CHAPTER 2

Classification of Bundles

In this chapter we prove Steenrod's classification theorem of principal G - bundles, and the corresponding classification theorem of vector bundles. This theorem states that for every group G, there is a "classifying space" BG with a well defined homotopy type so that the homotopy classes of maps from a space X, $[X, BG]$, is in bijective correspondence with the set of isomorphism classes of principal G - bundles, $Prin_G(X)$. We then describe various examples and constructions of these classifying spaces, and use them to study structures on principal bundles, vector bundles, and manifolds.

1. The homotopy invariance of fiber bundles

The goal of this section is to prove the following theorem, and to examine certain applications such as the classification of principal bundles over spheres in terms of the homotopy groups of Lie groups.

Theorem 2.1. *Let $p : E \to B$ be a fiber bundle with fiber F, and let $f_0 : X \to B$ and $f_1 : X \to B$ be homotopic maps.Then the pull - back bundles are isomorphic,*

$$f_0^*(E) \cong f_1^*(E).$$

The main step in the proof of this theorem is the basic *Covering Homotopy Theorem* for fiber bundles which we now state and prove.

Theorem 2.2. **Covering Homotopy theorem.** *Let $p_0 : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber, F, where B is normal and locally compact. Let h_0 be a bundle map*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h}_0} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[h_0]{} & Y \end{array}$$

Let $H : B \times I \to Y$ be a homotopy of h_0 (i.e $h_0 = H_{|_{B\times\{0\}}}$.) Then there exists a covering of the homotopy H by a bundle map

$$\begin{array}{ccc} E \times I & \xrightarrow{\tilde{H}} & Z \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow[H]{} & Y. \end{array}$$

Proof. We prove the theorem here when the base space B is compact. The natural extension is to when B has the homotopy type of a CW - complex. The proof in full generality can be found in Steenrod's book [**39**].

The idea of the proof is to decompose the homotopy H into homotopies that take place in local neighborhoods where the bundle is trivial. The theorem is obviously true for trivial bundles, and so the homotopy H can be covered on each local neighborhood. One then must be careful to patch the coverings together so as to obtain a global covering of the homotopy H.

Since the space X is compact, we may assume that the pull - back bundle $H^*(Z) \to B \times I$ has locally trivial neighborhoods of the form $\{U_\alpha \times I_j\}$, where $\{U_\alpha\}$ is a locally trivial covering of B (i.e there are local trivializations $\phi_{\alpha,\beta} : U_\alpha \times F \to p^{-1}(U_\alpha)$), and $I_1, \cdots, I_r$ is a finite sequence of open intervals covering $I = [0,1]$, so that each I_j meets only I_{j-1} and I_{j+1} nontrivially. Choose numbers

$$0 = t_0 < t_1 < \cdots < t_r = 1$$

so that $t_j \in I_j \cap I_{j+1}$. We assume inductively that the covering homotopy $\tilde{H}(x,t)$ has been defined $E \times [0, t_j]$ so as to satisfy the theorem over this part.

For each $x \in B$, there is a pair of neighborhoods (W, W') such that for $x \in W$, $\bar{W} \subset W'$ and $\bar{W}' \subset U_\alpha$ for some U_α. Choose a finite number of such pairs (W_i, W_i'), $(i = 1, \cdots, s)$ covering B. Then the Urysohn lemma implies there is a map $u_i : B \to [t_j, t_{j+1}]$ such that $u_i(\bar{W}_i) = t_{j+1}$ and $u_j(B - W_i') = t_j$. Define $\tau_0(x) = t_j$ for $x \in B$, and

$$\tau_i(x) = max(u_1(x), \cdots, u_i(x)), \quad x \in B, \quad i = 1, \cdots, s.$$

Then

$$t_j = \tau_0(x) \le \tau_1(x) \le \cdots \le t_s(x) = t_{j+1}.$$

Define B_i to be the set of pairs (x,t) such that $t_j \le t \le \tau_i(x)$. Let E_i be the part of $E \times I$ lying over B_i. Then we have a sequence of total spaces of bundles

$$E \times t_j = E_0 \subset E_1 \subset \cdots \subset E_s = E \times [t_j, t_{j+1}].$$

We suppose inductively that $\tilde{H}$ has been defined on E_{i-1} and we now define its extension over E_i.

By the definition of the τ's, the set $B_i - B_{i-1}$ is contained in $W_i' \times [t_j, t_{j+1}]$; and by the definition of the W's, $\bar{W}'_i \times [t_j, t_{j+1}] \subset U_\alpha \times I_j$ which maps via H to a locally trivial neighborhood, say V_k, for $q : Z \to Y$. Say $\phi_k : V_k \times F \to q^{-1}(V_k)$ is a local trivialization. In particular we can define $\rho_k : q^{-1}(V_k) \to F$ to be the inverse of ϕ_k followed by the projection onto F. We now define

$$\tilde{H}(e,t) = \phi_k(H(x,t), \rho(\tilde{H}(e, \tau_{i-1}(x)))$$

where $(e,t) \in E_i - E_{i-1}$ and $x = p(e) \in B$.

It is now a straightforward verification that this extension of $\tilde{H}$ is indeed a bundle map on E_i. This then completes the inductive step. □

We now prove theorem 2.1 using the covering homotopy theorem.

Proof. Let $p : E \to B$, and $f_0; X \to B$ and $f_1 : X \to B$ be as in the statement of the theorem. Let $H : X \times I \to B$ be a homotopy with $H_0 = f_0$ and $G_1 = f_1$. Now by the covering homotopy theorem there is a covering homotopy $\tilde{H} : f_0^*(E) \times I \to E$ that covers $H : X \times I \to B$. By definition this defines a map of bundles over $X \times I$, that by abuse of notation we also call $\tilde{H}$,

$$\begin{array}{ccc} f_0^*(E) \times I & \xrightarrow{\tilde{H}} & H^*(E) \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow[=]{} & X \times I. \end{array}$$

This is clearly a bundle isomorphism since it induces the identity map on both the base space and on the fibers. Restricting this isomorphism to $X \times \{1\}$, and noting that since $H_1 = f_1$, we get a bundle isomorphism

$$\begin{array}{ccc} f_0^*(E) & \xrightarrow[\cong]{\tilde{H}} & f_1^*(E) \\ \downarrow & & \downarrow \\ X \times \{1\} & \xrightarrow[=]{} & X \times \{1\}. \end{array}$$

This proves theorem 2.1 □

We now derive certain consequences of this theorem.

Corollary 2.3. *Let $p : E \to B$ be a principal G - bundle over a connected space B. Then for any space X the pull back construction gives a well defined map from the set of homotopy classes of maps from X to B to the set of isomorphism classes of principal G - bundles,*

$$\rho_E : [X, B] \to Prin_G(X).$$

Definition 2.1. A principal G - bundle $p : EG \to BG$ is called *universal* if the pull back construction

$$\rho_{EG} : [X, BG] \to Prin_G(X)$$

is a bijection for every space X. In this case the base space of the universal bundle BG is called a *classifying space* for G (or for principal G - bundles).

The main goal of this chapter is to prove that universal bundles exist for every group G, and that the classifying spaces are unique up to homotopy type.

Applying theorem 2.1 to vector bundles gives the following, which was claimed at the end of chapter 1.

