## CONTENTS

# **Homework Solutions**

## 1. HOMEWORK ASSIGNMENT # 1

1. Show that if $X_1$ and $X_2$ are compact connected surfaces, then $\chi(X_1\#X_2) = \chi(X_1) + \chi(X_2) - 2$.

Cover both surfaces by patterns of polygons, and let us write $V_i$ (resp. $E_i$ resp. $F_i$) for the number of vertices (resp. edges resp. faces) on $X_i$. Let us assume that we chose these pattern such that both contain a face which is a polygon with $k$ vertices and edges for some $k$. The point of this is that removing that face from both surfaces and identifying the resulting surfaces along the polygon results in the connected sum $X_1\#X_2$ and the patterns on both surfaces fit together to give a pattern of polygons on $X_1\#X_2$. Let us determine $V$ (= number of vertices), $E$ (= number of edges) and $F$ (= number of faces) of this pattern on $X_1\#X_2$:

**faces:** $F = F_1 + F_2 - 2$ (each face on $X_1$ or $X_2$ gives a face on $X_1\#X_2$ except the two polygons that we removed in order to form the connected sum).

**edges:** $E = E_1 + E_2 - k$ (each edge on $X_1$ or $X_2$ gives an edge on $X_1\#X_2$ except there are $k$ pairs of edges that get identified with each other when we form the connected sum).

**vertices:** $V = V_1 + V_2 - k$ (each vertex on $X_1$ or $X_2$ gives a vertex on $X_1\#X_2$ except there are $k$ pairs of vertices that get identified with each other when we form the connected sum).

This implies

$$\begin{aligned}\chi(X_1\#X_2) &= V - E + F \\ &= (V_1 + V_2 - k) - (E_1 + E_2 - k) + (F_1 + F_2 - 2) \\ &= V_1 - E_1 + F_1 + V_2 - E_2 + F_2 - 2 \\ &= \chi(X_1) + \chi(X_2) - 2\end{aligned}$$

2. Show that if $Y \to X$ is a $d$-fold covering of a closed 2-manifold $X$, then $\chi(Y) = d\chi(X)$.

The assumption that $p\colon Y \to X$ is a $d$-fold covering implies that $X$ has an open covering $U_i$, $i \in I$ such that for each $i$ the preimage $p^{-1}(U_i)$ is homeomorphic to $U_i \times \{1, \dots, d\}$ such that $p$ restricted to $p^{-1}(U_i)$ corresponds to the projection $U_i \times \{1, \dots, d\} \to U_i$ onto the first factor.

Choosing a fine enough pattern $P$ of polygons on $X$ we can assume that each edge and face of the pattern is contained in some $U_i$. This guarantees that the preimages of vertices, edges and faces give us vertices, edges and faces of a pattern $P'$ on $Y$ (this argument is hard to make precise mathematically since we provided only a heuristic definition of "pattern of polygons on a surface"). Clearly there number of vertices resp. edges resp. faces in $P'$ is $d$ times the corresponding number for $P$ and hence $\chi(Y) = d\chi(X)$.

## 2. HOMEWORK ASSIGNMENT # 2

1. We claim that the homomorphism $\mathbb{Z}/pq \to \mathbb{Z}/p \oplus \mathbb{Z}/q$ which sends $[k]$ to $[k,k]$ is an isomorphism if $p$, $q$ are relatively prime. The condition $\gcd(p,q) =$ implies that there are integers $a$, $b$ such that $ap + bq = 1$. Consider the homomorphism $\mathbb{Z}/p \oplus \mathbb{Z}/q \to \mathbb{Z}/pq$ given by $([m],[n]) \mapsto [bqm + apn]$. This is well-defined since it sends $[p]$ to $[bqp] = [0]$ and $[q]$ to $[apq] = [0]$. It is easy to check that this is an inverse.

If $p$, $q$ are not relatively prime, then the least common multiple of $p$ and $q$ is strictly smaller than $pq$. Hence multiplication by the least common multiple gives zero for every element in $\mathbb{Z}/p \oplus \mathbb{Z}/q$, but not $\mathbb{Z}/pq$ showing that these groups can't be isomorphic.

2. a) Calculate the homology groups of all compact connected surfaces.

Again using the Classification Theorem, it suffices to look at $S^2$ (homology calculated in class) and connected sums of tori and projective planes. In order to determine the homology groups, we need to choose a pattern of polygons for $T\#\dots\#T$ and $P\#\dots\#P$. To do so, it is convenient to think of these surfaces as polygons with edge identifications. I'd love to draw a picture of this, but unfortunately doing that in a TEX file is time consuming. Let's instead describe these pictures in words by reading off the edge labels going around the perimeter of the polygon in clockwise direction; let us agree that we write $a^{-1}$ instead of $a$ if the edge label is $a$, but the arrow of the edge in question is pointing in counterclockwise direction.

With these conventions, the torus $T$ is obtained from a square by labeling the edges $a$, $b$, $a^{-1}$, $b$, $b^{-1}$. In class we explained how to think of the connected sum $X\#Y$ of two surfaces given by polygons with edge identification again as being obtained from a polygon. Applying this to $\underbrace{T\#\dots\#T}_{g}$, we see that this surface can be obtained from a $4g$-gon with edges labeled

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

The resulting pattern on the surface has one vertex $v$, one face $f$ and $2g$ edges $a_i$, $b_i$, $1 \le i \le g$. Hence the chain complex associated to this pattern looks like

$$\mathbb{Z}v \xleftarrow{\partial_1} \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \oplus \dots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_g \xleftarrow{\partial_2} \mathbb{Z}f$$

Since there is only one vertex involved, we have $\partial_1(e) = v - v = 0$ for any edge $e$ and hence $\partial_1 \equiv 0$. To determine $\partial_2(f)$ we notice that every edge label occurs exactly twice, but with arrows pointing in opposite directions. Hence $\partial_2(f) = 0$ and $\partial_2 \equiv 0$ as well. It follows that the $q$-th homology group $H_q$ is isomorphic to the $q$-th chain group $C_q$ and we can read off the groups from the chain complex above:

$$H_q(\underbrace{T\#\ldots\#T}_{g}) = \begin{cases} \mathbb{Z} & q=0,2 \\ \mathbb{Z}^{2g} & q=1 \\ 0 & q\neq 0,1,2 \end{cases}$$

We proceed similarly to calculate the homology groups of $P\#\ldots\#P$. Since $P$ is obtained from a bigon with edges labeled $a$, $a$, the connected sum $\underbrace{P\#\ldots\#P}_{k}$ is obtained from a $2k$-gon with edges labeled

$$a_1, a_1, \ldots, a_k, a_k.$$

The associated chain complex then is

$$\begin{tikzcd} \mathbb{Z}v & \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_g \arrow[l, "\partial_1"'] & \mathbb{Z}f \arrow[l, "\partial_2"'] \end{tikzcd}$$

with $\partial_1(a_i)=0$ and $\partial_2(f)=2a_1+\cdots+2a_k$. It follows that $H_0=\mathbb{Z}$, $H_2=0$, and

$$H_1 = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k/\mathbb{Z}2(a_1+\cdots+a_k).$$

To identify this quotient group, we choose a different basis of the free abelian group $C_1$, namely $a_1, \ldots, a_{k-1}, c$, with $c = a_1+\cdots+a_k$. Then we see

$$H_1 = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_{k-1} \oplus \mathbb{Z}c/\mathbb{Z}2c \cong \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{k-1} \oplus \mathbb{Z}/2.$$

b) Can the Euler characteristic of a compact connected surface be expressed in terms of its homology groups?

Inspection of the Euler characteristic calculation (resp. homology group calculation) for a compact connected surfaces $X$ shows that

$$\chi(X) = \operatorname{rk} H_0(X) - \operatorname{rk} H_1(X) + \operatorname{rk} H_2(X),$$

where $\operatorname{rk} H_q(X)$ is the rank of the abelian group $H_q(X)$.

3. Show that the singular chain complex of a topological space X is in fact a chain complex; i.e., that $\partial_q \circ \partial_{q+1} = 0$, where $\partial_q : C_q(X) \to C_{q-1}(X)$ is the boundary map. It will suffice to show that $\partial_q \circ \partial_{q+1} = 0$ on the generators of $C_{q+1}$ since $\partial_q$ and $\partial_{q+1}$ are both homomorphisms.

Consider $\sigma : \Delta^{q+1} \to X$ in $C_{q+1}(X)$.

$$\begin{aligned}
&\partial_q \circ \partial_{q+1}(\sigma) \\
=&\partial_q \left( \sum_{j=0}^{q} (-1)^j \partial_{q+1}(\sigma) \circ [e_0, ..., \hat{e_j}, ..., e_{q+1}] \right) \\
=&\sum_{i=0}^{q} (-1)^i \left( \sum_{j=0}^{q+1} (-1)^j \sigma \circ [e_0, ..., \hat{e_j}, ..., e_{q+1}] \circ [e_0, ..., \hat{e_i}, ..., e_q] \right) \\
=&\sum_{0 \le i < j \le q+1} (-1)^{i+j} \sigma \circ [e_0, ..., \hat{e_i}, ..., \hat{e_j}, ..., e_{q+1}] \\
&+ \sum_{0 \le j < i \le q+1} (-1)^{i+j+1} \sigma \circ [e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_{q+1}] \\
=&\sum_{0 \le i < j \le q+1} (-1)^{i+j} \sigma \circ [e_0, ..., \hat{e_i}, ..., \hat{e_j}, ..., e_{q+1}] \\
&- \sum_{0 \le j < i \le q+1} (-1)^{i+j} \sigma \circ [e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_{q+1}] \\
=&0.
\end{aligned}$$

Hence, since $\partial_q \circ \partial_{q+1} = 0$ on the generators of $C_{q+1}(X)$, $\partial_q \circ \partial_{q+1} = 0$, and the singular chain complex of a topological space X is in fact a chain complex.

4. Let $X$ be a topological space with path components $X_\alpha$, $\alpha \in A$. Show that $H_q(X)$ is isomorphic to $\bigoplus_{\alpha \in A} H_q(X_\alpha)$.

Since a singular simplex always has path-connected image, $C_n(X)$ splits as the direct sum of its subgroups $C_n(X_\alpha)$. The boundary maps $\partial_n$ preserve this direct sum decomposition, taking $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, so $Z_n(X) = \ker \partial_n$ and $B_n(X) = \operatorname{Im} \partial_{n+1}$ split similarly as direct sums, hence the homology groups also split, $H_q(X) \cong \bigoplus_{\alpha \in A} H_q(X_\alpha)$.

