Algebraic Topology

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These are incomplete notes of a second semester basic topology course taught in the Sping 2013. A basic reference is Allen Hatcher's book [Ha].

Contents

1 Introduction

The following is an attempt at explaining what 'topology' is.

- Topology is the study of qualitative/global aspects of shapes, or – more generally – the study of qualitative/global aspects in mathematics.

A simple example of a 'shape' is a 2-dimensional surface in 3-space, like the surface of a ball, a football, or a donut. While a football is different from a ball (try kicking one...), it is *qualitatively* the same in the sense that you could squeeze a ball (say a balloon to make squeezing easier) into the shape of a football. While any surface is *locally homeomorphic* $\mathbb{R}^2$ (i.e., every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$) by definition of 'surface', the 'global shape' of two surfaces might be different meaning that they are not homeomorphic (e.g. the surface of a ball is not homeomorphic to the surface of a donut). The French mathematician Henry Poincaré (1854-1912) is regarded as one of the founders of topology, back then known as 'analysis situ'. He was interested in understanding qualitative aspects of the solutions of differential equation.

There are basically three flavors of topology:

1. Point set Topology: Study of general properties of topological spaces
2. Differential Topology: Study of manifolds (ideally: classification up to homeomorphism/diffeomorphism).
3. Algebraic topology: trying to distinguish topological spaces by assigning to them algebraic objects (e.g. a group, a ring, ...).

Let us go in more detail concerning algebraic topology, since that is the topic of this course. Before mentioning two examples of algebraic objects associated to topological spaces, let us make the purpose of assigning these algebraic objects clear: if X and Y are *homeomorphic* objects, we insist that the associated algebraic objects $A(X)$, $A(Y)$ are isomorphic. That means in particular, that if we find that $A(X)$ and $A(Y)$ are not isomorphic, then we can conclude that the spaces X and Y are not homeomorphic. In other words, the algebraic objects help us to *distinguish homeomorphism classes of topological spaces*.

Here are two examples of algebraic objects we can assign to topological spaces, which satisfy this requirement. We will discuss them in more detail below:

Homotopy groups To any topological space X equipped with a distinguished point $x_0 \in X$ (called the *base point*), we can associate groups $\pi_n(X, x_0)$ for $n = 1, 2, \dots$ called *homotopy groups* of X. These are *abelian* groups for $n \geq 2$.

Homology groups To any topological space X we can associate abelian groups $H_n(X)$ for $n = 0, 1, \dots$, called *homology groups of* X.

The advantages/disadvantages of homotopy versus homology groups are

- homotopy groups are easy to define, but extremely hard to calculate;
- homology groups are harder to define, but comparatively easier to calculate (with the appropriate tools in place, which will take us about half the semester)

Let us illustrate these statements in a simple example. We will show (in about a month) that the homology group of spheres look as follows:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

The homotopy groups of spheres are much more involved; for example:

k	1	2	3	4	5	6	7	8	9
$\pi_k(S^2, x_0)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$

It is perhaps surprising that these homotopy groups are *not known* for large k (not only in the sense that we don't have a 'closed formula' for these groups, but also in the sense that we don't have an algorithm that would crank out these groups one after another on a computer if we just give it enough time...). This holds not only for S^2, but for any sphere S^n (except $n = 1$). In fact, the calculation of the homotopy groups of spheres is something akin to the 'holy grail' of algebraic topology.

1.1 Homotopy groups

Suppose f and g are continuous maps from a topological space X to a topological space Y. Then true to the motto that in topology we are interested in 'qualitative' properties we shouldn't try to distinguish between f and g if they can be 'deformed' into each other in the sense that for each $t \in [0,1]$ there is a map $f_t\colon X \to Y$ such that $f_0 = f$ and $f_1 = g$, and such that the family of maps f_t 'depends continuously on t'. The following definition makes precise what is meant by 'depending continuously on t' and introduces the technical terminology 'homotopic' for the informal 'can be deformed into each other'.

Definition 1. Two continuous maps $f, g\colon X \to Y$ between topological spaces X, Y are *homotopic* if there is a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a *homotopy between f and g*. We will denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.

We note that if H is a homotopy, then we have a family of maps $f_t\colon X \to Y$ parametrized by $t \in [0,1]$ interpolating between f and g, given by $f_t(x) = H(t,x)$. Conversely, if $f_t\colon X \to Y$ is a family of maps parametrized by $t \in [0,1]$, then we can define a map $H\colon [0,1] \times X \to Y$ by the above formula. We note that the continuity requirement for H implies not only that each map f_t is continuous, but also implies that for fixed $x \in X$ the map $t \mapsto f_t(x)$ is continuous. In other words, our idea of requiring that f_t should 'depend continuously on t' is made precise by requiring continuity of H.

Examples of homotopic maps.

1. Any two maps $f, g\colon X \to \mathbb{R}$ are homotopic; in other words, $[X, \mathbb{R}]$ is a one point set. A homotopy $H\colon X \times [0,1] \to \mathbb{R}$ is given by $H(x,t) = (1-t)f(x) + tg(x)$. We note that for fixed x the map $[0,1] \to \mathbb{R}$ given by $t \mapsto (1-t)f(x) + tg(x)$ is the affine linear path (aka straight line) from $f(x)$ to $g(x)$. For this reason, the homotopy H is called a *linear homotopy.* The construction of a linear homotopy can be done more generally for maps $f, g\colon X \to Y$ if Y is a vector space, or a convex subspace of a vector space.