Corollary 2.4. *If $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic, they induce the same homomorphism of abelian monoids,*

$$f_0^* = f_1^* : Vect^*(Y) \to Vect^*(X)$$
$$Vect^*_{\mathbb{R}}(Y) \to Vect^*_{\mathbb{R}}(X)$$

and hence of K theories

$$f_0^* = f_1^* : K(Y) \to K(X)$$
$$KO(Y) \to KO(X)$$

Corollary 2.5. *If $f : X \to Y$ is a homotopy equivalence, then it induces isomorphisms*

$$f^* : Prin_G(Y) \xrightarrow{\cong} Prin_G(X)$$
$$Vect^*(Y) \xrightarrow{\cong} Vect^*(X)$$
$$K(Y) \xrightarrow{\cong} K(X)$$

Corollary 2.6. *Any fiber bundle over a contractible space is trivial.*

Proof. If X is contractible, it is homotopy equivalent to a point. Apply the above corollary. □

The following result is a classification theorem for bundles over spheres. It begins to describe why understanding the homotopy type of Lie groups is so important in Topology.

Theorem 2.7. *There is a bijective correspondence between principal bundles and homotopy groups*

$$Prin_G(S^n) \cong \pi_{n-1}(G)$$

where as a set $\pi_{n-1}G = [S^{n-1}, x_0; G, \{1\}]$*, which refers to (based) homotopy classes of basepoint preserving maps from the sphere* S^{n-1} *with basepoint* $x_0 \in S^{n-1}$*, to the group* G *with basepoint the identity* $1 \in G$.

Proof. Let $p : E \to S^n$ be a G - bundle. Write S^n as the union of its upper and lower hemispheres,

$$S^n = D^n_+ \cup_{S^{n-1}} D^n_-.$$

Since D^n_+ and D^n_- are both contractible, the above corollary says that E restricted to each of these hemispheres is trivial. Morever if we fix a trivialization of the fiber of E at the basepoint $x_0 \in S^{n-1} \subset S^n$, then we can extend this trivialization to both the upper and lower hemispheres. We may therefore write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

where θ is a clutching function defined on the equator, $\theta : S^{n-1} \to G$. That is, E consists of the two trivial components, $(D^n_+ \times G)$ and $(D^n_- \times G)$ where if $x \in S^{n-1}$, then $(x, g) \in (D^n_+ \times G)$ is identified with $(x, \theta(x)g) \in (D^n_- \times G)$. Notice that since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}G.$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow[\cong]{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

By restricting to the hemispheres, the isomorphism Ψ defines maps

$$\Psi_+ : D^n_+ \to G$$

and

$$\Psi_- : D^n_- \to G$$

which both map the basepoint $x_0 \in S^{n-1}$ to the identity $1 \in G$, and furthermore have the property that for $x \in S^{n-1}$,

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x),$$

or, $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. Now by considering the linear homotopy $\Psi_+(tx)\theta_1(x)\Psi_-(tx)^{-1}$ for $t \in [0, 1]$, we see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, where the two zeros in this description refer to the origins of D^n_+ and D^n_- respectively, i.e the north and south poles of the sphere S^n. Now since Ψ_+ and Ψ_- are defined on connected spaces, their images lie in a connected component of the group G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t = 0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t = 1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\Theta : Prin_G(S^n) \to \pi_{n-1}G$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

as seen in our discussion of clutching functions in chapter 1.

We now show that Θ is injective. That is, suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2 As above we write

$$E_1 = (D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G)$$

and

$$E_2 = (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

Let $H : S^{n-1} \times [-1, 1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator S^{n-1} in S^n with $S^{n-1} \times [-1, 1]$ Write $\mathcal{D}_+ = D^2_+ \cup \bar{\mathcal{N}}$ and $\mathcal{D}_- = D^2_- \cup \bar{\mathcal{N}}$ Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \bar{\mathcal{N}} \cong S^{n-1} \times [-1, 1].$$

Thus we may form the principal G - bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_- \times G$$

where by abuse of notation, H refers to the composition

$$\bar{\mathcal{N}} \cong S^{n-1} \times [-1, 1] \xrightarrow{H} G\cdot$$

We leave it to the interested reader to verify that E is isomorphic to both E_1 and E_2. This completes the proof of the theorem. □

2. Universal bundles and classifying spaces

The goal of this section is to study universal principal G - bundles, the resulting classification theorem, and the corresponding classifying spaces. We will discuss several examples including the universal bundle for any subgroup of the general linear group. We postpone the proof of the existence of universal bundles for all groups until the next section.

In order to identify universal bundles, we need to recall the following definition from homotopy theory.

DEFINITION 2.2. A space X is said to be *aspherical* if all of its homotopy groups are trivial,

$$\pi_n(X) = 0 \quad \text{for all } n \geq 0.$$

Equivalently, a space X is aspherical if every map from a sphere $S^n \to X$ can be extended to a map of its bounding disk, $D^{n+1} \to X$.

Note. A famous theorem of J.H.C. Whitehead states that if X has the homotopy type of a CW - complex, then X being aspherical is equivalent to X being contractible (see [**44**]).

The following is the main result of this section. It identifies when a principal bundle is universal.

THEOREM 2.8. *Let $p : E \to B$ be a principal G - bundle, where the total space E is aspherical. Then this bundle is universal* in the sense that if X is any space, the induced pull-back map

$$\psi : [X, B] \to Prin_G(X)$$
$$f \to f^*(E)$$

is a bijective correspondence.

For the purposes of these notes we will prove the theorem in the setting where the action of G on the total space E is *cellular*. That is, there is a CW - decomposition of the space E which, in an appropriate sense, is respected by the group action. There is very little loss of generality in these assumptions, since the actions of compact Lie groups on manifolds, and algebraic actions on projective varieties satisfy this property. For the proof of the theorem in its full generality we refer the reader to Steenrod's book [**39**], and for a full reference on equivariant CW - complexes and how they approximate a wide range of group actions, we refer the reader to [**24**]

In order to make the notion of cellular action precise, we need to define the notion of an *equivariant CW - complex,* or a G - CW - complex. The idea is the following. Recall that a CW - complex is a space that is made up out of disks of various dimensions whose interiors are disjoint. In particular it can be built up skeleton by skeleton, and the $(k+1)^{st}$ skeleton $X^{(k+1)}$ is constructed out of the k^{th} skeleton $X^{(k)}$ by attaching $(k+1)$ - dimensional disks via "attaching maps", $S^k \to X^{(k)}$.

A "G - CW - complex" is one that has a group action so that the orbits of the points on the interior of a cell are uniform in the sense that each point in a cell D^k has the same isotropy subgroup, say H, and the orbit of a cell itself is of the form $G/H \times D^k$. This leads to the following definition.

DEFINITION 2.3. A G - CW - complex is a space with G -action X which is topologically the direct limit of G - invariant subspaces $\{X^{(k)}\}$ called the equivariant skeleta,

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(k-1)} \subset X^{(k)} \subset \cdots X$$

where for each $k \geq 0$ there is a countable collection of k dimensional disks, subgroups of G, and maps of boundary spheres

$$\{D_j^k,\, H_j < G,\, \phi_j : \partial D_j^k \times G/H_j = S_j^{k-1} \times G/H_j \to X^{(k-1)} \quad j \in I_k\}$$

so that

(1) Each "attaching map" $\phi_j : S_j^{k-1} \times G/H_j \to X^{(k-1)}$ is G -equivariant, and

(2)
$$X^{(k)} = X^{k-1)} \bigcup_{\phi_j\, j \in I_j} (D_j^k \times G/H_j).$$

This notation means that each " disk orbit " $D_j^k \times G/H_j$ is attached to $X^{(k-1)}$ via the map $\phi_j : S_j^{k-1} \times G/H_j \to X^{(k-1)}$.

We leave the following as an exercise to the reader.

Exercise. Prove that when X is a G - CW complex the orbit space X/G has the an induced structure of a (non-equivariant) CW - complex.

Note. Observe that in a G -CW complex X with a *free* G action, all disk orbits are of the form $D^k \times G$, since all isotropy subgroups are trivial.

We now prove the above theorem under the assumption that the principal bundle $p : E \to B$ has the property that with respect to group action of G on E, then E has the structure of a G - CW - complex. The basespace is then given the induced CW - structure. The spaces X in the statement of the theorem are assumed to be of the homotopy type of CW - complexes.