## 3. HOMEWORK ASSIGNMENT # 3

1. Show that the Hurewicz map $h : \pi_1(X, x_0) \to H_1(X)$ given by $[\gamma] \to [[\gamma]]$ is a homomorphism. Let $[\gamma], [\gamma'] \in \pi_1(X, x_0)$, and let $\gamma\gamma'$ denote the concatenation of paths $\gamma$ and $\gamma'$. To show that h is a homomorphism, we need to verify that $h([\gamma][\gamma']) = h([\gamma]) + h([\gamma'])$; i.e., $h([\gamma\gamma']) = [[\gamma\gamma']] = [[\gamma]] + [[\gamma']]$. We will do so by showing that $\gamma + \gamma' - \gamma\gamma' \in B_1(X)$ which will then imply that $[[\gamma\gamma']] = [[\gamma + \gamma']] = [[\gamma]] + [[\gamma']]$, as desired. Consider the singular 2-simplex, $\sigma : \Delta^2 \to X$, defined as the composition

$$\begin{tikzcd}
\Delta^2 \arrow[r, "{[e_0, \frac{1}{2}(e_0+e_1), e_1]}"] & \Delta^1 \arrow[r, "\gamma\gamma'"] & X
\end{tikzcd}$$

Notice that $\partial_2(\sigma) = \gamma - \gamma\gamma' + \gamma'$, so $\gamma' - \gamma\gamma' + \gamma \in B_1(X)$ as desired. Thus, h is indeed a homomorphism.

2. Let $\Psi \colon C_1(X)/B_1(X) \to \pi_1(X, x_0)^{ab}$ be the map defined by $[[\gamma]] \mapsto [\bar{\lambda}_{\gamma(1)} \gamma \lambda_{\gamma(0)}]$ for any singular 1-simplex $\gamma$ (as in class we have chosen for every point $x \in X$ a path $\lambda_x$ from the basepoint $x_0$ to $x$). Show that the restriction of $\Psi$ to $H_1(X) \subset C_1(X)/B_1(X)$ provides an inverse to the map $\bar{h} \colon \pi_1^{ab}(X, x_0) \to H_1(X)$.

We need to show:

1. $\Psi$ is well-defined;
2. $\Psi|_{H_1(X)} \circ \bar{h} = id_{\pi_1^{ab}(X,x_0)}$;
3. $\bar{h} \circ \Psi|_{H_1(X)} = id_{H_1(X)}$.

We note that the sending a 1-simplex $\gamma$ to $[\bar{\lambda}_{\gamma(1)}\gamma\lambda_{\gamma(0)}] \in \pi_1^{ab}(X, x_0)$ extends uniquely to a linear map $\Psi\colon C_1(X) \to \pi_1^{ab}(X, x_0)$, since the range is an *abelian group*, and $C_1(X)$ is the *free* $\mathbb{Z}$-module generated by 1-simplices. Hence to prove (1), it suffices to show every 2-simplex $\sigma\colon \Delta^2 \to X$, the boundary $\partial_2\sigma$ maps to 0 under $\Psi$. We recall that $\partial_2\sigma = \gamma_0 - \gamma_1 + \gamma_2$, where $\gamma_i$ is the 1-simplex (aka path) $\sigma \circ [e_0, \widehat{e}_i, e_2]\colon \Delta^1 \to X$. Writing $y_i := \sigma(e_i)$, $\gamma_0$ is a path from $y_1$ to $y_2$, $\gamma_1$ is a path from $y_0$ to $y_2$ and $\gamma_2$ is a path from $y_0$ to $y_1$. Hence $\Psi$ sends $\partial_2\sigma$ to the product of the elements

$$[\bar{\lambda}_{y_1}\gamma_2\lambda_{y_0}] \qquad [\bar{\lambda}_{y_2}\gamma_0\lambda_{y_1}] \qquad \text{and} \qquad [\bar{\lambda}_{y_2}\gamma_1\lambda_{y_0}]^{-1}$$

in $\pi_1^{ab}(X, x_0)$. Hence it suffices to show that

$$\bar{\lambda}_{y_2}\gamma_1\lambda_{y_0} \qquad \text{is homotopic to} \qquad \bar{\lambda}_{y_2}\gamma_0\lambda_{y_1}\bar{\lambda}_{y_1}\gamma_2\lambda_{y_0}$$

This follows since $\lambda_{y_1}\bar{\lambda}_{y_1}$ is homotopic to the constant map at $y_1$, and hence $\bar{\lambda}_{y_2}\gamma_0\lambda_{y_1}\bar{\lambda}_{y_1}\gamma_2\lambda_{y_0}$ is homotopic to $\bar{\lambda}_{y_2}\gamma_0\gamma_2\lambda_{y_0}$ (both homotopies are relative endpoints). We observe that the simplex $\sigma$ provides a homotopy relative endpoints between $\gamma_0\gamma_2$ and $\gamma_1$ (both are paths from $y_0$ to $y_2$) which proves (1).

To prove (2), let $[\gamma : (I, \partial I) \to (X, x_0)] \in \pi_1^{ab}(X, x_0)$. Then

$$\begin{aligned}
\Psi|_{H_1(X)} \circ \bar{h}([\gamma]) &= \Psi|_{H_1(X)}([[\gamma]]) \\
&= [\lambda_{\gamma(0)}\gamma\overline{\lambda_{\gamma(1)}}] = [\lambda_{x_0}\gamma\overline{\lambda_{x_0}}] \\
&= [\lambda_{x_0}][\gamma][\overline{\lambda_{x_0}}] = [\lambda_{x_0}][\overline{\lambda_{x_0}}][\gamma] = [\gamma],
\end{aligned}$$

which shows $\Psi|_{H_1(X)} \circ \bar{h} = id_{\pi_1^{ab}(X,x_0)}$.

To prove that $\bar{h} \circ \Psi|_{H_1(X)} = id_{H_1(X)}$, it is useful to first calculate the left hand side not just for elements in $H_1(X)$, but for elements of $C_1(X)/B_1(X)$; the advantage is that $C_1(X)/B_1(X)$ is *generated by* 1-*simplices*. If $\gamma$ is a 1-simplex, i.e., a path (or more generally a 1-chain, i.e., linear combination of paths), let us denote by $[[\gamma]]$ the element represented by $\gamma$ in the quotient group $C_1(X)/B_1(X)$. We note that the considerations in part (a) show that if $\gamma$, $\gamma'$ are paths with $\gamma(1) = \gamma(0)$, then $[[\gamma\gamma']] = [[\gamma]] + [[\gamma']]$; moreover, $[[\bar{\gamma}]] = -[[\gamma]]$.

With these preliminaries, we can calculate for any path $\gamma$:

$$\begin{aligned}
\bar{h} \circ \Psi([[\gamma]]) &= \bar{h}([\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}] = [[\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}]] \\
&= [[\lambda_{\gamma(0)}]] + [[\gamma]] + [[\bar{\lambda}_{\gamma(1)}]] = [[\lambda_{\gamma(0)}]] + [[\gamma]] - [[\lambda_{\gamma(1)}]] \\
&= [[\gamma]] + [[\lambda(\partial(\gamma))]],
\end{aligned} \tag{1}$$

where $\lambda\colon C_0(X) \to C_1(X)$ is given on generators by $x \mapsto \lambda_x$ (and hence $\lambda$ maps $\partial\gamma = \gamma(1) - \gamma(0)$ to $\lambda_{\gamma(0)} - \lambda_{\gamma(1)}$). Since formula (3) holds for the generators $\gamma \in C_1(X)$, it hold for every element $z \in C_1(X)$; in particular, if $z$ is a *cycle*, we have $\bar{h} \circ \Psi([[z]]) = [[z]]$, which is what we wanted to prove.

3. a) Let $\mathsf{Top}_*$ be the category of *pointed topological spaces* whose objects are topological spaces $X$ equipped with a base point $x_0$, and whose morphisms are continuous maps $f\colon (X, x_0) \to (Y, y_0)$. Show that there is a functor $\pi_1\colon \mathsf{Top}_* \to \mathsf{Groups}$ which on objects assigns to $(X, x_0)$ its fundamental group $\pi_1(X, x_0)$.

We construct the functor on morphisms by sending a continuous map $f\colon (X,x_0) \to (Y,y_0)$ to the homomorphism

$$f_*\colon \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0) \qquad \text{defined by} \qquad [\gamma] \mapsto [f\circ\gamma]$$

We need to check that this prescription is compatible with identities and with composition. It is clear from the definition of $f_*$ that if $f$ is the identity of $X$, then $f_*$ is the identity on $\pi_1(X,x_0)$. Concerning composition, let $g\colon (Y,y_0) \to (Z,z_0)$ be a continuous map. Then

$$(g\circ f)_*([\gamma]) = [(g\circ f)\circ\gamma] = [g\circ(f\circ\gamma)] = g_*([f\circ\gamma]) = g_*(f_*([\gamma])),$$

as desired.

b) Show that the Hurewicz homomorphism $h\colon \pi_1(X,x_0) \to H_1(X)$ is a natural transformation.

We recall that $h$ being a natural transformation simply means that for each continuous map $f\colon (X,x_0) \to (Y,y_0)$, the diagram

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "f_*"] \arrow[d, "h"'] & \pi_1(Y,y_0) \arrow[d, "h"] \\
H_1(X) \arrow[r, "f_*"] & H_1(Y)
\end{tikzcd}$$

is commutative. So let $\gamma\colon (I,\partial I) \to (X,x_0)$ be a based loop in $X$. Then

$$h(f_*([\gamma])) = h([f\circ\gamma]) = [[f\circ\gamma]] = f_*([[\gamma]]) = f_*(h([\gamma])$$

which shows the commutativity of the diagram.

## 4. HOMEWORK ASSIGNMENT # 4

1. Let $A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$ be a short exact sequence of chain complexes. Show that the sequence

$$\ldots \longrightarrow H_q(A) \xrightarrow{f_*} H_q(B) \xrightarrow{g_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \ldots$$

is exact at $H_q(C)$ and $H_{q-1}(B)$.