2. A map $S^1 \to Y$ is a loop in the space Y. Physically, we can think of it as the trajectory of a particle that moves in the topological space Y, returning to its original position after some time. In general, there are maps $f, g\colon S^1 \to Y$ that are not homotopic. For example, given an integer $k \in \mathbb{Z}$, let

$$f_k\colon S^1 \to S^1 \qquad \text{be the map given by} \qquad f_k(z) = z^k.$$

   Physically that describes a particle that moves $|k|$ times around the circle, going counterclockwise for $k > 0$ and clockwise for $k < 0$. We will prove that f_k and f_ℓ are homotopic if and only if $k = \ell$. Moreover, we will show that *any* map $f\colon S^1 \to S^1$ is homotopic to f_k for some $k \in \mathbb{Z}$. In other words, we will prove that there is a bijection

$$\mathbb{Z} \leftrightarrow [S^1, S^1] \qquad \text{given by} \qquad k \mapsto f_k$$

   This fact will be used to prove the fundamental theorem of algebra.

Sometimes it is useful to consider pairs (X, A) of topological spaces, meaning that X is a topological space and A is a subspace of X. If (Y, B) is another pair, we write

$$f\colon (X, A) \longrightarrow (Y, B)$$

if f is a continuous map from X to Y which sends A to B. Two such maps $f, g\colon (X, A) \to (Y, B)$ are *homotopic* if there is a map

$$H\colon (X \times I, A \times I) \longrightarrow (Y, B)$$

with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We will use the notation $[(X, A), (Y, B)]$ for the set of homotopy classes of maps from (X, A) to (Y, B).

Definition 2. Let X be a topological space, and let x_0 be a point of X. Then the *n-th homotopy group of* (X, x_0) is by definition

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)].$$

Here $I^n := \underbrace{I \times \cdots \times I}_{n} \subset \mathbb{R}^n$ is the n-dimensional cube, and ∂I^n is its boundary.

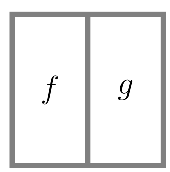
A map $f\colon (I, \partial) \to (X, x_0)$ is geometrically a path in X parametrized by the unit interval $I = [0, 1]$ with starting point $f(0) = x_0$ and endpoint $f(1) = x_0$. Such maps are also called *based loops*. Similarly, a map $f\colon (I^2, \partial I^2) \to (X, x_0)$ is geometrically a membrane in X parametrized by the square I^2, such that the boundary of the square maps to the base point x_0.

As suggested by the terminology of the above definition, the set $[(I^n, \partial I^n), (X, x_0)]$ has in fact the structure of a *group*. Given two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$, their product $f * g\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

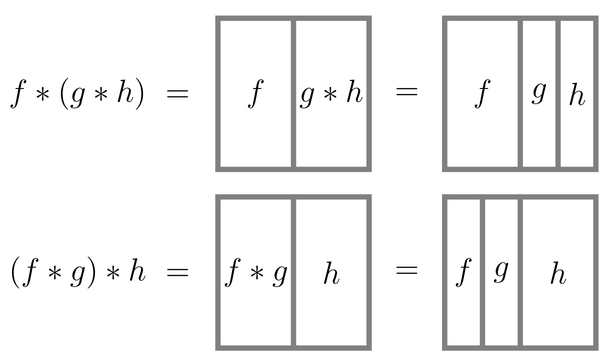
$$(f * g)(t_1, \dots, t_n) := \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{for } 0 \le t_1 \le \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{for } \frac{1}{2} \le t_1 \le 1 \end{cases}$$

We note that this is a well-defined map, since for $t_1 = \frac{1}{2}$ the points $(2t_1, t_2, \dots, t_n)$ and $(2t_1 - 1, t_2, \dots, t_n)$ both belong to the boundary ∂I^n, and hence both map to x_0 via f and g. Moreover, $f * g$ is continuous since its restriction to the closed subsets consisting of the points $t = (t_1, \dots, t_n)$ with $t_1 \le \frac{1}{2}$ resp. $t_1 \ge \frac{1}{2}$ is continuous. We will refer to $f * g$ as the *concatenation* of the maps f and g, since for $n = 1$ the map $f * g\colon I \to X$ is usually referred to as the concatenation of the paths f and g.

The following picture shows where $f * g$ maps points in the square I^2: if $t = (t_1, t_2)$ belongs to the left half of the square, it is mapped via f; points in the right half map via g (here we implicitly identify the left and right halves of the square again with I^2). In particular the boundaries of the two halves map to the base point x_0; this subset of I^2 is indicated by the gray lines in the picture.

$f * g =$ 

Next we want to address the question whether given $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$ the maps $f * (g * h)$ and $(f * g) * h$ agree. Thinking in terms of pictures, we have



which shows that these two maps do not agree. However, they are homotopic to each other. We leave it to the reader to provide a proof of this. This implies the third of the following equalities in $\pi_n(X, x_0)$; the others hold by definition:

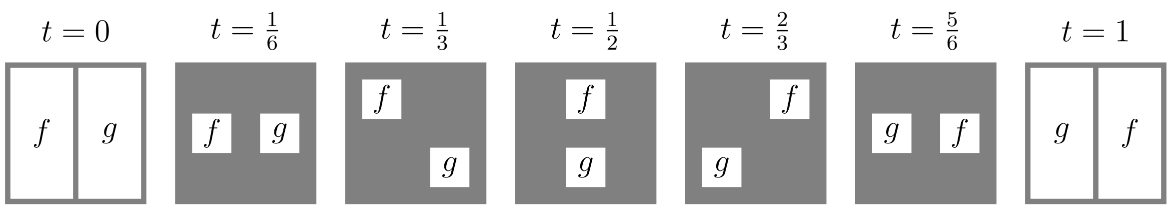
$$[f]([g][h]) = [f]([g * h]) = [f * (g * h)] = [(f * g) * h] = [f * g][h] = ([f][g])[h].$$

This shows that concatenation induces an associative product on $\pi_n(X, x_0)$. We leave it to the reader to show that this product gives $\pi_n(X, x_0)$ of a group where the unit element is represented by the constant map, and the inverse of an element $[f] \in \pi_n(X, x_0)$ is represented by $\bar{f}$, defined by $\bar{f}(t_1, \dots, t_n) := f(1 - t_1, t_2, \dots, t_n)$.