Proof. We first prove that the pull - back map

$$\psi : [X, B] \to Prin_G(X)$$

is surjective. So let $q : P \to X$ be a principal G - bundle, with P a G - CW - complex. We prove there is a G - equivariant map $h : P \to E$ that maps each orbit pG homeomorphically onto its image, $h(y)G$. We prove this by induction on the equivariant skeleta of P. So assume inductively that the map h has been constructed on the $(k-1)$ - skeleton,

$$h_{k-1} : P^{(k-1)} \to E.$$

Since the action of G on P is free, all the k - dimensional disk orbits are of the form $D^k \times G$. Let $D_j^k \times G$ be a disk orbit in the G-CW - structure of the k - skeleton $P^{(k)}$. Consider the disk $D_j^k \times \{1\} \subset D_j^k \times G$. Then the map h_{k-1} extends to $D_j^k \times \{1\}$ if and only if the composition

$$S_j^{k-1} \times \{1\} \subset S_j^{k-1} \times G \xrightarrow{\phi_j} P^{(k-1)} \xrightarrow{h_{k-1}} E$$

is null homotopic. But since E is aspherical, any such map is null homotopic and extends to a map of the disk, $\gamma : D_j^k \times \{1\} \to E$. Now extend γ equivariantly to a map $h_{k,j} : D_j^k \times G \to E$. By construction $h_{k,j}$ maps the orbit of each point $x \in D_j^k$ equivariantly to the orbit of $\gamma(x)$ in E. Since both orbits are isomorphic to G (because the action of G on both P and E are free), this map is a homeomorphism on orbits. Taking the collection of the extensions $h_{k,j}$ together then gives an extension

$$h_k : P^{(k)} \to E$$

with the required properties. This completes the inductive step. Thus we may conclude we have a G - equivariant map $h : P \to E$ that is a homeomorphism on the orbits. Hence it induces a map on the orbit space $f : P/G = X \to E/G = B$ making the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{h} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

Since h induces a homeomorphism on each orbit, the maps h and f determine a homeomorphism of principal G - bundles which induces an equivariant isomorphism on each fiber. This implies that h induces an isomorphism of principal bundles to the pull - back

$$\begin{array}{ccc} P & \xrightarrow[\cong]{h} & f^*(E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[=]{} & X. \end{array}$$

Thus the isomorphism class $[P] \in Prin_G(X)$ is given by $f^*(E)$. That is, $[P] = \psi(f)$, and hence

$$\psi : [X, B] \to Prin_G(X)$$

is surjective.

We now prove ψ is injective. To do this, assume $f_0 : X \to B$ and $f_1 : X \to B$ are maps so that there is an isomorphism

$$\Phi : f_0^*(E) \xrightarrow{\cong} f_1^*(E).$$

We need to prove that f_0 and f_1 are homotopic maps. Now by the cellular approximation theorem (see [**37**]) we can find cellular maps homotopic to f_0 and f_1 respectively. We therefore assume without loss of generality that f_0 and f_1 are cellular. This, together with the assumption that E is a G - CW complex, gives the pull back bundles $f_0^*(E)$ and $f_1^*(E)$ the structure of G -CW complexes.

Define a principal G - bundle $\mathcal{E} \to X \times I$ by

$$\mathcal{E} = f_0^*(E) \times [0, 1/2] \cup_\Phi f_1^*(E) \times [1/2, 1]$$

where $v \in f_0^*(E) \times \{1/2\}$ is identified with $\Phi(v) \in f_1^*(E) \times \{1/2\}$. $\mathcal{E}$ also has the structure of a G - CW - complex.

Now by the same kind of inductive argument that was used in the surjectivity argument above, we can find an equivariant map $H : \mathcal{E} \to E$ that induces a homeomorphism on each orbit, and that extends the obvious maps $f_0^*(E) \times \{0\} \to E$ and $f_1^*(E) \times \{1\} \to E$. The induced map on orbit spaces

$$F : \mathcal{E}/G = X \times I \to E/G = B$$

is a homotopy between f_0 and f_1. This proves the correspondence Ψ is injective, and completes the proof of the theorem. □

The following result establishes the homotopy uniqueness of universal bundles.

THEOREM 2.9. *Let $E_1 \to B_1$ and $E_2 \to B_2$ be universal principal G - bundles. Then there is a bundle map*

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{h}} & E_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow[h]{} & B_2 \end{array}$$

so that h is a homotopy equivalence.

PROOF. The fact that $E_2 \to B_2$ is a universal bundle means, by 2.8 that there is a "classifying map" $h : B_1 \to B_2$ and an isomorphism $\tilde{h} : E_1 \to h^*(E_2)$. Equivalently, $\tilde{h}$ can be thought of as a bundle map $\tilde{h} : E_! \to E_2$ lying over $h : B_1 \to B_2$. Similarly, using the universal property of $E_1 \to B_1$, we get a classifying map $g : B_2 \to B_1$ and an isomorphism $\tilde{g} : E_2 \to g^*(E_1)$, or equivalently, a bundle map $\tilde{g} : E_2 \to E_1$. Notice that the composition

$$g \circ f : B_1 \to B_2 \to B_1$$

is a map whose pull back,

$$\begin{aligned} (g \circ f)^*(E_1) &= g^*(f^*(E_1)) \\ &\cong g^*(E_2) \\ &\cong E_1. \end{aligned}$$

That is, $(g \circ f)^*(E_1) \cong id^*(E_1)$, and hence by 2.8 we have $g \circ f \simeq id : B_1 \to B_1$. Similarly, $f \circ g \simeq id : B_2 \to B_2$. Thus f and g are homotopy inverses of each other. □

Because of this theorem, the basespace of a universal principal G - bundle has a well defined homotopy type. We denote this homotopy type by BG, and refer to it as the *classifying space* of the group G. We also use the notation EG to denote the total space of a universal G - bundle.

We have the following immediate result about the homotopy groups of the classifying space BG.

COROLLARY 2.10. *For any group G, there is an isomorphism of homotopy groups,*

$$\pi_{n-1}G \cong \pi_n(BG).$$

PROOF. By considering 2.7 and 2.8 we see that both of these homotopy groups are in bijective correspondence with the set of principal bundles $Prin_G(S^n)$. To realize this bijection by a group homomorphism, consider the "suspension" of the group G, ΣG obtained by attaching two cones on G along the equator. That is,

$$\Sigma G = G \times [-1, 1]/ \sim$$

where all points of the form $(g, 1)$, $(h, -1)$, or $(1, t)$ are identified to a single point.

Notice that this suspension construction can be applied to any space with a basepoint, and in particular $\Sigma S^{n-1} \cong S^n$.

Consider the principal G bundle E over ΣG defined to be trivial on both cones with clutching function $id : G \times \{0\} \xrightarrow{=} G$ on the equator. That is, if $C_+ = G \times [0, 1]/ \sim \subset \Sigma G$ and $C_- = G \times [-1, 0] \subset \Sigma E$ are the upper and lower cones, respectively, then

$$E = (C_+ \times G) \cup_{id} (C_- \times G)$$

where $((g, 0), h) \in C_+ \times G$ is identified with $((g, 0)gh \in C_- \times G$. Then by 2.8 there is a classifying map

$$f : \Sigma G \to BG$$

such that $f^*(EG) \cong E$.