*Proof.* In order to show that the sequence is exact at $H_q(C)$ and $H_{q-1}(B)$, we need to verify the following: a) $Im\ g_* = ker\ \partial$ b) $Im\ f_* = ker\ g_*$ a) $Im\ g_* \subset ker\ \partial$ Recall that for $q \in \mathbb{Z}$, the chain map $g_*$ induces a map $g_* : H_q(B) \longrightarrow H_q(C)$ where $g_*([b]) = [g(b)]$. Let $g_*([b]) \in Im\ g_* \subset H_q(C)$, where $[b] \in H_q(B)$ (so $b \in Z_q(B)$), then $\partial g_*([b]) = \partial([g(b)])$. Recall the construction of $\partial$: $\partial$ assigns $c$, which is $g(b)$ in our case, to an $a \in A_{q-1}$ for which $f(a) = \partial(b')$, with $b'$ chosen so that $c = g(b) = g(b')$ for $b \in B_q$. Since we showed that the choice of such a $b'$ is irrelevant, we can then choose $b'$ to be $b$. Hence, $f(a) = \partial(b) = 0$ since $b \in Z_q(B)$. $f_q$ is injective (see the exactness of the given sequence), so we have that $a = 0$, whence $\partial g_*([b]) = \partial([g(b)]) = [0]$. Therefore, $Im\ g_* \subset ker\ \partial$ at $H_q(C)$. $Im\ g_* \supset ker\ \partial$ Let $[c] \in ker\ \partial \subset H_q(C)$. Using the notation from the construction of $\partial$, there exists a $b \in B_q$ such that $g(b) = c$ and an $a \in A_{q-1}$ for which $f(a) = \partial(b)$, and since $[c] \in ker\ \partial$, $\partial a' = a$ for some $a' \in A_q$. We notice that the element $b - f(a') \in B_q$ is a q-cycle since

$$\partial(b - f(a')) = \partial b - \partial f(a') = f(a) - f\partial a' = 0,$$

whence $[b - f(a')] \in H_q(B)$. Furthermore, by the exactness of the original sequence, $gf = 0$, so

$$g(b - f(a')) = g(b) - gf(a') = g(b) - 0 = g(b) = c.$$

Thus, $[c] = [g(b - f(a'))] = g_*([b - f(a')]) \in Im\ g_*$, and $Im\ g_* \supset ker\ \partial$. b) $Im\ f_* \subset ker\ g_*$ Let $f_*([a]) = [f(a)] \in Im\ f_* \subset H_{q-1}(B)$, where $a \in Z_{q-1}(A)$. Then, by the exactness of the original sequence, $gf = 0$, so $g_*([f(a)]) = [gf(a)] = [0]$. Hence, $Im\ f_* \subset ker\ g_*$. $Im\ f_* \supset ker\ g_*$ Let $[b] \in H_{q-1}(B)$ such that $g_*([b]) = [0]$; i.e., $g(b) \in Im(\partial : C_q \to C_{q-1})$. Therefore, $\exists c \in C_q$ such that $\partial(c) = g(b)$. Since $g$ is onto (by the exactness of the given sequence), $\exists b' \in B_q$ such that $g(b') = c$. The commutativity of the diagram (of the chain complexes, chain maps, and boundary maps) yields:

$$g\partial(b') = \partial g(b') = \partial(c) = g(b).$$

Since each $g$ is a homomorphism, we then have $g(b - \partial(b')) = 0$, so $b - \partial(b') \in ker\ g = Im\ f$. Thus, $\exists a \in A_{q-1}$ such that $f(a) = b - \partial(b')$. Again, by the commutativity of the diagram, $f(\partial a) = \partial f(a) = \partial b - \partial\partial b' = \partial b = 0$, so $a = 0$ since f is injective ($ker\ f = 0$). Thus, $a \in Z_{q-1}(A)$ and $[a] \in H_{q-1}(A)$. Moreover, since $b - f(a) = \partial(b')$, $f_*([a]) = [f(a)] = [b] \in H_{q-1}(B)$. Thus, $Im\ f_* \supset ker\ g_*$. To recap, we now have that:

$$... \longrightarrow H_q(A) \xrightarrow{f_*} H_q(B) \xrightarrow{g_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow ...$$

is exact at $H_q(C)$ and $H_{q-1}(B)$. □

2. Recall that the *reduced homology groups* $\tilde{H}_q(X)$ of a space $X$ are the homology groups of the *augmented chain complex*

$$...0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_3} ...$$

For a pair $(X, A)$ with $A \neq \emptyset$, we define $\tilde{H}_q(X, A) = H_q(X, A)$. a) Show that $\tilde{H}_q(X) \cong H_q(X)$ for $q \neq 0$, and that there is a short exact sequence

**:** (1) $$0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0.$$

Hint: one way of doing this is to construct a chain map from the augmented chain complex to the chain complex of X, try to fit it in a short exact sequence of chain complexes, and use the exact homology sequence of problem 1.

We note that equation (1) implies in particular that there is an isomorphism $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$, but this is not canonical.

*Proof.* Consider the chain complex $D_*$:

$$...0 \longleftarrow \mathbb{Z} = D_{-1} \longleftarrow 0 \longleftarrow 0 \longleftarrow 0....$$

Notice that the augmented chain complex, the original, and this new chain complex together form a commutative diagram for which each column is exact:

(2)
$$\begin{tikzcd}
... \arrow[d] & 0 \arrow[l] \arrow[d, "id_0"] & \mathbb{Z} \arrow[l] \arrow[d, "id_{\mathbb{Z}}"] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & ... \arrow[l] \\
... \arrow[d] & 0 \arrow[l] \arrow[d, "id_0"] & \mathbb{Z} \arrow[l] \arrow[d] & C_0(X) \arrow[l] \arrow[d, "id_{C_0(X)}"] & C_1(X) \arrow[l] \arrow[d, "id_{C_1(X)}"] & ... \arrow[l] \\
... & 0 \arrow[l] & 0 \arrow[l] & C_0(X) \arrow[l] & C_1(X) \arrow[l] & ... \arrow[l]
\end{tikzcd}$$

In other words, we have a short exact sequence of chain complexes:

$$D_* \longrightarrow \tilde{C}_*(X) \longrightarrow C_*(X)$$

where we denote the augmented chain complex $\tilde{C}_*$. Hence, problem 1. gives us a long exact sequence of homology groups. Substituting $H_q(\tilde{C}_*(X)) = \tilde{H}_q(X)$ and $H_q(C_*(X)) = H_q(X)$, we obtain the long exact sequence

$$\longrightarrow H_q(D_*) \longrightarrow \tilde{H}_q(X) \longrightarrow H_q(X) \xrightarrow{\partial} H_{q-1}(D_*) \longrightarrow \tilde{H}_{q-1}(X) \longrightarrow . \tag{3}$$

We note that the homology groups of $D_*$ are given by

$$H_q(D_*) = D_q = \begin{cases} \mathbb{Z} & q = -1 \\ 0 & q \neq -1 \end{cases}$$

It follows from the exact sequence (3) that the map $\tilde{H}_q(X) \to H_q(X)$ is an isomorphism for $q > 0$, since the groups in the sequence to the left and right of this map are zero. We also note that this is obviously true for $q < 0$, since both $\tilde{H}_q(X)$ and $H_q(X)$ are zero for $q < 0$. For $q = 0$ the terms $H_0(D_*)$ and $\tilde{H}_{q-1}(X)$ in the long exact sequence (3) are both zero, and hence we obtain the desired short exact sequence

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

□

b) Show that there is an exact sequence of reduced homology groups

$$\dots \longrightarrow \tilde{H}_q(A) \xrightarrow{i_*} \tilde{H}_q(X) \xrightarrow{j_*} \tilde{H}_q(X, A) \xrightarrow{\partial} \tilde{H}_{q-1}(A) \xrightarrow{i_*} \dots \tag{4}$$

*Proof.* Suppose $A \neq \emptyset$. Consider the following commutative diagram

(5)

$$\begin{tikzcd}
{} \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "id_{\mathbb{Z}}"] & C_0(A) \arrow[l] \arrow[d] & C_1(A) \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & C_0(X) \arrow[l] \arrow[d] & C_1(X) \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots & 0 \arrow[l] & 0 \arrow[l] & C_0(X, A) \arrow[l] & C_1(X, A) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

The three rows are the reduced singular chain complex of $A$ resp. the reduced singular chain complex of $X$ resp. the singular chain complex of the pair $(X, A)$. Each column is exact and hence we have the following short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{C}_*(A) \xrightarrow{i_*} \tilde{C}_*(X) \xrightarrow{j_*} C_*(X, A) \longrightarrow 0$$

The associated long exact homology sequence is the desired sequence (4). □

3. Prove the following statement which is known as the 5-*lemma*. Suppose we have a commutative diagram of abelian groups and group homomorphisms

(6)

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

such that the rows are exact sequences. Show that if the outer two vertical maps $h_1$, $h_2$, $h_4$, $h_5$ are all isomorphisms are slightly stronger than needed for the proof. What weaker assumptions will do?

*Proof.* We first prove that $h_3$ is injective. Let $a_3 \in A_3$ such that $h_3a_3 = 0 \in B_3$. Then by the commutativity of the diagram, we have $h_4f_3a_3 = g_3h_3a_3 = g_3(0) = 0$, so $f_3(a_3) \in \ker\ h_4 = 0$ since $h_4$ is injective. Hence, $a_3 \in \ker\ f_3 = \operatorname{im}\ f_2$, so there is an $a_2 \in A_2$ such that $f_2(a_2) = a_3$. Again, by the commutativity of the diagram, we have $g_2h_2(a_2) = h_3f_2(a_2) = h_3(a_3) = 0$, so $h_2(a_2) \in \ker\ g_2 = \operatorname{im}\ g_1$. Thus, there is an element $b_1 \in B_1$ such that $g_1(b_1) = h_2(a_2)$. Moreover, since $h_1$ is surjective, there is some $a_1 \in A_1$ with $h_1a_1 = b_1$. It follows that $h_2f_1a_1 = g_1h_1a_1 = g_1b_1 = h_2a_2$. Thus, since $h_2$ is injective, $a_2 = f_1a_1$, so $a_3 = f_2a_2 = f_2f_1a_1 = 0$. Therefore, $h_3$ is injectiv. To show surjectivity of $h_3$, let $b_3 \in B_3$. Since $h_4$ is surjective, there is some $a_4 \in A_4$ with $h_4a_4 = g_3b_3 \in A_4$ and by commutativity, $h_5f_4a_4 = g_4h_4a_4 = g_4g_3b_3 = 0$. Since $h_5$ is injective, this implies $f_4a_4 = 0$. By exactness of the top row at $A_4$, there is an $a_3 \in A_3$ such that $f_3(a_3) = a_4$. Hence,

$$g_3(h_3(a_3) - b_3) = h_4f_3a_3 - g_3b_3 = h_4a_4 - g_3b_3 = 0.$$

By the exactness of the lower row at $B_3$, this implies that there exists $b_2 \in B_2$ such that $g_2(b_2) = h_3(a_3) - b_3$. Since $h_2$ is surjective, there is some $a_2 \in A_2$ with $h_2a_2 = b_2$ and hence

$$h_3(a_3 - f_2a_2) = h_3a_3 - h_3f_2a_2 = h_3a_3 - g_2h_2a_2 = b_3,$$

which shows that $b_3$ is in the image of $h_3$. Since $b_3$ was arbitrary, this shows that $h_3$ is surjective.

