Next we observe that $D(f)$ can be covered by a finite number of the $D(h_i)$. Indeed, $D(f) \subseteq \bigcup D(h_i)$ if and only if $V((f)) \supseteq \bigcap V((h_i)) = V(\sum(h_i))$. By (2.1c) again, this is equivalent to saying $f \in \sqrt{\sum(h_i)}$, or $f^n \in \sum(h_i)$ for some n. This means that f^n can be expressed as a *finite* sum $f^n = \sum b_i h_i$, $b_i \in A$. Hence a finite subset of the h_i will do. So from now on we fix a finite set $h_1, \ldots, h_r$ such that $D(f) \subseteq D(h_1) \cup \ldots \cup D(h_r)$.

For the next step, note that on $D(h_i) \cap D(h_j) = D(h_i h_j)$ we have two elements of $A_{h_i h_j}$, namely a_i/h_i and a_j/h_j both of which represent s. Hence, according to the injectivity of ψ proved above, applied to $D(h_i h_j)$, we must have $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Hence for some n,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

Since there are only finitely many indices involved, we may pick n so large that it works for all i,j at once. Rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0.$$

Then replace each h_i by h_i^{n+1}, and a_i by $h_i^n a_i$. Then we still have s represented on $D(h_i)$ by a_i/h_i, and furthermore, we have $h_j a_i = h_i a_j$ for all i,j.

Now write $f^n = \sum b_i h_i$ as above, which is possible for some n since the $D(h_i)$ cover $D(f)$. Let $a = \sum b_i a_i$. Then for each j we have

$$h_j a = \sum_i b_i a_i h_j = \sum_i b_i h_i a_j = f^n a_j.$$

This says that $a/f^n = a_j/h_j$ on $D(h_j)$. So $\psi(a/f^n) = s$ everywhere, which shows that ψ is surjective, hence an isomorphism.

To each ring A we have now associated its spectrum (Spec A, $\mathcal{O}$). We would like to say that this correspondence is functorial. For that we need a suitable category of spaces with sheaves of rings on them. The appropriate notion is the category of locally ringed spaces.

Definition. A *ringed space* is a pair $(X,\mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism* of ringed spaces from $(X,\mathcal{O}_X)$ to $(Y,\mathcal{O}_Y)$ is a pair $(f,f^\#)$ of a continuous map $f:X \to Y$ and a map $f^\#:\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y. The ringed space $(X,\mathcal{O}_X)$ is a *locally ringed space* if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring. A *morphism* of locally ringed spaces is a morphism $(f,f^\#)$ of ringed spaces, such that for each point $P \in X$, the induced map (see below) of local rings $f_P^\#:\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a *local homomorphism* of local rings. We explain this last condition. First of all, given a point $P \in X$, the morphism of sheaves $f^\#:\mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$, for every open set V in Y. As V ranges over all open neighborhoods of $f(P)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of P. Taking direct limits, we obtain a map

$$\mathcal{O}_{Y,f(P)} = \varinjlim_V \mathcal{O}_Y(V) \to \varinjlim_V \mathcal{O}_X(f^{-1}V),$$