ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

In this note we outline a connection between the generalized cohomology theories of unoriented cobordism and (weakly-) complex cobordism and the theory of formal commutative groups of one variable [4], [5]. This connection allows us to apply Cartier's theory of typical group laws to obtain an explicit decomposition of complex cobordism theory localized at a prime p into a sum of Brown-Peterson cohomology theories [1] and to determine the algebra of cohomology operations in the latter theory.

1. Formal group laws. If R is a commutative ring with unit, then by a *formal* (commutative) *group law* over R one means a power series $F(X, Y)$ with coefficients in R such that

(i) $F(X, 0)=F(0, X)=X$,

(ii) $F(F(X, Y), Z)=F(X, F(Y, Z))$,

(iii) $F(X, Y)=F(Y, X)$. We let $I(X)$ be the "inverse" series satisfying $F(X, I(X))=0$ and let

$$\omega(X) = dX/F_2(X, 0)$$

be the normalized invariant differential form, where the subscript 2 denotes differentiation with respect to the second variable. Over $R\otimes \mathbf{Q}$, there is a unique power series $l(X)$ with leading term X such that

$$l(F(X, Y)) = l(X) + l(Y). \tag{1}$$

The series $l(X)$ is called the *logarithm* of F and is determined by the equations

$$\begin{aligned} l'(X)dX &= \omega(X), \\ l(0) &= 0. \end{aligned} \tag{2}$$

2. The formal group law of complex cobordism theory. By *complex cobordism theory* $\Omega^*(X)$ we mean the generalized cohomology theory associated to the spectrum MU. If E is a complex vector bundle of dimension n over a space X, we let $c_i^\Omega(E)\in\Omega^{2i}(X)$, $1\leqq i\leqq n$ be

[1] Alfred P. Sloan Foundation Fellow. This work was supported also by NSF GP9006.

the Chern classes of E in the sense of Conner-Floyd [3]. Since $\Omega^*(CP^\infty \times CP^\infty) = \Omega^*(pt)[[x, y]]$, where $x = c_1^\Omega(\mathcal{O}(1)) \otimes 1$, $y = 1 \otimes c_1^\Omega(\mathcal{O}(1))$ and $\mathcal{O}(1)$ is the canonical line bundle on CP^∞, there is a unique power series $F^\Omega(X, Y) = \sum a_{kl} X^k Y^l$ with $a_{kl} \in \Omega^{2-2k-2l}(pt)$ such that

$$c_1^\Omega(L_1 \otimes L_2) = F^\Omega(c_1^\Omega(L_1), c_1^\Omega(L_2)) \tag{3}$$

for any two complex line bundles with the same base. The power series F^Ω is a formal group law over $\Omega^{ev}(pt)$.

THEOREM 1. *Let E be a complex vector bundle of dimension n, let $f: PE' \to X$ be the associated projective bundle of lines in the dual E' of E, and let $\mathcal{O}(1)$ be the canonical quotient line bundle on PE'. Then the Gysin homomorphism $f_*: \Omega^q(PE') \to \Omega^{q-2n+2}(X)$ is given by the formula*

$$f_*(u(\xi)) = \operatorname{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^{n} F^\Omega(Z, I\lambda_j)} \cdot \tag{4}$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^\Omega(\mathcal{O}(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^Ω, and the λ_j are the dummy variables of which $c_q^\Omega(E)$ is the qth-elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7].

Applying the theorem to the map $f: CP^n \to pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1} \cdot \tag{5}$$

3. The universal nature of cobordism group laws.

THEOREM 2. *The group law F^Ω over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^Ω to F.*

PROOF. Let F_u over L be a universal formal group law [5] and let $h: L \to \Omega^{ev}(pt)$ be the unique ring homomorphism sending F_u to F^Ω. The law F_u over $L \otimes Q$ is universal for laws over Q-algebras. Such a law is determined by its logarithm series which can be any series with leading term X. Thus if $\sum p_n X^{n+1}/n+1$ is the logarithm of F_u, $L\otimes \mathbf{Q}$ is a polynomial ring over $\mathbf{Q}$ with generators p_i. By (5) $hp_i = P_i$, so as $\Omega^*(pt)\otimes \mathbf{Q} \cong \mathbf{Q}[P_1, P_2, \cdots]$, it follows that $h\otimes \mathbf{Q}$ is an isomorphism.