We see that we've used the assumptions that $h_4$, $h_2$ are injective, and that $h_1$ is surjective to show injectivity of $h_3$. Our proof that $h_3$ is surjective required the assumptions that $h_2$, $h_4$ are surjective, and that $h_5$ is injective. So it is sufficient to assume that $h_2$ and $h_4$ are isomorphisms, that $h_1$ is an epimorphism, and that $h_5$ is a monomorphism. ☐

## 5. HOMEWORK ASSIGNMENT # 5

1. Let $X$ be a topological space and let $\Sigma X$ be the *suspension of* $X$ which is defined as the quotient space $X \times [0,1]/\sim$, where the equivalence relation is generated by $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. Show that $\widetilde{H}_q(X) \cong \widetilde{H}_{q+1}(\Sigma X)$ (this is called the *suspension isomorphism*).

Hint: note that the suspension of $S^n$ is homeomorphic to $S^{n+1}$, and think of the suspension isomorphism as a generalization of the isomorphism $\widetilde{H}_q(S^n) \cong \widetilde{H}_{q+1}(S^{n+1})$ proved in class. For that proof the decomposition of $S^{n+1}$ into upper and lower hemisphere was important. Here the role of the upper/lower hemispheres is played by subspaces of $\Sigma X$ consisting of those points $[x,t] \in \Sigma X$ with $t \geq 1/2$ resp. $t \leq 1/2$.

*Proof.* Let $x^+ = [x,1] \in \Sigma X$, and $x_- = [x,0] \in \Sigma X$ (they are analogous to the north pole resp. south pole of $S^{n+1}$), and let $C^+X := \Sigma X \setminus \{x^-\}$ and $C^-X := \Sigma X \setminus \{x^+\}$.

**Claim 1:** The spaces $C^{\pm}X$ are contractible (i.e., homotopy equivalent to the one point space).

To prove that $C^+X$ is contractible, we will show that the inclusion map $i\colon \{x^+\} \to C^+X$ is the homotopy inverse to the constant map $r\colon C^+X \to \{x^+\}$ (i.e., that the compositions $r \circ i$ and $i \circ r$ are homotopic to the identities on $\{x^+\}$ resp. $C^+X$). We note that $r \circ i$ is *equal* to the identity on $\{x^+\}$; a homotopy between $i \circ r$ and the identity on $C^+X$ is provided by the map

$$H\colon [0,1] \times C^+X \longrightarrow C^+X \qquad (s,[x,t]) \mapsto [x, st + (1-s)].$$

The proof that $C^-X$ is contractible is similar.

**Step 1.** Consider the long exact homology sequence of the pair $(\Sigma, \Sigma^+X)$:

$$\begin{tikzcd} {} \arrow[r] & \widetilde{H}_q(C^+X) \arrow[r] & \widetilde{H}_q(\Sigma X) \arrow[r, "j_*"] & H_q(\Sigma X, C^+X) \arrow[r, "\partial"] & \widetilde{H}_{q-1}(C^+X) \arrow[r] & {} \end{tikzcd}$$

The contractibility of the space $C^+$ implies the vanishing of its reduced homology groups and hence it follows that the map $j_*$ in the above sequence is an isomorphism.

**Step 2.** Removing the point $x^+$ from the pair $(\Sigma X, C^+X)$ we obtain the pair

$$(\Sigma X \setminus \{x^+\}, C^+X \setminus \{x^+\}) = (C^-X, C^+X \cap C^-X)$$

The excision axiom then implies that the inclusion map

$$(C^-X, C^+X \cap C^-X) \longrightarrow (\Sigma X, C^+X)$$

induces an isomorphism on homology groups.

**Step 3.** Consider the long exact homology sequence of the pair $(C^-X, C)$, $C = C^+X \cap C^-X$:

$$\begin{tikzcd} {} \arrow[r] & \widetilde{H}_{q+1}(C^-X) \arrow[r] & H_{q+1}(C^-X, C) \arrow[r, "\partial"] & \widetilde{H}_q(C) \arrow[r] & \widetilde{H}_q(C^-X) \arrow[r] & {} \end{tikzcd}$$

The contractibility of $C^-X$ implies the vanishing of its reduced homology groups. Hence the connecting homomorphism $\partial$ in the above exact sequence is an isomorphism.

**Step 4.** We claim that the map $i\colon X \to C^+X \cap C^-X$ given by $i(x) = [x, \frac{1}{2}]$ is a homotopy equivalence. In particular, $i$ induces isomorphisms on homology groups.

Let $r\colon C^+X \cap C^-X \to X$ be defined by $[x,t] \mapsto x$ (note that this is well-defined and continuous). We note that $r \circ i$ is the identity on $X$; a homotopy between $i \circ r$ and the identity on $C^+X \cap C^-X$ is given by

$$H\colon [0,1] \times C^+X \cap C^-X \longrightarrow C^+X \cap C^-X \qquad (s,[x,t]) \mapsto [x, st + (1-s)\frac{1}{2}]$$

Composing the isomorphisms constructed in steps (1)-(4) we obtain the desired isomorphism

$$\begin{tikzcd} \widetilde{H}_{q+1}(\Sigma X) \arrow[r, "j_*", "\cong"'] & H_{q+1}(\Sigma X, C^+) & H_{q+1}(C^-, C) \arrow[l, "k_*"', "\cong"] \arrow[r, "\partial", "\cong"'] & \widetilde{H}_q(C) & \widetilde{H}_q(X) \arrow[l, "i_*"', "\cong"] \end{tikzcd}$$

□

2. Let $f\colon S^n \to S^n$ be a map and

$$\Sigma f\colon S^{n+1} \approx \Sigma S^n \to \Sigma S^n \approx S^{n+1}$$

its suspension. Show that $\deg(\Sigma f) = \deg(f)$. Hint: $f$ induces an endomorphism on all the homology groups you used in your proof of the suspension isomorphism $\widetilde{H}_q(S^n) \cong \widetilde{H}_{q+1}(\Sigma S^n)$.

*Proof.* We note that the map $\Sigma f\colon \Sigma S^n \to \Sigma S^n$ leaves the subspaces $C^\pm \subset \Sigma S^n$ used in problem #1 invariant. In particular, $\Sigma f$ induces selfmaps of the pairs $(\Sigma S^n, C^+)$ and $(C^-, C)$.

Abusing notation we will write $\Sigma f$ for all these selfmaps. These maps are compatible with the maps $j$, $k$ and $i$ in the sense that the following diagrams are commutative:

$$\begin{tikzcd}
\Sigma S^n \arrow[r, "j"] \arrow[d, "\Sigma f"'] & (\Sigma S^n, C^+) \arrow[d, "\Sigma f"'] & (C^-, C) \arrow[l, "k"'] \arrow[d, "\Sigma f"] \\
\Sigma S^n \arrow[r, "j"] & (\Sigma S^n, C^+) & (C^-, C) \arrow[l, "k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d, "\Sigma f"'] & S^n \arrow[l, "i"'] \arrow[d, "f"'] \\
C & S^n \arrow[l, "i"]
\end{tikzcd}$$

Now we consider the following diagram of homology groups

$$\begin{tikzcd}
\widetilde{H}_{n+1}(\Sigma S^n) \arrow[r, "j_*"] \arrow[d, "\Sigma f_*"'] & H_{n+1}(\Sigma S^n, C^+) \arrow[d, "\Sigma f_*"'] & H_{n+1}(C^-, C) \arrow[l, "k_*"'] \arrow[r, "\partial"] \arrow[d, "\Sigma f_*"'] & \widetilde{H}_n(C) \arrow[d, "\Sigma f_*"'] & \widetilde{H}_n(S^n) \arrow[l, "i_*"'] \arrow[d, "f_*"'] \\
\widetilde{H}_{n+1}(\Sigma S^n) \arrow[r, "j_*"] & H_{n+1}(\Sigma S^n, C^+) & H_{n+1}(C^-, C) \arrow[l, "k_*"'] \arrow[r, "\partial"] & \widetilde{H}_n(C) & \widetilde{H}_n(S^n) \arrow[l, "i_*"']
\end{tikzcd}$$

We note that the first, second and fourth square of this diagram are commutative, since the corresponding square of maps between topological spaces commute. The third square is commutative due to the 'naturality of the connecting homomorphism' (see next week's homework problems). We pick a generator $\alpha_n \in \widetilde{H}_n(S^n)$ and denote by $\alpha_{n+1} \in \widetilde{H}_{n+1}(\Sigma S^n)$ the generator corresponding to $\alpha_n$ via the isomorphism $\Psi\colon \widetilde{H}_{n+1}(\Sigma S^n) \to \widetilde{H}_n(S^n)$ given by the rows of the above diagram. We have

$$\begin{aligned}
f_* \circ \Psi(\alpha_{n+1}) &= f_*(\alpha_n) = \deg(f)\alpha_n \\
\Psi \circ \Sigma f_*(\alpha_{n+1}) &= \Psi(\deg(\Sigma f)\alpha_{n+1}) = \deg(\Sigma f)\alpha_n
\end{aligned}$$

The commutativity of the homology diagram above implies $f_* \circ \Psi = \Psi \circ \Sigma f_*$ and hence $\deg(f) = \deg(\Sigma f)$. ☐

3. Prove the following statements for the local degree.

a) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry (i.e., $f$ belongs to the orthogonal group), then $\deg(f, 0) = \deg(f_{|S^{n-1}})$.

b) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then

$$\deg(f, 0) = \begin{cases} +1 & \det(f) > 0 \\ -1 & \det(f) < 0 \end{cases}$$

c) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map which is differentiable at the point $x_0 \in \mathbb{R}^n$. Let $Df_{x_0}$ be the derivative at $x_0$ (which is a linear map $Df_{x_0}\colon \mathbb{R}^n \to \mathbb{R}^n$; the corresponding matrix is the Jacobian of $f$ at the point $x_0$). Show that if $Df_{x_0}$ is invertible, then

$$\deg(f, x_0) = \begin{cases} +1 & \det(Df_{x_0}) > 0 \\ -1 & \det(Df_{x_0}) < 0 \end{cases}$$

Hint: Try to compare $\deg(f, x_0)$ with $\deg(g, x_0)$, where $g$ is the best affine linear approximation to $f$ which is given by $g(x) = f(x) + Df_{x_0}(x)$ by showing that restricted to a small enough ball around $x$ there is a homotopy $f_t$, $0 \leq t \leq 1$ between these two maps such that $f_t^{-1}(f(x))$ consists of the point $x_0$ only.