Now for any space X, let ΩX be the *loop space* of X,

$$\Omega X = \{\gamma : [-1, 1] \to X \text{ such that } \gamma(-1) = \gamma(1) = x_0 \in X\}$$

where $x_0 \in X$ is a fixed basepoint. Then the map $f : \Sigma G \to BG$ determines a map (its adjoint)

$$\bar{f} : G \to \Omega BG$$

defined by $\bar{f}(g)(t) = f(g, t)$. But now the loop space ΩX of any connected space X has the property that $\pi_{n-1}(\Omega X) = \pi_n(X)$ (see the exercise below). We then have the induced group homomorphism

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

which induces the bijective correspondence described above. □

Exercises. 1. Prove that for any connected space X, there is an isomorphism

$$\pi_{n-1}(\Omega X) \cong \pi_n(X).$$

2. Prove that the composition

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

in the above proof yields the bijection associated with identifying both $\pi_{n-1}(G)$ and $\pi_n(BG)$ with $Prin_G(S^n)$.

We recall the following definition from homotopy theory.

Definition 2.4. An Eilenberg - MacLane space of type (G, n) is a space X such that

$$\pi_k(X) = \begin{cases} G & \text{if k = n} \\ 0 & \text{otherwise} \end{cases}$$

We write $K(G, n)$ for an Eilenberg - MacLane space of type (G, n). Recall that for $n \geq 2$, the homotopy groups $\pi_n(X)$ are abelian groups, so in this $K(G, n)$ only exists

Corollary 2.11. *Let π be a discrete group. Then the classifying space $B\pi$ is an Eilenberg - MacLane space $K(\pi, 1)$.*

Examples.

- $\mathbb{R}$ has a free, cellular action of the integers $\mathbb{Z}$ by

$$(t, n) \to t + n \quad t \in \mathbb{R},\, n \in \mathbb{Z}.$$

  Since $\mathbb{R}$ is contractible, $\mathbb{R}/\mathbb{Z} = S^1 = B\mathbb{Z} = K(\mathbb{Z}, 1)$.
- The inclusion $S^n \subset S^{n+1}$ as the equator is clearly null homotopic since the inclusion obviously extends to a map of the disk. Hence the direct limit space

$$\varinjlim_n S^n = \cup_n S^n = S^\infty$$

  is aspherical. Now $\mathbb{Z}_2$ acts freely on S^n by the antipodal map, as described in chapter one. The inclusions $S^n \subset S^{n+1}$ are equivariant and hence there is an induced free action of $\mathbb{Z}_2$ on S^∞. Thus the projection map

$$S^\infty \to S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$$

  is a universal principal $\mathbb{Z}_2 = O(1)$ - bundle, and so

$$\mathbb{RP}^\infty = BO(1) = B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1)$$

  .
- Similarly, the inclusion of the unit sphere in $\mathbb{C}^n$ into the unit sphere in $\mathbb{C}^{n+1}$ gives an the inclusion $S^{2n-1} \subset S^{2n+1}$ which is null homotopic. It is also equivariant with respect to the free $S^1 = U(1)$ - action given by (complex) scalar multiplication. Then the limit $S^\infty = \cup_n S^{2n+1}$ is aspherical with a free S^1 action. We therefore have that the projection

$$S^\infty \to S^\infty/S^1 = \mathbb{CP}^\infty$$

is a principal $S^1 = U(1)$ bundle. Hence we have

$$\mathbb{CP}^\infty = BS^1 = BU(1).$$

Moreover since S^1 is a $K(\mathbb{Z}, 1)$, then we have that

$$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

- The cyclic groups $\mathbb{Z}_n$ are subgroups of $U(1)$ and so they act freely on S^∞ as well. Thus the projection maps

$$S^\infty \to S^\infty/\mathbb{Z}_n$$

is a universal principal $\mathbb{Z}_n$ bundle. The quotient space $S^\infty/\mathbb{Z}_n$ is denoted $L^\infty(n)$ and is referred to as the infinite $\mathbb{Z}_n$ - lens space.

These examples allow us to give the following description of line bundles and their relation to cohomology. We first recall a well known theorem in homotopy theory. This theorem will be discussed further in chapter 4. We refer the reader to [**42**] for details.

Theorem 2.12. *Let G be an abelian group. Then there is a natural isomorphism*

$$\phi : H^n(K(G,n);G) \xrightarrow{\cong} Hom(G,G).$$

Let $\iota \in H^n(K(G,n);G)$ be $\phi^{-1}(id)$. This is called the fundamental class. Then if X has the homotopy type of a CW - complex, the mapping

$$[X, K(G,n)] \to H^n(X;G)$$
$$f \to f^*(\iota)$$

is a bijective correspondence.

With this we can now prove the following:

Theorem 2.13. *There are bijective correspondences which allow us to classify complex line bundles,*

$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{CP}^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z})$$

where the last correspondence takes a map $f : X \to \mathbb{CP}^\infty$ to the class

$$c_1 = f^*(c) \in H^2(X),$$

where $c \in H^2(\mathbb{CP}^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\zeta \in Vect^1(X)$ is called the first Chern class of ζ (or of the corresponding principal $U(1)$ - bundle).

PROOF. These correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X) \cong [X, BGL(1,\mathbb{C})]$, and that $\mathbb{CP}^\infty$ is a model for $BGL(1,\mathbb{C})$ as well as $BU(1)$. This is because, we can express $\mathbb{CP}^\infty$ in its homogeneous form as

$$\mathbb{CP}^\infty = \varinjlim_n (\mathbb{C}^{n+1} - \{0\})/GL(1,\mathbb{C}),$$

and that $\varinjlim_n (\mathbb{C}^{n+1} - \{0\})$ is an aspherical space with a free action of $GL(1,\mathbb{C}) = \mathbb{C}^*$. □

There is a similar theorem classifying real line bundles:

THEOREM 2.14. *There are bijective correspondences*

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{RP}^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2)$$

where the last correspondence takes a map $f : X \to \mathbb{RP}^\infty$ to the class

$$w_1 = f^*(w) \in H^1(X; \mathbb{Z}_2),$$

where $w \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$ is the generator. In the composition of these correspondences, the class $w_1 \in H^1(X; \mathbb{Z}_2)$ corresponding to a line bundle $\zeta \in Vect^1_{\mathbb{R}}(X)$ is called the first Stiefel - Whitney class of ζ (or of the corresponding principal $O(1)$ - bundle).

More Examples.

- Let $V_n(\mathbb{C}^N)$ be the Stieflel - manifold studied in the last chapter. We claim that the inclusion of vector spaces $\mathbb{C}^N \subset \mathbb{C}^{2N}$ as the first N - coordinates induces an inclusion $V_n(\mathbb{C}^N) \hookrightarrow V_n(\mathbb{C}^{2N})$ which is null homotopic. To see this, let $\iota : \mathbb{C}^n \to \mathbb{C}^{2N}$ be a fixed linear embedding, whose image lies in the last N - coordinates in $\mathbb{C}^{2N}$. Then given any $\rho \in V_n(\mathbb{C}^N) \subset V_n(\mathbb{C}^{2N})$, then $t \cdot \iota + (1-t) \cdot \rho$ for $t \in [0,1]$ defines a one parameter family of linear embeddings of $\mathbb{C}^n$ in $\mathbb{C}^{2N}$, and hence a contraction of the image of $V_n(\mathbb{C}^N)$ onto the element ι. Hence the limiting space $V_n(\mathbb{C}^\infty)$ is aspherical with a free $GL(n,\mathbb{C})$ - action. Therefore the projection

$$V_n(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/GL(n,\mathbb{C}) = Gr_n(\mathbb{C}^\infty)$$

is a universal $GL(n,\mathbb{C})$ - bundle. Hence the infinite Grassmannian is the classifying space

$$Gr_n(\mathbb{C}^\infty) = BGL(n,\mathbb{C})$$

and so we have a classification

$$Vect^n(X) \cong Prin_{GL(n,\mathbb{C})}(X) \cong [X, BGL(n,\mathbb{C})] \cong [X, Gr_n(\mathbb{C}^\infty)].$$

- A simlar argument shows that the infinite unitary Stiefel manifold, $V_n^U(\mathbb{C}^\infty)$ is aspherical with a free $U(n)$ - action. Thus the projection

$$V_n^U(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/U(n) = Gr_n(\mathbb{C}^\infty)$$

is a universal principal $U(n)$ - bundle. Hence the infinite Grassmanian $Gr_n(\mathbb{C}^\infty)$ is the classifying space for $U(n)$ bundles as well,

$$Gr_n(\mathbb{C}^\infty) = BU(n).$$

The fact that this Grassmannian is both $BGL(n, \mathbb{C})$ and $BU(n)$ reflects the fact that every n - dimensional complex vector bundle has a $U(n)$ - structure.