The group $\pi_1(X, x_0)$ is called the *fundamental group* of X, while the groups $\pi_n(X, x_0)$ for $n \geq 2$ are referred to as *higher homotopy groups*. Examples show that the fundamental group is in general not abelian. For example, the fundamental group of the "figure eight" is the free group generated by two elements. By contrast, for higher homotopy groups we have the following result.

Lemma 3. *For $n \geq 2$ the group $\pi_n(X, x_0)$ is abelian.*

Proof. We need to show that for maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ the concatenations $f * g$ and $g * f$ are homotopic to each other (as maps of pairs). Such a homotopy H is given by a continuous family of maps $H_t\colon (I^n, \partial I^n) \to (X, x_0)$ which agrees with $f * g$ for $t = 0$ and with $g * f$ for $t = 1$. Thinking of each such maps as a picture, like the one for $f * g$ above, such a homotopy H_t is a family of pictures parametrized by $t \in [0, 1]$. Interpreting t as "time", this means that the homotopy H_t is a *movie*! Here it is:



Here all points in the gray areas of the square map to the base point. So shrinking the rectangles inside of the square labeled f resp. g allows us to rotate them past each other, a move which is not possible for $n = 1$, but for all $n \geq 2$. □

1.2 The Euler characteristic of closed surfaces

The goal of this section is to discuss the Euler characteristic of closed surfaces, that is, compact manifolds without boundary of dimension 2. We begin by recalling the definition of manifolds.

Definition 4. A *manifold of dimension n* or *n-manifold* is a topological space X which is locally homeomorphic to $\mathbb{R}^n$, that is, every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. Moreover, it is useful and customary to require that X is Hausdorff (see Definition **??**) and second countable (see Definition **??**). A *manifold with boundary of dimension n* is defined by replacing $\mathbb{R}^n$ in the definition above by the half-space $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. If X is an n-manifold with boundary, its *boundary* ∂X consists of those points of X which via some homeomorphism $U \approx V \subset \mathbb{R}^n_+$ correspond to points in the hyperplane given by the equation $x_1 = 0$. The complement $X \setminus \partial X$ is called the *interior of* X. A *closed* n*-manifold* is a compact n-manifold without boundary.

Examples of manifolds of dimension 1**.** An open interval (a, b) is a 1-manifold. A closed interval $[a, b]$ is a 1-manifold with boundary $\{a, b\}$. A half-open interval $(a, b]$ is a 1-manifold with boundary $\{b\}$.

A non-example. The subspace $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$ of $\mathbb{R}^2$ consisting of the x-axis and y-axis is not a 1-dimensional manifold, since X is not locally homeomorphic to $\mathbb{R}^1$ at the origin $x = (0, 0)$. To prove this intuitively obvious fact, suppose that U is an open neighborhood of $(0, 0)$ which is homeomorphic to an open subset $V \subset \mathbb{R}$. Replacing U by the connected component of U containing $(0, 0)$, and V by the image of that component, we can assume that U and V are connected. This implies that V is an open interval. Restricting the homeomorphism $f : U \to V$, we obtain a homeomorphism $U \setminus \{(0, 0)\} \approx V \setminus f(0, 0)$. This is the desired contradiction, since $U \setminus \{(0, 0)\}$ has four connected components, while $V \setminus f(0, 0)$ has two.

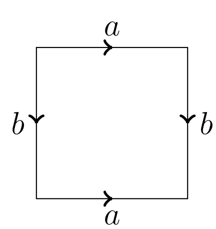
Examples of higher dimensional manifolds.

1. Any open subset $U \subset \mathbb{R}^n_+$ is an n-manifold whose boundary ∂U is the intersection of U with the hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$.

2. The n-sphere $S^n := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is an n-manifold.

3. The n-disk $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is an n-manifold with boundary $\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n \mid ||x|| = 1\}$.

4. The *torus* $T^2 := S^1 \times S^1$ is a manifold of dimension 2. There at least two other ways to describe the torus. The usual picture we draw describes the torus as a subspace of $\mathbb{R}^3$. It can also be constructed as a quotient space of the square I^2: we identify the two horizontal edges of the square to obtain a cylinder, and then the two boundary circles to obtain the torus T^2. From a formal point of view, the last sentence describes an equivalence relation $\sim$ on I^2, and the claim is that the quotient space $I^2/\sim$ is homeomorphic to $S^1 \times S^1$. It will be convenient to use pictures for this and similar quotient spaces. Here is the picture for the quotient space $I^2/\sim$ described above:



The definition of the Euler characteristic of a closed 2-manifold Σ will involve choosing a "pattern of polygons" on Σ. By this we mean a graph Γ (consisting of vertices and edges) on Σ, such that all connected components of the complement $\Sigma \setminus \Gamma$ are homeomorphic to open discs. For example, the boundary of the 3-dimensional cube is a 2-dimensional manifold homeomorphic to S^2. The 8 vertices and 12 edges of the cube form a graph Γ on S^2; the complement $S^2 \setminus \Gamma$ consists of the 6 faces of the cube.

Given a pattern of polygons Γ on a surface Σ, we define the integer

$$\chi(\Sigma, \Gamma) := \#V - \#E + \#F,$$

where V is the set of vertices, E is the set of edges, and F is the set of faces.

Lemma 5. *$\chi(\Sigma, \Gamma) = \chi(\Sigma, \Gamma')$ for any two choices of graphs Γ, Γ'.*

Before proving this lemma, let us illustrate the statement in the example of two patterns on the 2-sphere S^2:

1. Let Γ be the graph described above obtained by identifying S^2 with the boundary of the cube. Then $\chi(S^2, \Gamma) = 8 - 12 + 6 = 2$.