*Proof.* Let $g\colon S^{n-1} \to S^{n-1}$ be the restriction of $f$ to $S^{n-1}$. By the previous problem we know that $\deg(g) = \deg(\Sigma g)$. We note that $(\Sigma g)^{-1}(x_-) = \{x_-\}$, where $x_- = [x, 0] \in \Sigma S^{n-1}$. Hence by the theorem from class, we have $\deg(\Sigma g) = \deg(\Sigma g, x_-)$. To calculate that local degree, we can restrict $\Sigma g$ to a neighborhood of $x_-$, which we take to be $C^- = \{[x,t] \in \Sigma X \mid 0 \le t < 1\}$. We note that $C^-$ is homeomorphic to $\mathbb{R}^n$ via the homeomorphism

$$h\colon C^- \longrightarrow \mathbb{R}^n \qquad [x,t] \mapsto \left(\tan t\frac{\pi}{2}\right) x$$

Moreover, via this homeomorphism the map $\Sigma g$ corresponds to the original map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ in the sense that the diagram

$$\begin{tikzcd} C^- \arrow[r, "h"] \arrow[d, "\Sigma g"'] & \mathbb{R}^n \arrow[d, "f"] \\ C^- \arrow[r, "h"] & \mathbb{R}^n \end{tikzcd}$$

is commutative:

$$h(\Sigma g([x,t])) = h([f(x),t]) = \left(\tan t\frac{\pi}{2}\right) f(x) = f(\left(\tan t\frac{\pi}{2}\right) x) = f(h([x,t]))$$

Since $h(x_-) = 0$, it follows that $\deg(\Sigma g, x_-) = \deg(f, 0)$. This proves part (a).

To prove part (b), we note that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ belongs to the orthogonal group $O(n)$ of linear isometries, then $f$ restricts to a map $f_|\colon S^{n-1} \to S^{n-1}$. We've proved in class that $\deg(f_|) = \det(f)$, and together with part (a) this proves the desired statement if $f$ is an *isometry*. To prove the result for a general isomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, it suffices to show that there is a path $f(t)$ in the group $GL_n(\mathbb{R})$ of isomorphism of $\mathbb{R}^n$ which connects $f = f(0)$ with an element $f(1) \in O(n) \subset GL_n(\mathbb{R})$. We note that the sign of $\det(f(t))$ is independent of $t$, since it depends continuously on $t$, and $\det(f(t)) \neq 0$ for all $t \in [0,1]$.

To construct the path, we identify linear maps $\mathbb{R}^n \to \mathbb{R}^n$ with $n \times n$ matrices in the usual way. We will write a matrix $A$ in the form $A = (a_1, a_2, \dots, a_n)$, where $a_i \in \mathbb{R}^n$ are the column vectors of the matrix. We recall that $A$ belongs to $GL_n(\mathbb{R})$ if and only if the vectors $a_1, \dots, a_n$ are linearly independent; $A \in O(n)$ if and only if the vectors $a_i$ are unit vectors which are mutually perpendicular. Let $p_i\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the subspace spanned by $a_1, \dots, a_i$, and let $p_i^\perp\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the orthogonal complement of that subspace; in particular, we have $v = p_i^\perp v + p_i v$ for all $v \in \mathbb{R}^n$. Given a matrix $A$, let us define

$$A_t = (v_1, p_1^\perp v_2 + tp_1 v_2, \dots, p_{i-1}^\perp v_i + tp_{i-1} v_i, \dots, p_{n-1}^\perp v_n + tp_{n-1} v_n)$$

We note that $\det(A_t)$ is independent of $t$, since $\det(v_1, \dots, v_n) = 0$ depends linearly on each column vector and this determinant is zero if the vectors are linearly dependent. In particular, if $A = A_1$ invertible, then so is $A_t$. The column vectors of $B = A_0$ are mutually perpendicular, but not necessarily of unit length. Now for $B = (w_1, \dots, w_n)$ we define

$$B_t = ((1-t)w_1 + t\frac{w_1}{||w_1||}, \dots, (1-t)w_n + t\frac{w_n}{||w_n||}),$$

which is a path connecting $B = B_0$ with $B_1 \in O(n)$. Concatenating these paths, we obtain the desired path from $A$ to $B_0 \in O(n)$.

To prove part (c), we note that the assumption that $f$ is differentiable at $x_0$ means that $f(x_0 + h)$ can be written in the form

$$f(x_0 + h) = f(x_0) + Df_{x_0}(h) + e(h),$$

where the 'error term' $e(h)$ is $o(h)$ for $h \to 0$, which means that

$$\lim_{h\to 0} \frac{e(h)}{|h|} = 0. \tag{7}$$

We define

$$f_t(x_0 + h) = f(x_0) + Df_{x_0}(h) + te(h)$$

and want to argue that $f_t$ is a map of pairs

$$f_t\colon (B_\epsilon(x_0), B_\epsilon(x_0) \setminus \{x_0\}) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus f(x_0)),$$

for sufficiently small $\epsilon > 0$, where $B_\epsilon(x_0)$ is the ball of radius $\epsilon$ around $x_0$. In other words, we want to argue that $f_t^{-1}(f(x_0)) \cap B_\epsilon(x_0) = \{x_0\}$, or equivalently, that $Df_{x_0}(h) + te(h) \neq 0$ for all $h$ with $0 < h < \epsilon$. The idea is to show that for $0 < h < \epsilon$ the norm of $Df_{x_0}(h)$ is large compared to the norm of $te(h)$.

To make this precise, let $m := \min_{h\in S^{n-1}} ||Df_{x_0}(h)||$. We note that $m > 0$, since $m = ||Df_{x_0}(h_0)||$ for some $h_0 \in S^{n-1}$, and $Df_{x_0}(h_0) \neq 0$ due to our assumption that $Df_{x_0}$ is invertible. Now the statement (7) allows us to choose $\epsilon > 0$ such that $\frac{||e(h)||}{|h|} < m$ for $||h|| < \epsilon$. This implies that for $0 < ||h|| < \epsilon$ we have

$$||te(h)|| \leq ||e(t)|| < m||h|| \leq ||Df_{x_0}(\frac{h}{||h||})||\,||h|| = ||Df_{x_0}(h)||$$

and hence $Df_{x_0}(h) + te(h) \neq 0$ as desired. We conclude that $f = f_1$ is homotopic to $g = f_0$ as maps from $(B_\epsilon(x_0), B_\epsilon(x_0) \setminus \{x_0\})$ to $(\mathbb{R}^n, \mathbb{R}^n \setminus f(x_0))$ and hence $\deg(f, x_0) = \deg(g, x_0)$.

Finally, we want to compare $\deg(g, x_0)$ and $\deg(D, 0)$, where $D = Df_{x_0}$. Since $g(x) = f(x_0) + D(x - x_0)$, the map $g$ can be written in the form $g = T_{f(x_0)} \circ D \circ T_{-x_0}$, where $T_v\colon \mathbb{R}^n \to \mathbb{R}^n$ is the *translation map* given by $x \mapsto x + v$ for $v \in \mathbb{R}^n$. To relate the local degrees of $g$ and $D$ we note that the maps $D$, $T_v$ and $g$ extend to continuous selfmaps $\widehat{D}$, $\widehat{T}_v$, $\widehat{g}$ of the one-point compactification $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$. The space $\widehat{\mathbb{R}}^n$ is homeomorphic to $S^n$ via the stereographic projection, and hence we can consider the degree of these maps. We note that

$$\deg(\widehat{D}) = \deg(D, 0) \qquad \deg(\widehat{g}) = \deg(g, x_0)$$

by our theorem from class, since $\widehat{D}^{-1}(0) = D^{-1}(0) = \{0\}$ and $\widehat{g}^{-1}(f(x_0)) = g^{-1}(f(x_0)) = \{x_0\}$. Moreover, $\deg(\widehat{T}_v) = 1$, since the map

$$\widehat{R}^n \times [0,1] \to \widehat{\mathbb{R}}^n \qquad (x,t) \mapsto \widehat{T}_{tv}(x)$$

provides a homotopy between $\widehat{T}_v$ and $\widehat{T}_0 = 1_{\widehat{R}^n}$. It follows that

$$\deg(\widehat{g}) = \deg(\widehat{T}_{f(x_0)} \circ \widehat{D} \circ \widehat{T}_{-x_0}) = \deg(T_{f(x_0)}) \cdot \deg(\widehat{D}) \cdot \deg(\widehat{T}_{-x_0}) = \deg(\widehat{D})$$

which finishes the proof. □

## 6. HOMEWORK ASSIGNMENT # 6

1. Let $(X, V, A)$ be a triple of topological spaces (i.e., $A \subset V \subset X$). Show that there is a long exact sequence of homology groups

$$\dots \longrightarrow H_q(V, A) \longrightarrow H_q(X, A) \longrightarrow H_q(X, V) \xrightarrow{\partial} H_{q-1}(V, A) \longrightarrow \dots$$

*Proof.* Let $i\colon (V,A) \to (X,A)$ and $j\colon (X,A) \to (X,V)$ be the maps of pairs induced by the inclusion map $V \to X$ resp. the identity on $X$. The induced maps on singular $q$-chains

$$\begin{tikzcd}
C_q(V,A) \arrow[r, "i_q"] \arrow[d, equal] & C_q(X,A) \arrow[r, "j_q"] \arrow[d, equal] & C_q(X,V) \arrow[d, equal] \\
C_q(V)/C_q(A) & C_q(X)/C_q(A) & C_q(X)/C_q(V
\end{tikzcd}$$

form a short exact sequence since $j_q$ is an epimorphism whose kernel is equal to $C_q(V)/C_q(A) \subset C_q(X)/C_q(A)$.

This implies that

$$C_*(V,A) \xrightarrow{i_*} C_*(X,A) \xrightarrow{j_*} C_*(X,V)$$

is a short exact sequence of chain complexes which implies the desired long exact sequence of homology groups. $\square$

2. Let

$$\begin{tikzcd}
A_* \arrow[r, "f_*"] \arrow[d, "a_*"] & B_* \arrow[r, "g_*"] \arrow[d, "b_*"] & C_* \arrow[d, "c_*"] \\
A'_* \arrow[r, "f'_*"] & B'_* \arrow[r, "g'_*"] & C'_*
\end{tikzcd}$$

be a commutative diagram of chain complexes and chain maps whose rows are short exact. Show that the following diagram is commutative:

$$\begin{tikzcd}
H_q(C_*) \arrow[r, "\partial"] \arrow[d, "c_*"] & H_q(A_*) \arrow[d, "a_*"] \\
H_q(C'_*) \arrow[r, "\partial"] & H_{q-1}(A'_*)
\end{tikzcd}$$

This statement is referred to as the *naturality of the connecting homomorphism.* In particular, if $f\colon (X,A) \to (X',A')$ is a map between pairs of topological spaces, then the singular chain complexes of $A$, $X$, $(X,A)$, $A'$, $X'$, $(X',A')$ fit together in a diagram as above and we conclude that the diagram

$$\begin{tikzcd}
H_q(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{q-1}(A) \arrow[d, "f_*"] \\
H_q(X',A') \arrow[r, "\partial"] & H_{q-1}(A')
\end{tikzcd}$$

is commutative.