- We have similar universal $GL(n, \mathbb{R})$ and $O(n)$ - bundles:

$$V_n(\mathbb{R}^\infty) \to V_n(\mathbb{R}^\infty)/GL(n, \mathbb{R}) = Gr_n(\mathbb{R}^\infty)$$

and

$$V_n^O(\mathbb{R}^\infty) \to V_n^O(\mathbb{R}^\infty)/O(n) = Gr_n(\mathbb{R}^\infty).$$

Thus we have

$$Gr_n(\mathbb{R}^\infty) = BGL(n, \mathbb{R}) = BO(n)$$

and so this infinite dimensional Grassmannian classifies real n - dimensional vector bundles as well as principal $O(n)$ - bundles.

Now suppose $p : EG \to EG/G = BG$ is a universal G - bundle. Suppose further that $H < G$ is a subgroup. Then H acts freely on EG as well, and hence the projection

$$EG \to EG/H$$

is a universal H - bundle. Hence $EG/H = BH$. Using the infinite dimensional Stiefel manifolds described above, this observation gives us models for the classifying spaces for any subgroup of a general linear group. So for example if we have a subgroup (i.e a faithful representation) $H \subset GL(n, \mathbb{C})$, then

$$BH = V_n(\mathbb{C}^\infty)/H.$$

This observation also leads to the following useful fact.

Proposition 2.15. . *Let $p : EG \to BG$ be a universal principal G - bundle, and let $H < G$. Then there is a fiber bundle*

$$BH \to BG$$

with fiber the orbit space G/H.

PROOF. This bundle is given by

$$G/H \to EG \times_G G/H \to EG/G = BG$$

together with the observation that $EG \times_G G/H = EG/H = BH$. □

3. Classifying Gauge Groups

In this section we describe the classifying space of the group of automorphisms of a principal G - bundle, or the *gauge group* of the bundle. We describe the classifying space in two different ways: in terms of the space of connections on the bundle, and in terms of the mapping space of the base manifold to the classifying space BG. These constructions are important in Yang - Mills theory, and we refer the reader to [**3**] and [**11**] for more details.

Let A be a connection on a principal bundle $P \longrightarrow M$ where M is a closed manifold equipped with a Riemannian metric. The Yang - Mills functional applied to A, $\mathcal{YM}(A)$ is the square of the L^2 norm of the curvature,

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \|F_A\|^2 \, d(vol).$$

We view $\mathcal{YM}$ as a mapping $\mathcal{YM} : \mathcal{A}(P) \longrightarrow \mathbb{R}$. The relevance of the gauge group in Yang - Mills theory is that this is the group of symmetries of $\mathcal{A}$ that $\mathcal{YM}$ preserves.

DEFINITION 2.5. The gauge group $\mathcal{G}(P)$ of the principal bundle P is the group of bundle automorphisms of $P \longrightarrow M$. That is, an element $\phi \in \mathcal{G}(P)$ is a bundle isomorphism of P with itself lying over the identity:

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\phi} & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M. \end{array}$$

Equivalently, $\mathcal{G}(P)$ is the group $\mathcal{G}(P) = Aut_G(P)$ of G - equivariant diffeomorphisms of the space P.

The gauge group $\mathcal{G}(P)$ can be thought of in several equivalent ways. The following one is particularly useful.

Consider the conjugation action of the Lie group G on itself,

$$G \times G \longrightarrow G$$
$$(g, h) \longrightarrow ghg^{-1}.$$

This left action defines a fiber bundle

$$Ad(P) = P \times_G G \longrightarrow P/G = M$$

with fiber G. We leave the following as an exercise for the reader.

PROPOSITION 2.16. *The gauge group of a principal bundle $P \longrightarrow M$ is naturally isomorphic (as topological groups) the the group of sections of $Ad(P)$, $C^\infty(M; Ad(P))$.*

The gauge group $\mathcal{G}(P)$ acts on the space of connections $\mathcal{A}(P)$ by the pull - back construction. More generally, if $f : P \to Q$ is any smooth map of principal G - bundles and A is a connection on Q, then there is a natural pull back connection $f^*(A)$ on Q, defined by pulling back the equivariant splitting of τQ to an equivariant splitting of τP in the obvious way. The pull - back construction for automorphisms $\phi : P \longrightarrow P$ defines an action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$.

We leave the proof of the following is an exercise for the reader.

PROPOSITION 2.17. *Let P be the trivial bundle $M \times G \to M$. Then the gauge group $\mathcal{G}(P)$ is given by the function space from M to G,*

$$\mathcal{G}(P) \cong C^\infty(M; G).$$

Furthermore if $\phi : M \to G$ is identified with an element of $\mathcal{G}(P)$, and $A \in \Omega^1(M; \mathfrak{g})$ is identified with an element of $\mathcal{A}(G)$, then the induced action of ϕ on G is given by

$$\phi^*(A) = \phi^{-1} A \phi + \phi^{-1} d\phi.$$

It is not difficult to see that in general the gauge group $\mathcal{G}(P)$ does not act freely on the space of connections $\mathcal{A}(P)$. However there is an important subgroup $\mathcal{G}_0(P) < \mathcal{G}(P)$ that does. This is the group of *based* gauge transformations. To define this group, let $x_0 \in M$ be a fixed basepoint, and let P_{x_0} be the fiber of P at x_0.

DEFINITION 2.6. The based gauge group $\mathcal{G}_0(P)$ is a subgroup of the group of bundle automorphisms $\mathcal{G}(P)$ which pointwise fix the fiber $P_{x_)}$. That is,

$$\mathcal{G}_0(P) = \{\phi \in \mathcal{G}(P) : \text{if}\, v \in P_{x_0} \text{then}\, \phi(v) = v\}.$$

THEOREM 2.18. *The based gauge group $\mathcal{G}_0(P)$ acts freely on the space of connections $\mathcal{A}(P)$.*

PROOF. Suppose that $A \in \mathcal{A}(P)$ is a fixed point of $\phi \in \mathcal{G}_0(P)$. That is, $\phi^*(A) = A$. We need to show that $\phi = 1$.

The equivariant splitting ω_A given by a connection A defines a notion of parallel transport in P along curves in M (see [**16**]) . It is not difficult to see that the statement $\phi^*(A) = A$ implies that application of the automorphism ϕ commutes with parallel transport. Now let $w \in P_x$ be a point in the fiber of an element $x \in M$. Given curve γ in M between the basepoint x_0 and x one sees that

$$\phi(w) = T_\gamma(\phi(T_{\gamma^{-1}}(w))$$

where T_γ is parallel transport along γ. But since $T_{\gamma^{-1}}(w) \in P_{x_0}$ and $\phi \in \mathcal{G}_0(P)$,

$$\phi(T_{\gamma^{-1}}(w)) = w.$$

Hence $\phi(w) = w$, that is, $\phi = 1$. □

Remark. Notice that this argument actually says that if $A \in \mathcal{A}(P)$ is the fixed point of *any* gauge transformation $\phi \in \mathcal{G}(P)$, then ϕ is determined by its action on a single fiber.