2. Let Γ' be the graph obtained by identifying S^2 with the boundary of the tetrahedron. Then $\chi(S^2, \Gamma') = 4 - 6 + 4 = 2$.

Proof. □

2 Appendix on Pointset Topology

This appendix is a quick introduction into point set topology. An excellent source for more detailed information is the book [Mu] by James Munkres.

2.1 Metric spaces and topological spaces

We recall that a map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ between Euclidean spaces is *continuous* if and only if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \in X \quad d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon, \tag{6}$$

where $d(x,y) = |x - y| \in \mathbb{R}_{\geq 0}$ is the distance of two points x, y in some Euclidean space.

Example 7. (Examples of continuous maps.)

1. The addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 + x_2$;

2. The multiplication map $m\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 x_2$;

The proofs that these maps are continuous are simple estimates that you probably remember from calculus. Since the continuity of *all* the maps we'll look at in these notes is proved by expressing them in terms of the maps a and m, we include the proofs of continuity of a and m for completeness.

Proof. To prove that the addition map a is continuous, suppose $x = (x_1, x_2) \in \mathbb{R}^2$ and $\epsilon > 0$ are given. We claim that for $\delta := \epsilon/2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(a(x), a(y)) < \epsilon$ and hence a is a continuous function. To prove the claim, we note that

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

and hence $|x_1 - y_1| \leq d(x, y)$, $|x_1 - y_1| \leq d(x, y)$. It follows that

$$d(a(x), a(y)) = |a(x) - a(y)| = |x_1 + x_2 - y_1 - y_2| \leq |x_1 - y_1| + |x_2 - y_2| \leq 2d(x, y) < 2\delta = \epsilon.$$

To prove that the multiplication map m is continuous, we claim that for

$$\delta := \min\{1, \epsilon/(|x_1| + |x_2| + 1)\}$$

and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(m(x), m(y)) < \epsilon$ and hence m is a continuous function. The claim follows from the following estimates:

$$\begin{aligned} d(m(y), m(x)) &= |y_1y_2 - x_1x_2| = |y_1y_2 - x_1y_2 + x_1y_2 - x_1x_2| \\ &\leq |y_1y_2 - x_1y_2| + |x_1y_2 - x_1x_2| = |y_1 - x_1||y_2| + |x_1||y_2 - x_2| \\ &\leq d(x, y)(|y_2| + |x_1|) \leq d(x, y)(|x_2| + |y_2 - x_2| + |x_1|) \\ &\leq d(x, y)(|x_1| + |x_2| + 1) < \delta(|x_1| + |x_2| + 1) \leq \epsilon \end{aligned}$$

□

Lemma 8. *The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ has the following properties:*

1. $d(x, y) = 0$ *if and only if* $x = y$;
2. $d(x, y) = d(y, x)$ *(symmetry);*
3. $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Definition 9. A *metric space* is a set X equipped with a map

$$d\colon X \times X \to \mathbb{R}_{\geq 0}$$

with properties (1)-(3) above. A map $f\colon X \to Y$ between metric spaces X, Y is

an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$;

continuous if condition (6) is satisfied.

Two metric spaces X, Y are *isometric* (resp. *homeomorphic*) if there are isometries (resp. continuous maps) $f\colon X \to Y$ and $g\colon Y \to X$ which are inverses of each other.

Example 10. An important class of examples of metric spaces are subsets of $\mathbb{R}^n$. Here are particular examples we will be talking about during the semester:

1. The *n-disk* $D^n := \{x \in \mathbb{R}^n \mid |x| \le 1\} \subset \mathbb{R}^n$, and more generally, the *n-disk of radius* r $D^n_r := \{x \in \mathbb{R}^n \mid |x| \le r\}$. We note that D^2_r is homeomorphic to D^2 for all r, but D^2_r is isometric to D^2 if and only if $r = 1$. (To see that D^n_r is not isometric to D^n_s we note if a metric space X is isometric to a metric space Y, then $\operatorname{diam}(X) = \operatorname{diam}(Y)$, where $\operatorname{diam}(X)$, the *diameter of* X is defined by $\operatorname{diam}(X) := \sup\{d(x, y) \mid x, y \in X\} \in \mathbb{R}_{\ge 0} \cup \{\infty\}$. It is easy to see that $\operatorname{diam}(D^n_r) = 2r$.)

2. The *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \subset \mathbb{R}^{n+1}$.

3. The *torus* $T = \{v \in \mathbb{R}^3 \mid d(v, C) = r\}$ for $0 < r < 1$. Here $C = \{(x, y, 0) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$ is the standard circle, and $d(x, C) = \inf_{y \in C} d(x, y)$ is the distance between x and C.

4. The *general linear group*

$$\begin{aligned} GL_n(\mathbb{R}) &= \{\text{vector space isomorphisms } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &\leftrightarrow \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) \neq 0\} \\ &= \{\text{invertible } n \times n\text{-matrices}\} \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} = \mathbb{R}^{n^2} \end{aligned}$$

   Here the bijection sends $f\colon \mathbb{R}^n \to \mathbb{R}^n$ to $(f(e_1), \dots, f(e_n))$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.

5. The *special linear group*

$$SL_n(\mathbb{R}) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

6. The *orthogonal group*

$$\begin{aligned} O(n) &= \{\text{linear isometries } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{n^2} \end{aligned}$$

   We recall that a collection of vectors $v_i \in \mathbb{R}^n$ is *orthonormal* if $|v_i| = 1$ for all i, and v_i is perpendicular to v_j for $i \neq j$.

7. The *special orthogonal group*

$$SO(n) = \{(v_1, \dots, v_n) \in O(n) \mid \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

8. The *Stiefel manifold*

$$\begin{aligned} V_k(\mathbb{R}^n) &= \{\text{linear isometries } f \colon \mathbb{R}^k \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{kn} \end{aligned}$$

Example 11. The following maps between metric spaces are continuous. While it is possible to prove their continuity using the definition of continuity, it will be much simpler to prove their continuity by 'building' these maps using compositions and products from the continuous maps a and m of Example 7. We will do this below in Lemma 22.