*Proof.* Let $c$ be a $q$-cycle in the chain complex $C_*$ and let us denote as usual by $[c] \in H_q(C_*)$ the homology class it represents. We recall that $\partial([c]) = [a]$, if there is some $b \in B_q$ with $g_q b = c$ and $\partial^A_q b = f_{q-1} a$ (where we write $\partial^A$ for the boundary map in the chain complex $A_*$). We note that the commutativity of the diagram above implies that

$$g'_q(b_q b) = c_q c \quad \text{and} \quad f'_{q-1} a_{q-1} a = b_{q-1} f_{q-1} a = b_{q-1} \partial^A_q b = \partial^{A'}_q b_q b.$$

Here the last equation follows from the fact that $b_*$ is a chain map. It follows that $\partial\colon H_q(C'_*) \to H_{q-1}(A'_*)$ maps $[c_q c] = c_*([c])$ to $[a_q a] = a_*([a])$. $\square$

3. a) Show that any non-constant polynomial $f(z)$, viewed as a map $f\colon \mathbb{C} \to \mathbb{C}$ always extends to a continuous map $\widehat{f}\colon \widehat{C} \approx S^2 \to \widehat{C} \approx S^2$ of the one-point compactification $\widehat{C} = \mathbb{C} \cup \{\infty\} \approx S^2$.

b) Show that the degree of $\widehat{f}$ equals the degree of $f$ as a polynomial. Hint: first show this for $f(z) = z^n$ and then reduce to this case.

c) Conclude the Fundamental Theorem of Algebra that $f(z)$ has some zero.

*Proof.* **Part (a).** It will be useful to have a criterion for when a continuous map $f\colon X \times T \to Y$ extends to a continuous map $\widehat{f}\colon \widehat{X} \times T \to \widehat{Y}$, where $\widehat{X} = X \cup \{x_0\}$ (resp. $\widehat{Y} = Y \cup \{y_0\}$) is the one-point compactification of $X$ (resp. $Y$). We will apply this for $X = Y = \mathbb{C}$, $T = \{\text{pt}\}$ to show that a non-constant polynomial $f$ induces a map $\widehat{f}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and to $T = [0,1]$ to show that suitable homotopies between polynomials $f$ and $g$ gives a homotopy between $\widehat{f}$ and $\widehat{g}$.

**Warning.** The function $H(z,t) = tz^n + (1-t)z^{n-1}$ is a path of non-constant polynomials, i.e., a homotopy between the polynomials $f(z) = z^n$ and $g(z) = z^{n_1}$. It is tempting to argue that $H$ gives a homotopy between $\widehat{f}$ and $\widehat{g}$. We emphasize that **there can't be a homotopy between $\widehat{f}$ and $\widehat{g}$**, since their degrees are not equal: $\deg(\widehat{f}) = n$, $\deg(\widehat{g}) = n-1$. This shows that we need to be extremely careful when arguing that a continuous map $H\colon \mathbb{C} \times [0,1] \to \mathbb{C}$ extends to a continuous map $\widehat{H}\colon \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$.

**Lemma 1.** *Let $T$, $X$, $Y$ be topological spaces and let $\widehat{X} = X \cup \{x_0\}$, $\widehat{Y} = Y \cup \{y_0\}$ be the one-point compactifications of $X$ and $Y$. Let $f\colon X \times T \to Y$ be a continuous map, and define*

$$\widehat{f}\colon \widehat{X} \times T \to \widehat{Y} \qquad \text{by} \qquad \widehat{f}(x,t) = \begin{cases} f(x,t) & x \in X \\ y_0 & x = x_0 \end{cases}$$

*Suppose that for any compact subset $L \subset Y$ and any $t \in T$, there exists a compact subset $K \subset X$ and open neighborhood $U \subset T$ of $t$ such that $f^{-1}(L) \cap X \times U \subset K \times U$. Then $\widehat{f}$ is continuous.*

*Proof.* It suffices to prove continuity of $\widehat{f}$ at all points of the form $(x_0, t)$, i.e., to show that the preimage of an open neighborhood of $\widehat{f}(x_0,t) = y_0$ contains an open neighborhood of $(x_0,t)$. We recall that an open subset of the one-point compactification $\widehat{Y}$ containing $y_0$ is of the form $y_0 \cup (Y \setminus L)$ where $L \subset Y$ is compact. Our assumptions imply that the open neighborhood $(\{x_0\} \cup (X \setminus K)) \times U$ of $(x_0,t)$ is contained in $f^{-1}(y_0 \cup (Y \setminus L))$. ☐

We recall that according to the Heine-Borel Theorem a subset $K \subset \mathbb{C}$ is compact if and only ifit is closed and bounded. So the lemma implies that a continuous map $f\colon \mathbb{C} \to \mathbb{C}$ extends to a continuous map $\widehat{f}$ provided that for all $r > 0$ there is some $s > 0$ such that $|z| > s$ implies $|f(z)| > r$. Let us assume that $f(z)$ is a polynomial of degree $n$, i.e.,

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \qquad \text{with} \qquad a_i \in \mathbb{C}, a_n \neq 0.$$

Then

$$|f(z)| = \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| |z|^n$$

Since $\lim_{z\to\infty} \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} = 0$, there is some $s_0 > 0$ such that $|z| > s_0$ implies

$$|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}| < \frac{|a_n|}{2}$$

and hence

$$|f(z)| \leq (|a_n| - \frac{1}{2}|a_n|)|z|^n = \frac{1}{2}|a_n||z|^n$$

In particular, setting $s = \max\{s_0, \left(\frac{2r}{|a_n|}\right)^{1/n}$, then $|z| > s$ implies $|f(z)| > r$, which proves part (a).

More generally, if instead of a single polynomial $f(z)$ we have a family of polynomials $f_t(z)$ whose coefficients are continuous functions $a_i(t)$ of a parameter $t \in T$ with the property that $a_n(t) \neq 0$ for all $t \in T$, then we can not only obtain the above estimate for any fixed $t_0 \in T$, but rather the statement $|z| > s$ implies $|f_t(z)| > r$ follows for every $t$ in a suitable neighborhood $U$ of $t_0$. This implies that we obtain a continuous map $\widehat{\mathbb{C}} \times T \to \widehat{\mathbb{C}}$.

**Part (b).** To calculate the degree of $p$ for part b), we first assume $p(z) = z^n$, and calculate the degree of $p$ by our formula

$$\deg(p) = \sum_{z \in p^{-1}(z_0)} \deg(p, z)$$

for any point $z_0 \in \mathbb{C}$ for which $p^{-1}(z_0)$ is a finite set. E.g., for $z_0 = 1$, the preimage consists of the $n$-th roots of unity $\{e^{2\pi i/n}\}$. We note that the derivative $p'(z) = nz^{n-1}$ is non-zero at any $z \neq 0$ and hence by the Inverse Function Theorem, $p$ is a local diffeomorphism at any point $z \neq 0$. This implies that the local degree $\deg(p, z)$ is defined for any $z \neq 0$ and is equal to $\pm 1$. Moreover, since $p$ is a *holomorphic* function, its derivative $(Dp)_z \colon \mathbb{R}^2 = \mathbb{C} \to \mathbb{R}^2 = \mathbb{C}$ for any $z \in \mathbb{C}$ is a *complex* linear map, given in fact by multiplication by the complex number $p'(z)$. So if $p'(z) = a + ib$ with $a, b \in \mathbb{R}$, then the matrix corresponding to $(Dp)_z$ is $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, and hence

$$\det(Dp)_z = \det \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) = a^2 + b^2 \geq 0$$

Hence $\deg(p, z) = 1$ for any $z \neq 0$. In particular, the degree of $p(z) = z^n$ is $n$, the number of $n$-th roots of unity.

Next assume that $g(z)$ is a monic polynomial of degree $n$, i.e., that $a_n = 1$. Then

$$H \colon \mathbb{C} \times [0,1] \longrightarrow \mathbb{C} \qquad (z, t) \mapsto z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$$

is a homotopy between the monic polynomial $g$ and $h(z) = z^n$. Our considerations above show that $\widehat{H}$ is a homotopy between $\widehat{g}$ and $\widehat{h}$ which implies $\deg(g) = \deg(h) = n$. If $f(z) = a_n z^n + \cdots + a_0$ is any polynomial of degree $n$, we can write $a_n = e^b$ for some $b \in \mathbb{C}$. Then

$$H'(z, t) = e^{tb} z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

is a between $f(z)$ and the monic polynomial $g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Again, $H'$ induces a homotopy $\widehat{H}'$ between $\widehat{f}$ and $\widehat{g}$ which implies $\deg(\widehat{f}) = \deg(\widehat{g}) = n$ and proves part (b).

**Part (c).** The assumption that $f(z)$ is non-constant implies that $\deg(\widehat{f}) = \deg(f) > 0$. Hence $f \colon \mathbb{C} \to \mathbb{C}$ is surjective; in particular, its set of zeroes $f^{-1}(0)$ is non-empty. ☐

## 7. HOMEWORK ASSIGNMENT # 7

1. Show that the complex projective space $\mathbb{CP}^n$ is a CW complex with one cell of dimension $2i$ for $0 \le i \le n$.

*Proof.* It suffices to show that $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell of dimension $2n$. Define

$$\Phi\colon D^{2n} \longrightarrow \mathbb{C}P^n$$
$$z = (z_0, \ldots, z_{n-1}) \mapsto [z_0, z_1, \ldots, z_{n-1}, \sqrt{1 - ||z||^2}],$$

and let $\varphi\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ be the natural projection map given by $(z_0, \ldots, z_{n-1}) \mapsto [z_0, z_1, \ldots, z_{n-1}]$. We note that these maps are compatible in the sense that the following diagram is commutative

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "\varphi"] \arrow[d, "j"] & \mathbb{C}P^{n-1} \arrow[d, "i"] \\
D^{2n} \arrow[r, "\Phi"] & \mathbb{C}P^n,
\end{tikzcd}$$

where $i$, $j$ are the obvious inclusion maps. It follows that the map

$$\mathbb{C}P^{n-1} \cup_\varphi D^{2n} \xrightarrow{i \cup \Phi} \mathbb{C}P^n$$

is well-defined and continuous. We note that this map is surjective, since if $[z_0, \ldots, z_n] \in \mathbb{C}P^n$ with $z_n \neq 0$, then multiplying all components by the unit complex number $z_n^{-1}||z_n||$, we can assume w.l.o.g. that $z_n$ is a positive real number. Injectivity is obvious and it follows that $f = i \cup \Phi$ is a continuous bijection. It follows that $f$ is a homeomorphism since the domain of $f$ is compact and its range is Hausdorff. ☐