Let $\mathcal{B}(P)$ and $\mathcal{B}_0(P)$ be the orbit spaces of connections on P up to guage and based gauge equivalence respectively,

$$\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P) \quad \mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P).$$

Now it is straightforward to check directly that the Yang - Mills functional in invariant under gauge transformations. Thus it yields maps

$$\mathcal{YM} : \mathcal{B}(P) \to \mathbb{R} \quad \text{and} \quad \mathcal{YM} : \mathcal{B}_0(P) \to \mathbb{R}.$$

It is therefore important to understand the homotopy types of these orbit spaces. Because of the freeness of the action of $\mathcal{G}_0(P)$, the homotopy type of the orbit space $\mathcal{G}_0(P)$ is easier to understand.

We end this section with a discussion of its homotopy type. Since the space of connections $\mathcal{A}(P)$ is affine, it is contractible. Moreover it is possible to show that the free action of the based gauge group $\mathcal{G}_0(P)$ has local slices (see [**11**]). Thus we have $\mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P)$ is the classifying space of the based gauge group,

$$\mathcal{B}_0(P) = B\mathcal{G}_0(P).$$

But the classifying spaces of the gauge groups are relatively easy to understand. (see [**3**].)

THEOREM 2.19. *Let $G \longrightarrow EG \longrightarrow BG$ be a universal principal bundle for the Lie group G (so that EG is aspherical). Let $y_0 \in BG$ be a fixed basepoint. Then there are homotopy equivalences*

$$B\mathcal{G}(P) \simeq Map^P(M, BG) \quad \textit{and} \quad \mathcal{B}_0(P) \simeq B\mathcal{G}_0(P) \simeq Map_0^P(M, BG)$$

where $Map(M, BG)$ is the space of all continuous maps from M to BG and $Map_0(M, BG)$ is the space of those maps that preserve the basepoints. The superscript P denotes the path component of these mapping spaces consisting of the homotopy class of maps that classify the principal G - bundle P.

Proof. Consider the space of all G - equivariant maps from P to EG, $Map^G(P, EG)$. The gauge group $\mathcal{G}(P) \cong Aut^G(P)$ acts freely on the left of this space by composition. It is easy to see that $Map^G(P, EG)$ is aspherical, and its orbit space is given by the space of maps from the G - orbit space of P $(= M)$ to the G - orbit space of EG (= BG),

$$Map^G(P, EG)/\mathcal{G}(P) \cong Map^P(M, BG).$$

This proves that $Map(M, BG) = B\mathcal{G}(P)$. Similarly $Map_0^G(P, EG)$, the space of G - equivariant maps that send the fiber P_{x_0} to the fiber EG_{y_0}, is an aspherical space with a free $\mathcal{G}_0(P)$ action, whose orbit space is $Map_0^P(M, BG)$. Hence $Map_0^P(M, BG) = B\mathcal{G}_0(P)$. □

4. Existence of universal bundles: the Milnor join construction and the simplicial classifying space

In the last section we proved a "recognition principle" for universal principal G bundles. Namely, if the total space of a principal G - bundle $p : E \to B$ is aspherical, then it is universal. We also proved a homotopy uniqueness theorem, stating among other things that the homotopy type of the base space of a universal bundle, i.e the classifying space BG, is well defined. We also described many examples of universal bundles, and particular have a model for the classifying space BG, using Stiefel manifolds, for every subgroup of a general linear group.

The goal of this section is to prove the general existence theorem. Namely, for every group G, there is a universal principal G - bundle $p : EG \to BG$. We will give two constructions of the universal bundle and the corresponding classifying space. One, due to Milnor [**30**] involves taking the "infinite join" of a group with itself. The other is an example of a simplicial space, called the simplicial bar construction. It is originally due to Eilenberg and MacLane [**12**]. These constructions are essentially equivalent and both yield G - CW - complexes. Since they are so useful in algebraic topology and combinatorics, we will also take this opportunity to introduce the notion of a general simplicial space and show how these classifying spaces are important examples.

4.1. The join construction. The "join" between two spaces X and Y, written $X * Y$ is the space of all lines connecting points in X to points in Y. The following is a more precise definition:

Definition 2.7. The join $X * Y$ is defined by

$$X * Y = X \times I \times Y / \sim$$

where $I = [0, 1]$ is the unit interval and the equivalence relation is given by $(x, 0, y_1) \sim (x, 0, y_2)$ for any two points y_1, $y_2 \in Y$, and similarly $(x_1, 1, y) \sim (x_2, 1, y)$ for any two points x_1, $x_2 \in X$.

A point $(x, t, y) \in X * Y$ should be viewed as a point on the line connecting the points x and y. Here are some examples.

Examples.

- Let y be a single point. Then $X * y$ is the cone $CX = X \times I / X \times \{1\}$.
- Let $Y = \{y_1, y_2\}$ be the space consisting of two distinct points. Then $X * Y$ is the suspension ΣX discussed earlier. Notice that the suspension can be viewed as the union of two cones, with vertices y_1 and y_2 respectively, attached along the equator.
- **Exercise.** Prove that the join of two spheres, is another sphere,

$$S^n * S^m \cong S^{n+m+1}.$$

- Let $\{x_0, \cdots, x_k\}$ be a collection of $k + 1$ - distinct points. Then the k - fold join $x_0 * x_1 * \cdots * x_k$ is the convex hull of these points and hence is by the k - dimensional simplex Δ^k with vertices $\{x_0, \cdots, x_k\}$.

Observe that the space X sits naturally as a subspace of the join $X * Y$ as endpoints of line segments,

$$\iota : X \hookrightarrow X * Y$$
$$x \to (x, 0, y).$$

Notice that this formula for the inclusion makes sense and does not depend on the choice of $y \in Y$. There is a similar embedding

$$j : Y \hookrightarrow X * Y$$
$$y \to (x, 1, y).$$

Lemma 2.20. *The inclusions $\iota : X \hookrightarrow X * Y$ and $j : Y \hookrightarrow X * Y$ are null homotopic.*

Proof. Pick a point $y_0 \in Y$. By definition, the embedding $\iota : X \to X * Y$ factors as the composition

$$\iota : X \hookrightarrow X * y_0 \subset X * Y$$
$$x \to (x, 0, y_0).$$

But as observed above, the join $X * y_0$ is the cone on X and hence contractible. This means that ι is null homotopic, as claimed. The fact that $j : Y \hookrightarrow X * Y$ is null homotopic is proved in the same way. □

Now let G be a group and consider the iterated join

$$G^{*(k+1)} = G * G * \cdots * G$$

where there are $k+1$ copies of the group element. This space has a free G action given by the diagonal action

$$g \cdot (g_0, t_1, g_1, \cdots, t_k, g_k) = (gg_0, t_1, gg_1, \cdots, t_k, gg_k).$$

Exercise. 1. Prove that there is a natural G - equivariant map

$$\Delta^k \times G^{k+1} \to G^{*(k+1)}$$

which is a homeomorphism when restricted to $\tilde{\Delta}^k \times G^{k+1}$ where $\tilde{\Delta}^k \subset \Delta^k$ is the interior. Here G acts on $\Delta^k \times G^{k+1}$ trivially on the simplex Δ^k and diagonally on G^{k+1}.

2. Use exercise 1 to prove that the iterated join $G^{*(k+1)}$ has the structure of a G - CW - complex.

Define $\mathcal{J}(G)$ to be the infinite join

$$\mathcal{J}(G) = \lim_{k\to\infty} G^{*(k+1)}$$

where the limit is taken over the embeddings $\iota : G^{*(k+1)} \hookrightarrow G^{*(k+2)}$ Since these embedding maps are G -equivariant, we have an induced G - action on $\mathcal{J}(G)$.

THEOREM 2.21. *The projection map*

$$p : \mathcal{J}(G) \to \mathcal{J}(G)/G$$

is a universal principal G - bundle.