By Lazard [5, Theorem II], L is a polynomial ring over $\mathbf{Z}$ with infinitely many generators; in particular L is torsion-free and hence h is injective. To prove surjectivity we show $h(L)$ contains generators for $\Omega^*(pt)$. First of all $hp_n = P_n \in h(L)$ because $p_n \in L$ as it is the nth coefficient of the invariant differential of F_u. Secondly we must consider elements of the form $[M_n]$ where M_n is a nonsingular hypersurface of degree $k_1, \cdots, k_r$ in $CP^{n_1}\times \cdots \times CP^{n_r}$. Let π be the map of this multiprojective space to a point. Then $[M_n] = \pi_* c_1^\Omega(L_1^{k_1}\otimes \cdots \otimes L_r^{k_r})$, where L_j is the pull-back of the canonical line bundle on the jth factor. The Chern class of this tensor product may be written using the formal group law F^Ω in the form $\sum \pi^* a_{i_1\cdots i_r} z_1^{i_1} \cdots z_r^{i_r}$, where $0 \leqq i_j \leqq n_j$, $1 \leqq j \leqq r$, where $z_i = c_1^\Omega(L_i)$, and where $a_{i_1\cdots i_r} \in h(L)$. Since

$$\pi_* z_1^{i_1} \cdots z_r^{i_r} = \prod_{j=1}^{r} P_{n_j - i_j}$$

also belongs to $h(L)$, it follows that $[M_n] \in h(L)$. Thus h is an isomorphism and the theorem is proved.

We can also give a description of the unoriented cobordism ring using formal group laws. Let $\eta^*(X)$ be the unoriented cobordism ring of a space X, that is, its generalized cohomology with values in the spectrum MO. There is a theory of Chern (usually called Whitney) classes for real vector bundles with $c_i(E) \in \eta^i(X)$. The first Chern class of a tensor product of line bundles gives rise to a formal group law F^η over the commutative ring $\eta^*(pt)$. Since the square of a real line bundle is trivial, we have the identity

$$F^\eta(X, X) = 0. \tag{6}$$

THEOREM 3. *The group law F^η over $\eta^*(pt)$ is a universal formal (commutative) group law over a ring of characteristic two satisfying* (6).

4. **Typical group laws (after Cartier [2]).** Let F be a formal group law over R. Call a power series $f(X)$ with coefficients in R and without constant term a *curve* in the formal group defined by the law. The set of curves forms an abelian group with addition $(f +^F g)(X) = F(f(X), g(X))$ and with operators

$$([r]f)(X) = f(rX) \qquad r \in R$$
$$(V_n f)(X) = f(X^n) \qquad n \geqq 1$$
$$(F_n f)(X) = \sum_{i=1}^{n} {}^F f(\zeta_i X^{1/n}) \qquad n \geqq 1,$$

where the ζ_i are the nth roots of 1. The set of curves is filtered by the order of a power series and is separated and complete for the filtration.

If R is an algebra over $\mathbf{Z}_{(p)}$, the integers localized at the prime p, then a curve is said to be *typical* if $F_q f = 0$ for any prime $q \neq p$. If R is torsion-free then it is the same to require that the series $l(f(X))$ over $R \otimes \mathbf{Q}$ has only terms of degree a power of p, where l is the logarithm of F. The group law F is said to be a *typical law* if the curve $\gamma_0(X) = X$ is typical. There is a canonical change of coordinates rendering a given law typical. Indeed let c_F be the curve

$$c_F^{-1} = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n \gamma_0 \tag{7}$$

where the sum as well as division by n prime to p is taken in the filtered group of curves and where μ is the Möbius function. Then the group law $(c_F * F)(X, Y) = c_F(F(c_F^{-1}X, c_F^{-1}Y))$ is typical.

5. **Decomposition of $\Omega^*_{(p)}$.** For the rest of this paper p is a fixed prime. Let $\Omega^*_{(p)}(X) = \Omega^*(X) \otimes \mathbf{Z}_{(p)}$ and let $\xi = c_F \Omega$. Then $\xi(Z)$ is a power series with leading term Z with coefficients in $\Omega^*_{(p)}(pt)$, so there is a unique natural transformation $\hat{\xi}$: $\Omega^*_{(p)}(X) \rightarrow \Omega^*_{(p)}(X)$ which is stable, a ring homomorphism, and such that

$$\hat{\xi} c_1^{\Omega}(L) = \xi(c_1^{\Omega}(L))$$

for all line bundles L.

THEOREM 4. *The operation $\hat{\xi}$ is homogeneous, idempotent, and its values on $\Omega^*_{(p)}(pt)$ are*

$$\hat{\xi}(P_n) = P_n \quad \textit{if } n = p^a - 1 \textit{ for some } a \geqq 0,$$
$$= 0 \quad \textit{otherwise.}$$

Let $\Omega T^*(X)$ be the image of $\hat{\xi}$. Then there are canonical ring isomorphisms

$$\Omega T^*(pt) \underset{\Omega^*_{(p)}(pt)}{\otimes} \Omega^*_{(p)}(X) \cong \Omega T^*(X), \tag{8}$$

$$\Omega^*_{(p)}(pt) \underset{\Omega T^*(pt)}{\otimes} \Omega T^*(X) \cong \Omega^*_{(p)}(X). \tag{9}$$

ΩT^* is the generalized cohomology theory associated to the Brown-Peterson spectrum [1] *localized at p.*

It is also possible to apply typical curves to unoriented cobordism theory where the prime involved is $p=2$. One defines similarly an idempotent operator ξ whose image now is $H^*(X, \mathbf{Z}/2\mathbf{Z})$; there is also a canonical ring isomorphism

$$\eta^*(pt) \otimes H^*(X, \mathbf{Z}/2\mathbf{Z}) \simeq \eta^*(X)$$

analogous to (9).