1. Every polynomial function $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuous. We recall that a polynomial function is of the form $f(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$ for $a_{i_1, \dots, i_n} \in \mathbb{R}$.

2. Let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices. Then the map

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

given by matrix multiplication is continuous. Here we use the fact that a map to the product $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R} \times \dots \times \mathbb{R}$ is continuous if and only if each component map is continuous (see Lemma 21), and each matrix entry of AB is a polynomial and hence a continuous function of the matrix entries of A and B. Restricting to the invertible matrices $GL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$, we see that the multiplication map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

is continuous. The same holds for the subgroups $SO(n) \subset O(n) \subset GL_n(\mathbb{R})$.

3. The map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous (this is a homework problem). The same statement follows for the subgroups of $GL_n(\mathbb{R})$.

Definition 12. Let X be a metric space. A subset $U \subset X$ is *open* if for every point $x \in U$ there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Here $B_\epsilon(x) = \{y \in X \mid d(y, x) < \epsilon\}$ is the *ball of radius ϵ around x*.

Lemma 13. *A map $f \colon X \to Y$ between metric space is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset $V \subset Y$.*

Proof: homework

Lemma 14. *Let X be a metric space, and $\mathcal{U}$ be the collection of open subsets of X. Then $\mathcal{U}$ has the following properties:*

1. *X and $\emptyset$ belong to $\mathcal{U}$.*

2. *The union of a collection in $\mathcal{U}$ belongs to $\mathcal{U}$.*

3. *The intersection of a finite collection of subsets in $\mathcal{U}$ belongs to $\mathcal{U}$.*

Definition 15. A *topology* on a set X is a collection $\mathcal{U}$ of subsets of X satisfying the properties of the previous lemma. A *topological space* is a pair $(X, \mathcal{U})$ consisting of a set X and a topology $\mathcal{U}$ on X. If $(X, \mathcal{U})$ is a topological space, a subset $U \subset X$ is *open* if U belongs to $\mathcal{U}$; it is *closed* if its complement $X \setminus U$ belongs to $\mathcal{U}$.

Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be topological spaces. A map $f\colon X \to Y$ is *continuous* if and only if $f^{-1}(U) \in \mathcal{V}$ for every $U \in \mathcal{U}$. It is easy to see that any composition of continuous maps is continuous.

Examples of topological spaces.

1. Let X be a metric space. Then $\mathcal{U} = \{\text{open subsets of } X\}$ is a topology on X, the *metric topology*.

2. Let X be a set. Then $\mathcal{U} = \{\text{all subsets of } X\}$ is a topology, the *discrete topology*. We note that *any* map $f\colon X \to Y$ to a topological space Y is continuous. We will see later that the only continuous maps $\mathbb{R}^n \to X$ are the constant maps.

3. Let X be a set. Then $\mathcal{U} = \{\emptyset, X\}$ is a topology, the *indiscrete topology*.

2.2 Constructions with topological spaces

The subspace topology. Let X be a topological space, and $A \subset X$ a subset. Then

$$\mathcal{U} = \{A \cup U \mid U \underset{open}{\subset} X\}$$

is a topology on A called the *subspace topology*.

Lemma 16. *Let X be a metric space and $A \subset X$. Then the metric topology on A agrees with the subspace topology on A (as a subset of X equipped with the metric topology).*

Lemma 17. *Let X, Y be topological spaces and let A be a subset of X equipped with the subspace topology. Then the inclusion map $i\colon A \to X$ is continuous and a map $f\colon Y \to A$ is continuous if and only if the composition $i \circ f\colon Y \to X$ is continuous.*

Basis for a topology. Sometimes it is convenient to define a topology $\mathcal{U}$ on a set X by first describing a smaller collection $\mathcal{B}$ of subsets of X, and then defining $\mathcal{U}$ to be those subsets of X that can be written as *unions* of subsets belonging to $\mathcal{B}$. We've done this already when defining the metric topology: Let X be a metric space and let $\mathcal{B}$ be the collection of subsets of X of the form $B_\epsilon(x) := \{y \in X \mid d(y,x) < \epsilon\}$ (we call these subsets *balls* in X). A subset of X is open (in the sense of Definition 12) if and only if it is a union of balls in X.

Lemma 18. *Let $\mathcal{B}$ be a collection of subsets of a set X satisfying the following conditions*

1. *Every point $x \in X$ belongs to some subset $B \in \mathcal{B}$.*
2. *If $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$ there is some $B \in \mathcal{B}$ with $x \in B$ and $B \subset B_1 \cap B_2$.*

Then $\mathcal{U} := \{$unions of subsets belonging to $\mathcal{B}\}$ is a topology on X.

If the above conditions are satisfied, the collection $\mathcal{B}$ is called a *basis for the topology* $\mathcal{U}$ or $\mathcal{B}$ *generates the topology* $\mathcal{U}$. It is easy to check that the collection of balls in a metric space satisfies the above conditions and hence the collection of open subsets is a topology as claimed by Lemma 14.

The Product topology

Definition 19. The *product topology* on the Cartesian product $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$ of topological spaces X, Y is the topology with basis

$$\mathcal{B} = \{U \times V \mid U \underset{open}{\subset} X,\ V \underset{open}{\subset} Y\}$$

The collection $\mathcal{B}$ obviously satisfies property (1) of a basis; property (2) holds since $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$. We note that the collection $\mathcal{B}$ is *not* a topology since the union of $U \times V$ and $U' \times V'$ is typically not a Cartesian product (e.g., draw a picture for the case where $X = Y = \mathbb{R}$ and U, U', V, V' are open intervals).