PROOF. By the above exercise the space $\mathcal{J}(G)$ has the structure of a G - CW - complex with a free G - action. Therefore by the results of the last section the projection $p : \mathcal{J}(G) \to \mathcal{J}(G)/G$ is a principal G - bundle. To see that $\mathcal{J}(G)$ is aspherical, notice that since S^n is compact, any map $\alpha : S^n \to \mathcal{J}(G)$ is homotopic to one that factors through a finite join (that by abuse of notation we still call α), $\alpha : S^n \to G^{*(n+1)} \hookrightarrow \mathcal{J}(G)$. But by the above lemma the inclusion $G^{*(n+1)} \subset \mathcal{J}(G)$ is null homotopic, and hence so is α. Thus $\mathcal{J}(G)$ is aspherical. By the results of last section, this means that the projection $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is a universal G - bundle. □

4.2. Simplicial spaces and classifying spaces. We therefore now have a universal bundle for every topological group G. We actually know a fair amount about the geometry of the total space $EG = \mathcal{J}(G)$ which, by the above exercise can be described as the union of simplices, where the k - simplices are parameterized by $k+1$ -tuples of elements of G,

$$EG = \mathcal{J}(G) = \bigcup_k \Delta^k \times G^{k+1}/\sim$$

and so the classifying space can be described by

$$BG = \mathcal{J}(G)/G \cong \bigcup_k \Delta^k \times G^k/\sim$$

It turns out that in these constructions, the simplices are glued together along faces, and these gluings are parameterized by the $k+1$ - product maps $\partial_i : G^{k+2} \to G^{k+1}$ given by multiplying the i^{th} and $(i+1)^{st}$ coordinates.

Having this type of data (parameterizing spaces of simplices as well as gluing maps) is an example of an object known as a *"simplicial set"* which is an important combinatorial object in topology. We now describe this notion in more detail and show how these universal G - bundles and classifying spaces can be viewed in these terms.

Good references for this theory are [**9**], [**26**].

The idea of simplicial sets is to provide a combinatorial technique to study cell complexes built out of simplices; i.e simplicial complexes. A simplicial complex X is built out of a union of simplices, glued along faces. Thus if X_n denotes the indexing set for the n - dimensional simplices of X, then we can write

$$X = \bigcup_{n \geq 0} \Delta^n \times X_n/\sim$$

where Δ^n is the standard n - simplex in $\mathbb{R}^n$;

$$\Delta^n = \{(t_1, \cdots, t_n) \in \mathbb{R}^n : 0 \leq t_j \leq 1, \text{ and } \sum_{i=1}^n t_i \leq 1\}.$$

The gluing relation in this union can be encoded by set maps among the X_n's that would tell us for example how to identify an $n-1$ simplex indexed by an element of X_{n-1} with a particular face of an n - simplex indexed by an element of X_n. Thus in principal simplicial complexes can be studied purely combinatorially in terms of the sets X_n and set maps between them. The notion of a *simplicial set* makes this idea precise.

DEFINITION 2.8. A *simplicial set* X_* is a collection of sets

$$X_n, \quad n \geq 0$$

together with set maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

for $0 \leq i, j \leq n$ called **face** and **degeneracy** maps respectively. These maps are required to satisfy the following compatibility conditions

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{for } i < j$$
$$s_i s_j = s_{j+1} s_i \quad \text{for } i < j$$

and

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{for } i < j \\ 1 & \text{for } i = j, \, j+1 \\ s_j \partial_{i-1} & \text{for } i > j+1 \end{cases}$$

As mentioned above, the maps ∂_i and s_j encode the combinatorial information necessary for gluing the simplices together. To say precisely how this works, consider the following maps between the standard simplices:

$$\delta_i : \Delta^{n-1} \longrightarrow \Delta^n \quad \text{and} \quad \sigma_j : \Delta^{n+1} \longrightarrow \Delta^n$$

for $0 \leq i, j \leq n$ defined by the formulae

$$\delta_i(t_1, \cdots, t_{n-1}) = \begin{cases} (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) & \text{for } i \geq 1 \\ (1 - \sum_{q=1}^{n-1} t_q, t_1, \cdots, t_{n-1}) & \text{for } i = 0 \end{cases}$$

and

$$\sigma_j(t_1, \cdots, t_{n+1}) = \begin{cases} (t_1, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}) & \text{for } i \geq 1 \\ (t_2, \cdots, t_{n+1}) & \text{for } i = 0 \,. \end{cases}$$

δ_i includes Δ^{n-1} in Δ^n as the i^{th} face, and σ_j projects, in a linear fashion, Δ^{n+1} onto its j^{th} face.

We can now define the space associated to the simplicial set X_* as follows.

Definition 2.9. The *geometric realization* of a simplicial set X_* is the space

$$\|X_*\| = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

where if $t \in \Delta^{n-1}$ and $x \in X_n$, then

$$(t, \partial_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{n+1}$ and $x \in X_n$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

In the topology of $\|X_*\|$, each X_n is assumed to have the discrete topology, so that $\Delta^n \times X_n$ is a discrete set of n - simplices.

Thus $\|X_*\|$ has one n - simplex for every element of X_n, glued together in a way determined by the face and degeneracy maps.

Example. Consider the simplicial set $\mathbf{S}_*$ defined as follows. The set of n - simplices is given by

$$\mathbf{S}_n = \mathbb{Z}/(n+1), \text{ generated by an element } \tau_n.$$

The face maps are given by

$$\partial_i(\tau_n^r) = \begin{cases} \tau_{n-1}^r & \text{if } r \leq i \leq n \\ \tau_{n-1}^{r-1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

The degeneracies are given by

$$s_i(\tau_n^r) = \begin{cases} \tau_{n+1}^r & \text{if } r \leq i \leq n \\ \tau_{n+1}^{r+1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

Notice that there is one zero simplex, two one simplices, one of them the image of the degeneracy $s_0 : \mathbf{S}_0 \longrightarrow \mathbf{S}_1$, and the other nondegenerate (i.e not in the image of a degeneracy map). Notice also that all simplices in dimensions larger than one are in the image of a degeneracy map. Hence we have that the geometric realization

$$\|\mathbf{S}_*\| = \Delta^1/0 \sim 1 = S^1.$$

Let X_* be any simplicial set. There is a particularly nice and explicit way for computing the homology of the geometric realization, $H_*(\|X_*\|)$.

Consider the following chain complex. Define $C_n(X_*)$ to be the free abelian group generated by the set of n - simplices X_n. Define the homomorphism

$$d_n : C_n(X_*) \longrightarrow C_{n-1}(X_*)$$

by the formula

$$d_n([x]) = \sum_{i=0}^{n} (-1)^i \partial_i([x])$$

where $x \in X_n$.

Proposition 2.22. *The homology of the geometric realization $H_*(\|X_*\|)$ is the homology of the chain complex*

$$\longrightarrow \cdots \xrightarrow{d_{n+1}} C_n(X_*) \xrightarrow{d_n} C_{n-1}(X_*) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} C_0(X_*).$$

PROOF. It is straightforward to check that the geometric realization $\|X_*\|$ is a CW - complex and that this is the associated cellular chain complex. □

Besides being useful computationally, the following result establishes the fact that all CW complexes can be studied simplicially.