6. **Operations in ΩT^*.** If $\pi: \Omega^*_{(p)} \to \Omega T^*$ is the surjection induced by ξ, then π carries the Thom class in $\Omega^*_{(p)}(MU)$ into one for ΩT^*. As a consequence ΩT^* has the usual machinery of characteristic classes with $c_i^{\Omega T}(E) = \pi c_i^{\Omega}(E)$ and $F^{\Omega T} = \pi F^{\Omega}$. Let $t = (t_1, t_2, \cdots)$ be an infinite sequence of indeterminates and set

$$\phi_t(X) = \sum_{n \geq 0}^{F^{\Omega T}} t_n X^{p^n} \qquad t_0 = 1$$

where the superscript on the summation indicates that the sum is taken as curves in the formal group defined by $F^{\Omega T}$. There is a unique stable multiplicative operation $(\phi_t^{-1})^\wedge: \Omega^*(X) \to \Omega T^*(X)[t_1, t_2, \cdots]$ such that

$$(\phi_t^{-1})^\wedge c_1^{\Omega}(L) = \phi_t^{-1}(c_1^{\Omega T}(L))$$

for all line bundles L. This operation can be shown using (8) to kill the kernel of π and hence it induces a stable multiplicative operation

$$r_t: \Omega T^*(X) \to \Omega T^*(X)[t_1, t_2, \cdots].$$

Writing

$$r_t(x) = \sum_{\alpha} r_\alpha(x) t^\alpha \qquad \text{if } x \in \Omega T^*(X)$$

where the sum is taken over all sequences $\alpha = (\alpha_1, \alpha_2, \cdots)$ of natural numbers all but a finite number of which are zero, we obtain stable operations

$$r_\alpha: \Omega T^*(X) \to \Omega T^*(X).$$

THEOREM 5. (i) *r_α is a stable operation of degree $2\sum_i \alpha_i(p^i - 1)$. Every stable operation may be uniquely written as an infinite sum*

$$\sum_{\alpha} u_\alpha r_\alpha \qquad u_\alpha \in \Omega T^*(pt)$$

and every such sum defines a stable operation.

(ii) *If* $x, y \in \Omega T^*(X)$, *then*

$$r_\alpha(xy) = \sum_{\beta+\gamma=\alpha} r_\beta(x) r_\gamma(y).$$

(iii) *The action of* r_α *on* $\Omega T^*(pt)$ *is given by*

$$r_t(P_{p^n-1}) = \sum_{h=0}^{n} p^{n-h} P_{p^h-1} t_{n-h}^{p^h}.$$

(iv) *If* $t' = (t_1', t_2', \cdots)$ *is another sequence of indeterminates, then the compositions* $r_\alpha \circ r_\beta$ *are found by comparing the coefficients of* $t^\alpha t'^\beta$ *in*

$$r_t \circ r_{t'} = \sum_\gamma \Phi(t, t')^\gamma r_\gamma$$

where $\Phi = (\Phi_1(t_1; t_1'), \Phi_2 = (t_1, t_2; t_1', t_2'), \cdots)$ *is the sequence of polynomials with coefficients in* $\Omega T^*(pt)$ *in the variables* t_i *and* t_i' *obtained by solving the equations*

$$\sum_{h=0}^{N} p^{N-h} P_{p^h-1} \Phi_{N-h}^{p^h} = \sum_{k+m+n=N} p^{m+n} P_{p^k-1} t_m^{p^k} t_n'^{p^{k+m}}.$$

This theorem gives a complete description of the algebra of operations in ΩT^*. The situation is similar to that for Ω^* except the set of $\boldsymbol{Z}_{(p)}$-linear combinations of the r_α's is not closed under composition.

Bibliography

1. E. H. Brown, Jr. and F. P. Peterson, *A spectrum whose* Z_p *cohomology is the algebra of reduced p-th powers*, Topology **5** (1966), 149–154.
2. P. Cartier, *Comptes rendus*, **265** (1967), 49, 129.
3. P. E. Conner and E. E. Floyd, *The relation of cobordism to K-theories*, Lectures Notes in Math., vol. 28, 1966, Springer-Verlag, Berlin.
4. A. Fröhlich, *Formal groups*, Lecture Notes in Math., vol. 74, 1968, Springer-Verlag, Berlin.
5. M. Lazard, *Sur les groupes de Lie formels à un paramètre*, Bull. Soc. Math. France **83** (1955), 251–274.
6. S. P. Novikov, *The methods of algebraic topology from the viewpoint of cobordism theories*, Izv. Akad. Nauk SSSR **31** (1967), 855–951.
7. J. T. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. **1** (1968), 149–159.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139