Lemma 20. *The product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (with each factor equipped with the metric topology) agrees with the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.*

Proof: homework.

Lemma 21. *Let X, Y_1, Y_2 be topological spaces. Then the projection maps $p_i\colon Y_1 \times Y_2 \to Y_i$ is continuous and a map $f\colon X \to Y_1 \times Y_2$ is continuous if and only if the* component maps

$$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{p_i} Y_i$$

are continuous for $i = 1, 2$.

Proof: homework

Lemma 22. *1. Let $A \subset \mathbb{R}^n$ and let $f, g\colon A \to \mathbb{R}$ be continuous maps. Then $f+g$ and $f \cdot g$ continuous maps from A to $\mathbb{R}$. If $g(x) \neq 0$ for all $x \in A$, then also f/g is continuous.*

2. Any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.

3. The multiplication map $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is continuous.

Proof. To prove part (1) we note that the map $f+g\colon A \to \mathbb{R}$ can be factored in the form

$$A \xrightarrow{f\times g} \mathbb{R}\times\mathbb{R} \xrightarrow{a} \mathbb{R}$$

The map $f \times g$ is continuous by Lemma 21 since its component maps f, g are continuous; the map a is continuous by Example 7, and hence the composition $f+g$ is continuous. The argument for $f \cdot g$ is the same, with a replaced by m. To prove that f/g is continuous, we factor it in the form

$$A \xrightarrow{\quad f\times g \quad} \mathbb{R}\times\mathbb{R}^\times \xrightarrow{p_1\times(I\circ p_2)} \mathbb{R}\times\mathbb{R}^\times \xrightarrow{\quad m \quad} \mathbb{R},$$

where p_1 (resp. p_2) is the projection to the first (resp. second) factor of $\mathbb{R}\times\mathbb{R}^\times$ and $I\colon \mathbb{R}^\times \to \mathbb{R}^\times$ is the inversion map $x \mapsto x^{-1}$. By Lemma 21 the p_i's are continuous, in calculus we learned that I is continuous, and hence again by Lemma 21 the map $p_1 \times (I \circ p_2)$ is continuous.

To prove part (2), we note that the constant map $\mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto a$ is obviously continuous, and that the projection map $p_i\colon \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_i$ is continuous by Lemma 21. Hence by part (1) of this lemma, the monomial function $x \mapsto a x_1^{i_1} \cdots x_n^{i_n}$ is continuous. Any polynomial function is a sum of monomial functions and hence continuous.

For the proof of (3), let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices and let

$$\mu\colon M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

be the map given by matrix multiplication. By Lemma 21 the map μ is continuous if and only if the composition

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \xrightarrow{\mu} M_{n\times n}(\mathbb{R}) \xrightarrow{p_{ij}} \mathbb{R}$$

is continuous for all $1 \leq i, j \leq n$, where p_{ij} is the projection map that sends a matrix A to its entry $A_{ij} \in \mathbb{R}$. Since the $p_{ij}(\mu(A, B)) = (A \cdot B)_{ij}$ is a *polynomial* in the entries of the matrices A and B, this is a continuous map by part (2) and hence μ is continuous.

Restricting μ to invertible matrices, we obtain the multiplication map

$$\mu_| \colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

that we want to show is continuous. We will argue that in general if $f\colon X \to Y$ is a continuous map with $f(A) \subset B$ for subsets $A \subset X$, $B \subset Y$, then the *restriction* $f_{|A}\colon A \to B$ is continuous. To prove this, consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_{|A}} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

where i, j are the obvious inclusion maps. These inclusion maps are continuous w.r.t. the subspace topology on A, B by Lemma 17. The continuity of f and i implies the continuity of $f \circ i = j \circ f_{|A}$ which again by Lemma 17 implies the continuity of $f_{|A}$. $\square$

Quotient topology. Let X be a topological space, let $\sim$ be an equivalence relation on X, let $X/\sim$ be the set of equivalence classes and let

$$p\colon X \to X/\sim \qquad x \mapsto [x]$$

be the projection map that sends a point $x \in X$ to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is the collection of subsets $\mathcal{U} = \{U \subset X/\sim \mid p^{-1}(U) \text{ is an open subset of } X\}$. The set $X/\sim$ equipped with the quotient topology is called the *quotient space*.

Lemma 23. *The projection map $p\colon X \to X/\sim$ is continuous and a map $f\colon X/\sim \to Y$ to a topological space Y is continuous if and only if the composition $p \circ f\colon X \to Y$ is continuous.*

Example 24.

1. Let A be a subset of a topological space X. Define a equivalence relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. We use the notation X/A for the quotient space $X/\sim$.
   (a) We claim that the quotient space $[-1,+1]/\{\pm 1\}$ is homeomorphic to S^1 via the map $f\colon [-1,+1]/\{\pm 1\} \to S^1$ given by $[t] \mapsto e^{\pi i t}$. Here we use that a continuous bijection $f\colon X \to Y$ from a compact space to a Hausdorff space is a homeomorphism (see Corollary 32).
   (b) More generally, D^n/S^{n-1} is homeomorphic to S^n. (proof: homework)
2. quotients of the square by various equivalence relations gives: torus, Klein bottle, real projective plane $D^2/\sim = S^2/\sim$. We can obtain a surface of genus 2 from an 8-gon with suitable boundary identifications (first redraw 8-gon as a union of squares with a corner chipped off; identifying boundaries on each square leads to punctured torus).
3. The real projective space

$$\mathbb{RP}^n := \{\text{1-dimensional subspaces of } \mathbb{R}^{n+1}\} = S^n/v \sim \pm v$$

   Homework: $\mathbb{RP}^1 \approx S^1$; $\mathbb{RP}^3 \approx SO(3)$

4. The complex projective space
$$\mathbb{CP}^n := \{1\text{-dimensional subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/v \sim zv, \qquad z \in S^1$$
homework: $\mathbb{CP}^1 \approx S^2$

