and invariant zero have been extended to this case [98]. A study of module results with respect to zeros for SISO discrete systems over a fixed finite field, their related periodic behavior, and their connection to algebraic coding theory has been produced [99]. The concept of the zero module for multivariable systems over rings has also been investigated [100].

Recent investigations have also introduced the idea of zeros for nonlinear systems. Suppose we have a nonlinear, single-input, single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x). \end{aligned} \tag{65}$$

By coordinate transformations [101], it is possible under reasonable assumptions to transform (65) into a normal form

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, 2, \ldots, r-1, \\ \dot{z}_r &= b(z_1, \ldots, z_r, \eta) + a(z_1, \ldots, z_r, \eta)u, \\ \dot{\eta} &= q(z_1, \ldots, z_r, \eta). \end{aligned} \tag{66}$$

The zero dynamics of (65) are described by

$$\dot{\eta} = q(0, \ldots, 0, \eta), \tag{67}$$

where the vector η has dimension equal to the difference between that of x and r, and where r is called the relative degree of the system. These zero dynamics are closely related to certain linear ideas of zeros, and play a role analogous to that of zeros in the solution of control problems.

Finally, in the linear case, there are generalizations and extensions of the notion of zero [80]. The idea is based upon the use of rings as scalars multiplying a commutative group of vectors. One can visualize this quite easily by thinking of vector spaces in which it is not always possible to divide by a scalar. In such a situation, one speaks of a module, instead of a vector space. The module is well suited to discuss poles and zeros, inasmuch as modules, poles, and zeros are intrinsically related to rings (e.g., $k[z],\ O_\infty,\ O_{ps}$).

With the use of modules, we can extend the notion of zero to include generalizations having a character different from, and yet somewhat similar to, the ideas of Sections III and VI. These extensions are "invisible" unless we have both multiple inputs and multiple outputs, and in that case only when the matrix of transfer functions is not square, or has a zero determinant. It is these types of zeros that account for the special treatment of MCZP in Section V.

In a module approach to zeros, they are regarded as "spaces" instead of "points"; and in this sense the new type of zero is a vector space over the field k.

VIII. CONCLUSIONS

This paper has discussed the performance of feedback systems in terms of poles and zeros. An historical account of the development of system zeros has been provided, with permission, in the words of the concept originator, H. H. Rosenbrock. Consensus zero definitions have been presented, and a discussion of current trends and developments in zero theory and application has been provided. A control zero principle (CZP) and a control pole principle (CPP) have been developed to explain the role of poles and zeros in feedback system performance. With the aid of two new results, these principles have been extended to very general multivariable cases. Zeros at infinity, and their relation to controller causality, have been examined, and the approach to CZP and CPP also sketched. Finally, extensions of the idea of zero to time-varying systems, to nonlinear systems, and to zero spaces or modules have been briefly laid out.

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