In this section we will develop the basic properties of quasi-coherent and coherent sheaves. In particular we will introduce the important "twisting sheaf" $\mathcal{O}(1)$ of Serre on a projective scheme.

We will start by defining sheaves of modules on a ringed space.

Definitions. Let $(X,\mathcal{O}_X)$ be a ringed space (see §2). A *sheaf of* $\mathcal{O}_X$*-modules* (or simply an $\mathcal{O}_X$*-module*) is a sheaf $\mathscr{F}$ on X, such that for each open set $U \subseteq X$, the group $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathscr{F}(U) \rightarrow \mathscr{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$. A *morphism* $\mathscr{F} \rightarrow \mathscr{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathscr{F}(U) \rightarrow \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Note that the kernel, cokernel, and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. If $\mathscr{F}'$ is a subsheaf of $\mathcal{O}_X$-modules of an $\mathcal{O}_X$-module $\mathscr{F}$, then the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is an $\mathcal{O}_X$-module. Any direct sum, direct product, direct limit, or inverse limit of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{O}_X$-modules, we denote the group of morphisms from $\mathscr{F}$ to $\mathscr{G}$ by $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$, or sometimes $\operatorname{Hom}_X(\mathscr{F},\mathscr{G})$ or $\operatorname{Hom}(\mathscr{F},\mathscr{G})$ if no confusion can arise. A sequence of $\mathcal{O}_X$-modules and morphisms is *exact* if it is exact as a sequence of sheaves of abelian groups.

If U is an open subset of X, and if $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $\mathscr{F}|_U$ is an $\mathcal{O}_X|_U$-module. If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{O}_X$-modules, the presheaf

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathscr{F}|_U,\mathscr{G}|_U)$$

is a sheaf, which we call the *sheaf* $\mathscr{H}om$ (Ex. 1.15), and denote by $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$. It is also an $\mathcal{O}_X$-module.

We define the *tensor product* $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ of two $\mathcal{O}_X$-modules to be the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$. We will often write simply $\mathscr{F} \otimes \mathscr{G}$, with $\mathcal{O}_X$ understood.

An $\mathcal{O}_X$-module $\mathscr{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if X can be covered by open sets U for which $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In that case the *rank* of $\mathscr{F}$ on such an open set is the number of copies of the structure sheaf needed (finite or infinite). If X is connected, the rank of a locally free sheaf is the same everywhere. A locally free sheaf of rank 1 is also called an *invertible sheaf*.

A *sheaf of ideals* on X is a sheaf of modules $\mathscr{I}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set U, $\mathscr{I}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Let $f:(X,\mathcal{O}_X) \rightarrow (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces (see §2). If $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathscr{F}$ is an $f_*\mathcal{O}_X$-module. Since we have the morphism $f^\#:\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ of sheaves of rings on Y, this gives $f_*\mathscr{F}$ a natural structure of $\mathcal{O}_Y$-module. We call it the *direct image* of $\mathscr{F}$ by the morphism f.

Now let $\mathscr{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathscr{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1} (Ex. 1.18) we have a morphism