5. The Grassmann manifold $G_k(\mathbb{R}^{n+k}) := \{k\text{-dimensional subspaces of } \mathbb{R}^{n+k}\}$. There is a surjective map
$$V_k(\mathbb{R}^{n+k}) = \{\text{isometries } f\colon \mathbb{R}^k \to \mathbb{R}^{n+k}\} \twoheadrightarrow G_k(\mathbb{R}^{n+k}) \qquad f \mapsto \operatorname{im}(f)$$
Two isometries f, f' have the same image if and only if there is some isometry $g\colon \mathbb{R}^k \to \mathbb{R}^k$ such that $f' = f \circ g$. In other words, we get a bijection $V_k(\mathbb{R}^{n+k})/\sim \leftrightarrow G_k(\mathbb{R}^{n+k})$ if we define an equivalence relation $\sim$ on the Stiefel manifold by $f \sim f'$ if and only if there is some isometry $g\colon \mathbb{R}^k \to \mathbb{R}^k$ such that $f' = f \circ g$. This the quotient topology on $V_k(\mathbb{R}^{n+k})/\sim$ then gives $G_k(\mathbb{R}^{n+k})$ a topology (note that for $k = 1$, $V_k(\mathbb{R}^{n+k}) = S^n$, and this agrees with how we put a topology on the projective space $\mathbb{RP}^n = G_1(\mathbb{R}^{n+1})$.

6. If X is a topological space and a group H acts X (say from the right via $X \times H \to X$, $(x, h) \mapsto xh$; requirement: $(xh)h' = x(hh')$ for $x \in X$, $h, h' \in H$). The group action defines an equivalence relation $\sim$ on X via $x' \sim x$ if and only if there is some $h \in H$ such that $x' = xh$. Equivalence classes are called the *orbits* of the action; the quotient space $X/\sim$ is the *orbit space*, denoted X/H.
   (a) $G_k(\mathbb{R}^{n+k}) = V_k(\mathbb{R}^{n+k})/O(k)$
   (b) homogeneous spaces G/H for topological groups G. Explanation: a *topological group* is a group G equipped with a topology such that the multiplication map $G \times G \to G$ and the inversion map $G \to G$, $g \mapsto g^{-1}$ are continuous. A subgroup $H \le G$ act on G via the multiplication map $G \times H \to G$, $(g, h) \mapsto gh$. The orbit space is denoted G/H (or $H\backslash G$ if we use the corresponding left H-action on G), and is called *homogeneous space*. Warning: there is difference between the homogeneous space G/H and the quotient space of G obtained by collapsing the subspace H to a point (Example 24 (1)), which we also would denote by G/H (unfortunately, both notations are standard; fortunately, it is usually clear from the context which version of G/H we are talking about, since the homogeneous space makes only sense if H is a subgroup of a topological group G).

We want to show that many topological spaces we've discussed so far are actually homogeneous spaces. To do that we use the following result.

Proposition 25. (Recognition principle for homogeneous spaces) *Let G be a compact topological group that acts continuously and transitively on a topological space X. Then X is homeomorphic to the homogeneous space G/H where $H = \{g \in G \mid gx_0 = x_0\}$ is the isotropy subgroup of some point $x_0 \in X$.*

Proof. Let

$$f\colon G/H \longrightarrow X \quad \text{be defined by} \quad [g] \mapsto gx_0$$

This map is *surjective* by the transitivity assumption; it is *injective* since if $gx_0 = g'x_0$, then $x_0 = g^{-1}g'x_0$ and hence $h := g^{-1}g'$ belongs to the isotropy subgroup H. This implies $g' = gh$, and hence $[g'] = [g] \in G/H$.

To show that f is continuous it suffices to show that the composition $f \circ p\colon G \to X$, $g \mapsto gx_0$ is continuous. To see this, we factor $f \circ p$ in the form

$$G = G \times \{x_0\} \hookrightarrow G \times X \xrightarrow{\mu} X$$

where μ is the action map □

Examples of homogeneous spaces.

1. spheres $S^n \approx O(n+1)/O(n)$
   (take the action $O(n+1) \times S^n \to S^n$, $(f, v) \mapsto f(v)$ and $x_0 = (0, \dots, 0, 1) \in S^n$)

2. Stiefel manifold $V_k(\mathbb{R}^{n+k}) \approx O(n+k)/O(n)$
   (take the action $O(n+k) \times V_k(\mathbb{R}^{n+k}) \to V_k(\mathbb{R}^{n+k})$, $(g, f) \mapsto g \circ f$ and $x_0\colon \mathbb{R}^k \to \mathbb{R}^{n+k}$, $v \mapsto (0, v)$).

3. Grassmann manifold $G_k(\mathbb{R}^{n+k}) \approx O(n+k)/O(n) \times O(k)$ (homework problem).

2.3 Properties of topological spaces

Definition 26. Let X be a topological space, $x_i \in X$, $i = 1, 2, \dots$ a sequence in X and $x \in X$. Then x *is the limit of the x_i's* if for all open subsets $U \subset X$ containing x there is some N such that $x_i \in U$ for all $i \geq N$.

Caveat: If X is a topological space with the indiscrete topology, *every point* is the limit of every sequence. The limit is *unique* if the topological space has the following property:

Definition 27. A topological space X is *Hausdorff* if for every $x, y \in X$, $x \neq y$, there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$.

Note: if X is a metric space, then the metric topology on X is Hausdorff (since for $x \neq y$ and $\epsilon = d(x, y)/2$, the balls $B_\epsilon(x)$, $B_\epsilon(y)$ are disjoint open subsets).

Warning: The notion of *Cauchy sequences* can be defined in metric spaces, but not in general for topological spaces (even when they are Hausdorff).