2. See Hatcher's book, pages 140, 141.

For the calculation of the homology groups of the product spaces in problems (3) and (4) use the following facts. If $X$, $Y$ are finite CW complexes, then the product $X \times Y$ again has a CW structure whose cells correspond to products of cells of $X$ and $Y$. More precisely, if $e^m_\alpha$ is an $m$-cell of $X$, and $e^n_\beta$ is an $n$-cell of $Y$, then these determine a $m+n$-cell of $X \times Y$ denoted $e^m_\alpha \times e^n_\beta$. In particular, $C^{CW}_q(X \times Y)$ is the free abelian group generated by products cells $e^m_\alpha \times e^n_\beta$ with $m + n = q$. The cellular boundary map is determined by the formula

$$\partial(e^m_\alpha \times e^n_\beta) = (\partial e^m_\alpha) \times e^n_\beta + (-1)^m e^m_\alpha \times \partial(e^n_\beta). \tag{8}$$

3. Calculate $H_*(S^m \times S^n)$ for $m \ge n \ge 1$.

To calculate the homology of $S^m \times S^n$, let us furnish the sphere $S^m$ with the CW structure consisting of one 0-cell $e^0$ and one $m$-cell $e^m$. Then the product CW structure on $S^m \times S^n$ has four cells $e^0 \times e^0$, $e^m \times e^0$, $e^0 \times e^n$, $e^m \times e^n$ of dimension 0, $m$, $n$ and $m + n$. The cellular boundary map in the cellular chain complex of $S^m$ and $S^n$ is zero, and hence the 'product rule' (8) implies that the cellular boundary map for $S^m \times S^n$ is trivial. Hence $H_q(S^m \times S^n) \cong C^{CW}_q(S^m \times S^n)$ is a direct sum of as many copies of $\mathbb{Z}$ as there are cells of dimension $q$. In particular, for $m, n > 0$ and $m \neq n$ we have

$$H_q(S^m \times S^n) \cong \begin{cases} \mathbb{Z} & q = 0, m, n, m+n \\ 0 & \text{otherwise} \end{cases}$$

If $n > 0$ we have:

$$H_q(S^n \times S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & q = n \\ \mathbb{Z} & q = 0, 2n \\ 0 & \text{otherwise} \end{cases}$$

4. Calculate the homology groups of a product of two Moore spaces $M(\mathbb{Z}/p, m) \times M(\mathbb{Z}/q, n)$ for $m \geq n \geq 1$.

*Proof.* The Moore space $M(\mathbb{Z}/p, m)$ has three cells $e^0$, $e^m$, $e^{m+1}$ whose dimensions are indicated by the superscripts. Hence the product

$$M(\mathbb{Z}/p, m) \times M(\mathbb{Z}/q, n)$$

has nine cells. To simplify matters, we note that the cell $e^0$ does not 'interact' with the cells $e^m$, $e^{m+1}$ in the cellular chain complex of $M(\mathbb{Z}/p, m)$ in the sense that $C_*^{CW}(M(\mathbb{Z}/p, m))$ is a direct sum of two chain complexes; the first summand is generated by $e^0$, the other is generated by $e^m$ and $e^{m+1}$. It follows that the chain complex of the product can be written as a direct sum of *four* chain complexes:

$$\begin{aligned} A_* &= \langle e^0 \times e^0 \rangle \\ B_* &= \langle e^0 \times e^n, e^0 \times e^{n+1} \rangle \\ C_* &= \langle e^m \times e^0, e^{m+1} \times e^0 \rangle \\ D_* &= \langle e^m \times e^n, e^m \times e^{n+1}, e^{m+1} \times e^n, e^{m+1} \times e^{n+1} \rangle \end{aligned}$$

The boundary maps of these chain complexes are readily determined by the product formula (8). Also the homology groups of $A_*$, $B_*$ and $C_*$ are straightforward to determine and so we only state the result:

$$H_i(A_*) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H_i(B_*) = \begin{cases} \mathbb{Z}/q & i = n \\ 0 & i \neq n \end{cases}$$

$$H_i(C_*) = \begin{cases} \mathbb{Z}/p & i = m \\ 0 & i \neq m \end{cases}$$

To calculate the homology groups of the chain complex $D_*$, we first determine the boundary maps of $D_*$:

$$\begin{aligned} d(e^m \times e^n) &= 0 \\ d(e^{m+1} \times e^n) &= p(e^m \times e^n) \\ d(e^m \times e^{n+1}) &= (-1)^m q(e^m \times e^n) \\ d(e^{m+1} \times e^{n+1}) &= p(e^m \times e^{n+1}) + (-1)^{m+1} q(e^{m+1} \times e^n) \end{aligned}$$

To determine the homology groups it is convenient to use a different basis for the chain group $D_{m+n+1}$. Let $g = \gcd pq$, and write $p = gp'$, $q = gq'$. Then $p'$ and $q'$ are relatively prime and hence there are integers $a$, $b$ such that $ap' + bq' = 1$. We note that the element

$d(e^{m+1} \times e^{n+1})$ is divisible by $g$. Let $e_1 \in D_{m+n+1}$ be the element obtained by that division, i.e.,

$$e_1 := p'(e^m \times e^{n+1}) + (-1)^{m+1} q'(e^{m+1} \times e^n)$$

and define

$$e_2 := (-1)^m b(e^m \times e^{n+1}) + a(e^{m+1} \times e^n)$$

This is taylor-made so that the base change matrix

$$\begin{pmatrix} p' & (-1)^m b \\ (-1)^{m+1} q' & a \end{pmatrix}$$

has determinant one which implies that $\{e_1, e_2\}$ is in fact a new basis for $D_{m+n+1}$. The advantage of this new basis is that the boundary map of the chain complex $D_*$ has the following simple form

$$\begin{aligned} d(e^{m+1} \times e^{n+1}) &= g e_1 \\ d(e_1) &= 0 \\ d(e_2) &= (-1)^m b d(e^m \times e^{n+1}) + a d(e^{m+1} \times e^n) \\ &= (-1)^m b(-1)^m q(e^m \times e^n) + ap(e^m \times e^n) \\ &= (bq + ap)(e^m \times e^n) = g(e^m \times e^n) \end{aligned}$$

It follows that

$$H_q(D_*) = \begin{cases} \mathbb{Z}/g & q = m+n, m+n+1 \\ 0 & \text{otherwise} \end{cases}$$

□

## 8. Homework Assignment # 8

1. Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained by attaching a Möbius band to $\mathbb{RP}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

*Proof.* First we need to find an open cover $\{A, B\}$ of the space $X$ under consideration. We try to find this cover in such a way that $A$ is homotopy equivalent to the Möbiusband and that $B$ is homotopy equivalent to $\mathbb{RP}^2$. Let us write $M$ for the Möbiusband and

$$p\colon M \amalg \mathbb{RP}^2 \to X = (M \amalg \mathbb{RP}^2)/\sim$$

for the projection map. We note that simply taking $A = p(M)$ and $B = p(\mathbb{RP}^2)$ is undesirable for *two* reasons:

(1) These subsets aren't *open* in $X$. We recall that a subset $U$ in the quotient space $X$ is open if and only if$p^{-1}(U)$ is an open subset of $M \amalg \mathbb{RP}^2$, and note that $p^{-1}(A)$ consists of the disjoint union of $M$ and $S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^2$ which is not an open subset of $M \amalg \mathbb{RP}^2$.

(2) The subspace $A \subset X$ is *not* homeomorphic to the Möbiusband, but rather to a quotient space of $M$ obtained by identifying antipodal points on the boundary circle of $M$.

Both problems can be solved by choosing $A$ a little smaller and $B$ a little larger than the choice above. Let $M_1$ be the interior of $M$, and let $M_2$ be the "outer half" of $M$. Explicitly, if $M = [0,1] \times [-1,+1]/\sim$ with identification $(0,y) \sim (1,-y)$, then

$$M_1 = [0,1] \times (-1,+1)/\sim \qquad M_2 = [0,1] \times ([-1,-1/2) \cup (1/2,1])/\sim$$

We set $A := p(M_1)$ and $B := p(M_2 \cup \mathbb{RP}^2)$, and note that $A$, $B$ are open subsets of $X$ since $p^{-1}(A) = M_1$ and $p^{-1}(B) = M_2 \amalg \mathbb{RP}^2$ are open subsets. Also, $A \cup B = X$ since $M_1 \cup M_2 = M$.

We observe that $A$ is homeomorphic to $M_1$ which has the central circle

$$C_1 := ([0,1] \times \{0\}/\sim) \subset ([0,1] \times [-1,+1]/\sim) = M$$

as deformation retract. In particular, $\widetilde{H}_1(A) \cong \mathbb{Z}$ and $\widetilde{H}_q(A) = 0$ for $q \neq 1$. Similarly, $A \cap B$ is homeomorphic to $M_1 \cap M_2$ which has the circle

$$C_2 := [0,1] \times \{\pm 3/4\} sim \subset M$$

as deformation retract, and hence $\widetilde{H}_1(A \cap B) \cong \mathbb{Z}$ and $\widetilde{H}_q(A \cap B) = 0$ for $q \neq 0$. Finally, $B$ has $\mathbb{RP}^2 \subset X$ as a deformation retract. The homotopy between the identity on $B$ and the retraction to $\mathbb{RP}^2$ is the identity on $\mathbb{RP}^2$ and given on $M_2$ by

$$H\colon I \times M_2 \to M_2 \qquad (t,[x,y]) \mapsto [x, ty + (1-t)y/|y|]$$

It follows that $\widetilde{H}_1(B) \cong \mathbb{Z}$ and $\widetilde{H}_q(B) = 0$ for $q \neq 0$.

Let us write $i^A\colon A \cap B \to A$ and $i^B\colon A \cap B \to B$ for the inclusion maps. Then

$$S^1 = C_2 \hookrightarrow A \cap B \xrightarrow{i^A} A \xrightarrow{r} C_1 = S^1$$

is the map $z \mapsto z^2$ which has degree 2. It follows that $i^A_*\colon \widetilde{H}_1(A \cap B) = \mathbb{Z} \to \widetilde{H}_1(A) = \mathbb{Z}$ is multiplication by 2. The map

$$S^1 = C_2 \hookrightarrow A \cap B \xrightarrow{i^B} B \xrightarrow{r} \mathbb{RP}^2$$

is just the usual inclusion of $S^1 = \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2$ and hence the induced map on $\widetilde{H}_1$ is the projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$.