THEOREM 2.23. *Every CW complex has the homotopy type of the geometric realization of a simplicial set.*

PROOF. Let X be a CW complex. Define the singular simplicial set of X , $\mathcal{S}(X)_*$ as follows. The n simplices $\mathcal{S}(X)_n$ is the set of singular n - simplices,

$$\mathcal{S}(X)_n = \{c : \Delta^n \longrightarrow X\}.$$

The face and degeneracy maps are defined by

$$\partial_i(c) = c \circ \delta_i : \Delta^{n-1} \longrightarrow \Delta^n \longrightarrow X$$

and

$$s_j(c) = c \circ \sigma_i : \Delta^{n+1} \longrightarrow \Delta^n \longrightarrow X.$$

Notice that the associated chain complex to $\mathcal{S}(X)_*$ as in 2.22 is the singular chain complex of the space X. Hence by 2.22 we have that

$$H_*(\|\mathcal{S}(X)\|) \cong H_*(X).$$

This isomorphism is actually realized by a map of spaces

$$E : \|\mathcal{S}(X)_*\| \longrightarrow X$$

defined by the natural evaluation maps

$$\Delta^n \times \mathcal{S}(X)_n \longrightarrow X$$

given by

$$(t, c) \longrightarrow c(t).$$

It is straightforward to check that the map E does induce an isomorphism in homology. In fact it induces an isomorphism in homotopy groups. We will not prove this here; it is more technical and we refer the reader to [M] for details. Note that it follows from the homological isomorphism by the Hurewicz theorem if we knew that X was simply connected. A map between spaces that induces an isomorphism in homotopy groups is called a *weak homotopy equivalence*. Thus *any* space is weakly homotopy equivalent to a CW - complex (i.e the geometric realization of its singular simplicial set). But by the Whitehead theorem, two CW complexes that are weakly homotopy equivalent are homotopy equivalent. Hence X and $\|\mathcal{S}(X)_*\|$ are homotopy equivalent. □

We next observe that the notion of simplicial set can be generalized as follows. We say that X_* is a **simplicial space** if it is a simplicial set (i.e it satisfies definition 2.8) where the sets X_n are topological spaces and the face and degeneracy maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

are continuous maps. The definition of the geometric realization of a simplicial space X_*, $\|X_*\|$, is the same as in 2.9 with the proviso that the topology of each $\Delta^n \times X_n$ is the product topology. Notice that since the "set of n - simplices" X_n is actually a space, it is not necessarily true that $\|X_*\|$ is a CW complex. However if in fact each X_n is a CW complex and the face and degeneracy maps are cellular, then $\|X_*\|$ does have a natural CW structure induced by the product CW - structures on $\Delta^n \times X_n$.

Notice that this simplicial notion generalizes even further. For example a **simplicial group** would be defined similarly, where each X_n would be a group and the face and degeneracy maps are group homomorphisms. Simplicial vector spaces, modules, etc. are defined similarly. The categorical nature of these definitions should by now be coming clear. Indeed most generally one can define a **simplicial object in a category** $\mathcal{C}$ using the above definition where now the X_n's are assumed to be objects in the category and the face and degenarcies are assumed to be morphisms. If the category $\mathcal{C}$ is a subcategory of the category of sets then geometric realizations can be defined as in 2.9 For example the geometric realization of a simplicial (abelian) group turns out to be a topological (abelian) group.(Try to verify this for yourself!)

We now use this simplicial theory to construct universal principal G - bundles and classifying spaces.

Let G be a topological group and let $\mathcal{E}G_*$ be the simplicial space defined as follows. The space of n - simplices is given by the $n+1$ - fold cartesian product

$$\mathcal{E}G_n = G^{n+1}.$$

The face maps $\partial_i : G^{n+1} \longrightarrow G^n$ are given by the formula

$$\partial_i(g_0, \cdots, g_n) = (g_0, \cdots, \hat{g}_i, \cdots, g_n).$$

The degeneracy maps $s_j : G^{n+1} \longrightarrow G^{n+2}$ are given by the formula

$$s_j(g_0, \cdots, g_n) = (g_0, \cdots, g_j, g_j, \cdots, g_n).$$

Exercise. Show that the geometric realization $\|\mathcal{E}G_*\|$ is aspherical. Hint. Let $\|\mathcal{E}G_*\|^{(n)}$ be the n^{th} - skeleton,

$$\|\mathcal{E}G_*\|^{(n)} = \bigcup_{p=0}^{n} \Delta^p \times G^{p+1}.$$

Then show that the inclusion of one skeleton in the next $\|\mathcal{E}G_*\|^{(n)} \hookrightarrow \|\mathcal{E}G_*\|^{(n+1)}$ is null - homotopic. One way of doing this is to establish a homeomorphism between $\|\mathcal{E}G_*\|^{(n)}$ and n - fold join $G * \cdots * G$. See [M] for details.

Notice that the group G acts freely on the right of $\|\mathcal{E}G_*\|$ by the rule

$$\|\mathcal{E}G_*\| \times G = \left(\bigcup_{p\geq 0} \Delta^p \times G^{p+1}\right) \times G \longrightarrow \|\mathcal{E}G_*\| \tag{4.1}$$

$$(t;(g_0,\cdots,g_p)) \times g \longrightarrow (t;(g_0 g,\cdots,g_p g)).$$

Thus we can define $EG = \|\mathcal{E}G_*\|$. The projection map

$$p : EG \to EG/G = BG$$

is therefore a universal principal G - bundle.

This description gives the classfiying space BG an induced simplicial structure described as follows.

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Let $\mathcal{B}G_*$ be the simplicial space whose n - simplices are the cartesian product

$$\mathcal{B}G_n = G^n. \tag{4.2}$$

The face and degeneracy maps are given by

$$\partial_i(g_1,\cdots,g_n) = \begin{cases} (g_2,\cdots,g_n) & \text{for } i = 0 \\ (g_1,\cdots,g_i g_{i+1},\cdots g_n) & \text{for } 1 \leq i \leq n-1 \\ (g_1,\cdots,g_{n-1}) & \text{for } i = n. \end{cases}$$

The degeneracy maps are given by

$$s_j(g_1,\cdots,g_n) = \begin{cases} (1,g_1,\cdots,g_n) & \text{for } j = 0 \\ (g_1,\cdots g_j,1,g_{j+1},\cdots,g_n) & \text{for } j \geq 1. \end{cases}$$

The simplicial projection map

$$p : \mathcal{E}G_* \longrightarrow \mathcal{B}G_*$$

defined on the level of n - simplicies by

$$p(g_0,\cdots,g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1},\cdots,g_{n-1}g_n^{-1})$$

is easily checked to commute with face and degeneracy maps and so induces a map on the level of geometric realizations

$$p : EG = \|\mathcal{E}G_*\| \longrightarrow \|\mathcal{B}G_*\|$$

which induces a homemorphism

$$BG = EG/G \xrightarrow{\cong} \|\mathcal{B}G_*\|.$$

Thus for any topological group this construction gives a simplicial space model for its classifying space. This is referred to as the **simplicial bar construction.** Notice that when G is discrete the bar construction is a CW complex for the classifying space $BG = K(G,1)$ and 2.22 gives a particularly nice complex for computing its homology. (The homology of a $K(G,1)$ is referred to as the homology of the group G.)

The n - chains are the group ring

$$C_n(\mathcal{B}G_*) = \mathbb{Z}[G^n] \cong \mathbb{Z}[G]^{\otimes n}$$

and the boundary homomorphisms

$$d_n : \mathbb{Z}[G]^{\otimes n} \longrightarrow \mathbb{Z}[G]^{\otimes n-1}$$

are given by

$$d_n(a_1 \otimes \cdots \otimes a_n) = (a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i (a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n (a_1 \otimes \cdots \otimes a_{n-1}).$$

This complex is called the **bar complex** for computing the homology of a group and was discovered by Eilenberg and MacLane in the mid 1950's.

We end this chapter by observing that the bar construction of the classifying space of a group did not use the full group structure. It only used the existence of an associative multiplication with unit. That is, it did not use the existence of inverse. So in particular one can study the classifying space BA of a monoid A. This is an important construction in algebraic - K - theory.

5. Some Applications

In a sense much of what we will study in the next chapter are applications of the classification theorem for principal bundles. In this section we describe a few immediate applications.