Lemma 28. *Let X be a topological space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.*

Proof. Assume $x \notin A$. Then x is a point in the open subset $X \setminus A$ and hence by the definition of limit, all but finitely many elements x_n must belong to $X \setminus A$, contradicting our assumptions. □

Definition 29. An *open cover* of a topological space X is a collection of open subsets of X whose union is X. If for every open cover of X there is a finite subcollection which also covers X, then X is called *compact*.

Some books (like Munkres' *Topology*) refer to open covers as *open coverings*, while newer books (and wikipedia) seem to prefer to above terminology, probably for the same reasons as me: to avoid confusions with *covering spaces*, a notion we'll introduce soon.

Now we'll prove some useful properties of compact spaces and maps between them, which will lead to the important Corollaries 34 and 32.

Lemma 30. *If $f\colon X \to Y$ is a continuous map and X is compact, then the image $f(X)$ is compact.*

In particular, if X is compact, then any quotient space $X/\sim$ is compact, since the projection map $X \to X/\sim$ is continuous with image $X/\sim$.

Proof. To show that $f(X)$ is compact assume that $\{U_a\}$, $a \in A$ is an open cover of the subspace $f(X)$. Then each U_a is of the form $U_a = V_a \cap f(X)$ for some open subset $V_a \in Y$. Then $\{f^{-1}(V_a)\}$, $a \in A$ is an open cover of X. Since X is compact, there is a finite subset A' of A such that $\{f^{-1}(V_a)\}$, $a \in A'$ is a cover of X. This implies that $\{U_a\}$, $a \in A'$ is a finite cover of $f(X)$, and hence $f(X)$ is compact. □

Lemma 31. *1. If K is a closed subspace of a compact space X, then K is compact.*

2. If K is compact subspace of a Hausdorff space X, then K is closed.

Proof. To prove (1), assume that $\{U_a\}$, $a \in A$ is an open covering of K. Since the U_a's are open w.r.t. the subspace topology of K, there are open subsets V_a of X such that $U_a = V_a \cap K$. Then the V_a's together with the open subset $X \setminus K$ form an open covering of X. The compactness of X implies that there is a finite subset $A' \subset A$ such that the subsets V_a for $a \in A'$, together with $X \setminus K$ still cover X. It follows that U_a, $a \in A'$ is a finite cover of K, showing that K is compact.

The proof of part (2) is a homework problem. □

Corollary 32. *If $f\colon X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. We need to show that the map $g\colon Y \to X$ inverse to f is continuous, i.e., that $g^{-1}(U) = f(U)$ is an open subset of Y for any open subset U of X. Equivalently (by passing to complements), it suffices to show that $g^{-1}(C) = f(C)$ is a closed subset of Y for any closed subset C of C.

Now the assumption that X is compact implies that the closed subset $C \subset X$ is compact by part (1) of Lemma 31 and hence $f(C) \subset Y$ is compact by Lemma 30. The assumption that Y is Hausdorff then implies by part (2) of Lemma 31 that $f(C)$ is closed. □

Lemma 33. *Let K be a compact subset of $\mathbb{R}^n$. Then K is* bounded*, meaning that there is some $r > 0$ such that K is contained in the open ball $B_r(0) := \{x \in \mathbb{R}^n \mid d(x, 0) < r\}$.*

Proof. The collection $B_r(0) \cap K$, $r \in (0, \infty)$, is an open cover of K. By compactness, K is covered by a *finite* number of these balls; if R is the maximum of the radii of these finitely many balls, this implies $K \subset B_R(0)$ as desired. □

Corollary 34. *If $f : X \to \mathbb{R}$ is a continuous function on a compact space X, then f has a maximum and a minimum.*

Proof. $K = f(X)$ is a compact subset of $\mathbb{R}$. Hence K is bounded, and thus K has an infimum $a := \inf K \in \mathbb{R}$ and a supremum $b := \sup K \in \mathbb{R}$. The infimum (resp. supremum) of K is the limit of a sequence of elements in K; since K is closed (by Lemma 31 (2)), the limit points a and b belong to K by Lemma 28. In other words, there are elements $x_{min}, x_{max} \in X$ with $f(x_{min}) = a \leq f(x)$ for all $x \in X$ and $f(x_{max}) = b \geq f(x)$ for all $x \in X$. □

In order to use Corollaries 32 and 34, we need to be able to show that topological spaces we are interested in, are in fact compact. Note that this is *quite difficult* just working from the definition of compactness: you need to ensure that *every* open cover has a finite subcover. That sounds like a lot of work...

Fortunately, there is a very simple classical characterization of compact subspaces of Euclidean spaces:

Theorem 35. (Heine-Borel Theorem) *A subspace $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

We note that we've already proved that if $K \subset \mathbb{R}^n$ is compact, then K is a closed subset of $\mathbb{R}^n$ (Lemma 31(2)), and K is bounded (Lemma 33).

There two important ingredients to the proof of the converse, namely the following two results:

Lemma 36. *A closed interval $[a, b]$ is compact.*

This lemma has a short proof that can be found in any pointset topology book, e.g., [Mu].

Theorem 37. *If $X_1, \ldots, X_n$ are compact topological spaces, then their product $X_1 \times \cdots \times X_n$ is compact.*

For a proof see e.g. [Mu, Ch. 3, Thm. 5.7]. The statement is true more generally for a product of *infinitely many* compact space (as discussed in [Mu, p. 113], the correct definition of the product topology for infinite products requires some care), and this result is called *Tychonoff's Theorem*, see [Mu, Ch. 5, Thm. 1.1].

Proof of the Heine-Borel Theorem. Let $K \subset \mathbb{R}^n$ be closed and bounded, say $K \subset B_r(0)$. We note that $B_r(0)$ is contained in the n-fold product

$$P := [-r, r] \times \cdots \times [-r, r] \subset \mathbb{R}^n$$

which is compact by Theorem 37. So K is a closed subset of P and hence compact by Lemma 31(1). □

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