Now we have all the information we need to analyse what happens in the Meyer-Vietoris sequence

$$\widetilde{H}_q(A \cap B) \xrightarrow{i^A_* \oplus i^B_*} \widetilde{H}_q(A) \oplus \widetilde{H}_q(B) \longrightarrow \widetilde{H}_q(X)$$
$$\longrightarrow \widetilde{H}_{q-1}(A \cap B) \xrightarrow{i^A_* \oplus i^B_*} \widetilde{H}_{q-1}(A) \oplus \widetilde{H}_{q-1}(B)$$

This sequence implies that $\widetilde{H}_2(X)$ (resp. $\widetilde{H}_1(X)$) is isomorphic to the kernel (resp. cokernel) of the map

$$i^A_* \oplus i^B_*\colon H_1(A \cap B) = \mathbb{Z} \xrightarrow{2 \oplus 1} \mathbb{Z} \oplus \mathbb{Z}/2 = H_1(A) \oplus H_1(B)$$

This map is injective and hence $H_2(X) = 0$. To determine its cokernel, let $a = (1,0)$ and $b = (0,1)$ in $\mathbb{Z} \oplus \mathbb{Z}/2$. Then the cokernel is the $\mathbb{Z}$-module generated by $a$ and $b$ with the relations $2b = 0$ and $2a + b = 0$, or equivalently, the $\mathbb{Z}$-module generated by $a$ with relation $4a = 0$. It follows that $H_1(X) \cong \mathbb{Z}/4$. □

2. Let $M$ be a module over a ring $R$. We recall that a *free resolution of* $M$ is an exact sequence of $R$-modules and $R$-module homomorphisms

$$\begin{tikzcd} 0 & M \arrow[l] & M_0 \arrow[l, "\epsilon"'] & M_1 \arrow[l, "d_1"'] & M_2 \arrow[l, "d_2"'] & \dots \arrow[l] \end{tikzcd}$$

such that all the $M_q$'s are *free* $R$-modules.

(a) Let $(M_*, \epsilon^M)$ be a free resolution of an $R$-module $M$ and let $(N_*, \epsilon^N)$ be a free resolution of an $R$-module $N$. Show that if $f\colon M \to N$ is an $R$-linear map, then there are $R$-linear maps $f_q\colon M_q \to N_q$ such that the diagram

$$\begin{tikzcd} M \arrow[d, "f"] & M_0 \arrow[l, "\epsilon^M"'] \arrow[d, "f_0"] & M_1 \arrow[l] \arrow[d, "f_1"] & M_2 \arrow[l] \arrow[d, "f_2"] & \cdots \arrow[l] \\ N & N_0 \arrow[l, "\epsilon^N"'] & N_1 \arrow[l] & N_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

is commutative. Hint: construct the $R$-linear maps $f_q$ inductively using the following property of a free module: if $g\colon A \to B$ is an $R$-module map whose domain $A$ is a *free* module, then $g$ factors through any $R$-linear surjection $h\colon C \to B$; i.e., there is an $R$-linear map $\widehat{g}\colon A \to C$ making the following diagram commutative:

$$\begin{tikzcd} & C \arrow[d, "h"] \\ A \arrow[ur, "\widehat{g}"] \arrow[r, "g"] & B \end{tikzcd}$$

(b) Show that the $R$-linear chain map $f_*\colon M_* \to N_*$ constructed in part (a) is unique up to $R$-linear chain homotopies, i.e., if $f'_*\colon M_* \to N_*$ is another solution to (a), show that there is a chain homotopy $T$ between them.

*Proof.* Since $M_0$ is a free $R$-module, the module map $f \circ \epsilon^M\colon M_0 \to N$ factors through the surjective map $\epsilon^N\colon N_0 \to N$; i.e., there is an $R$-linear map $f_0\colon M_0\colon N_0$ making the first square commutative. We will construct the $f_q$'s by induction. Let us assume that we already constructed $R$-linear maps $f_0, \dots, f_q$ making all diagrams to the left of $f_q$ commutative. We note that this implies in particular that $f_q$ maps $\ker(M_q \to M_{q-1})$ to $\ker(N_q \to N_{q-1})$. Now we want to construct $f_{q+1}$ such that the following diagram commutes:

$$\begin{tikzcd} \ker(M_q \to M_{q-1}) \arrow[d, "f_q"] & M_{q+1} \arrow[l, "d^M_{q+1}"'] \arrow[d, "f_{q+1}"] \\ \ker(N_q \to N_{q-1}) & N_{q+1} \arrow[l, "d^N_{q+1}"'] \end{tikzcd}$$

This map exists since $M_{q+1}$ is free and the map $d^N_{q+1}$ is surjective by exactness of the resolution $N_*$ at $N_q$.

To prove part (b), assume that $f'_*\colon M_* \to N_*$ is another chain map lifting the map $f$. Our goal is to construct a chain homotopy $T$ between them; i.e., we want $R$-linear maps $T_q\colon M_q \to N_{q+1}$ with

$$d^N_{q+1} T_q + T_{q-1} d^M_q = f_q - f'_q \tag{9}$$

where the modules $M_q$, $N_q$ are interpreted as the trivial modules for $q < 0$. We will construct the $T_q$'s inductively. To construct $T_0$, we note that

$$\epsilon^N \circ f_0 = f \circ \epsilon^M = \epsilon^N \circ f_0'$$

implies that the range of $f_0 - f_0'$ is contained in $\ker \epsilon^N$, and hence there is a map $T_0$ making the diagram

$$\begin{tikzcd}
M_0 \arrow[d, "f_0 - f_0'"'] \arrow[dr, "T_0"] & \\
\ker \epsilon^N & N_1 \arrow[l, "d_1^N"]
\end{tikzcd}$$

commutative since $M_0$ is free and the horizontal map is surjective.

Now let us assume that we have constructed $T_0, \dots, T_{k-1}$ satisfying equation (9) for $q < k$. To construct $T_k$, we consider this equation for $q = k$ and put the term $T_{k-1}d_k^M$ on the right side of the above equation and try to solve for $T_k$. We note that the image of

$$g := f_k - f_k' - T_{k-1}d_k^M \colon M_k \longrightarrow N_k$$

is contained in the kernel of $d_k^N$ since

$$\begin{aligned}
d_k^N(f_k - f_k' - T_{k-1}d_k^M) &= f_{k-1}d_k^M - f_{k-1}'d_k^M - d_k^N T_{k-1}d_k^M \\
=& f_{k-1}d_k^M - f_{k-1}'d_k^M - (T_{k-2}d_{k-1}^M d_k^M - f_{k-1}d_k^M - f_{k-1}'d_k^M) = 0
\end{aligned}$$

Here the first equation holds since $f_*$, $f_*'$ are chain maps, and the second equation follows from the inductive assumption.

Now we can construct $T_k$ making the diagram

$$\begin{tikzcd}
M_k \arrow[d, "g"'] \arrow[dr, "T_k"] & \\
\ker d_k^N & N_{k+1} \arrow[l, "d_{k+1}^N"]
\end{tikzcd}$$

commutative, since $M_k$ is free and $d_{k+1}^N$ is surjective onto the kernel of $d_k^N$ by the exactness of $N_*$.

□

3. Calculate the abelian groups $\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z})$ and $\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z}/t)$ for all $q = 0, 1, 2 \dots$.

*Proof.* The sequence

$$0 \longleftarrow M = \mathbb{Z}/s \longleftarrow M_0 = \mathbb{Z} \xleftarrow{s} M_1 = \mathbb{Z} \longleftarrow M_2 = 0 \longleftarrow$$

is a free resolution of $\mathbb{Z}/s$, where the map $s \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ stands for multiplication by $s$.

Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the resolution $M_*$ we obtain the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{s^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$$

We note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, where the isomorphism is given by $f \mapsto f(1)$. This implies that the above cochain complex is isomorphic to

$$\mathbb{Z} \xrightarrow{s} \mathbb{Z}$$

and hence

$$\operatorname{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z}) = H^q(\operatorname{Hom}_{\mathbb{Z}}(M_*, \mathbb{Z})) = \begin{cases} \mathbb{Z}/s & q = 1 \\ 0 & q \neq 1 \end{cases}$$

For $N = \mathbb{Z}/t$ we obtain the cochain complex

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/t) \xrightarrow{s^*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/t)$$

We note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/t)$ is isomorphic to $\mathbb{Z}/t$, where the isomorphism is given by $f \mapsto f(1)$. This implies that the above cochain complex is isomorphic to

$$\mathbb{Z}/t \xrightarrow{s} \mathbb{Z}/t$$

We note that the cokernel of the map $s$ is $\mathbb{Z}$ modulo the ideal generated by $t$ and $s$. This is equal to the ideal generated by $g = \gcd(s, t)$ and hence the cokernel of the map $s\colon \mathbb{Z}/t \to \mathbb{Z}/t$ is isomorphic to $\mathbb{Z}/g$. Considering the order of the image, the domain and the kernel of the map $s$, we see that $\ker s$ has order $g$; as a subgroup of the cyclic group $\mathbb{Z}/t$ it is isomorphic to $\mathbb{Z}/g$. We conclude:

$$\operatorname{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z}/t) = H^q(\operatorname{Hom}_{\mathbb{Z}}(M_*, \mathbb{Z}/t)) = \begin{cases} \mathbb{Z}/g & q = 0, 1 \\ 0 & q \neq 0, 1 \end{cases}$$

where $g = \gcd(s, t)$. □

## 9. HOMEWORK ASSIGNMENT # 9

1. Show that for $k, l \geq 1$

$$\widetilde{H}_q((S^k \times D^l)/(S^k \times S^{l-1})) = \begin{cases} \mathbb{Z} & q = l, k + l \\ 0 & q \neq l, k + l \end{cases}$$

*Proof.* We claim that $(S^k \times D^l, S^k \times S^{l-1})$ is a good pair. This follows by crossing with $S^k$ if we can show that $(D^l, S^{l-1})$ is a good pair, i.e., $S^{l-1}$ is a closed subset of $D^l$ (this is clear) and that $S^{l-1}$ is a deformation retract of some neighborhood of $S^{l-1}$. To prove the latter, we note that $D^l \setminus \{0\}$ is an open neighborhood of $S^{l-1}$, and $r\colon D^l \setminus \{0\} \to S^{l-1}$ given by $x \mapsto x/||x||$ is a retraction. The composition of $r$ with the inclusion map $i\colon S^{l-1} \to D^l \setminus \{0\}$ is homotopic to the identity map relative $S^{l-1}$ via the homotopy

$$H\colon D^l \setminus \{0\} \times [0, 1] \longrightarrow D^l \setminus \{0\} \qquad (x, t) \mapsto (1 - t)\frac{x}{||x||} + tx.$$

We recall that the homology groups of good pairs $(X, A)$ can be identified with the homology groups of the quotient $X/A$. Hence the long exact homology sequence of the good pair $(S^k \times D^l, S^k \times S^{l-1})$ leads to the following exact sequence:

$$\widetilde{H}_q(S^k \times S^{l-1}) \xrightarrow{i_*} \widetilde{H}_q(S^k \times D^l) \longrightarrow \widetilde{H}_q((S^k \times D^l/S^k \times S^{l-1})$$
$$\xrightarrow{\partial} \widetilde{H}_{q-1}(S^k \times S^{l-1}) \xrightarrow{i_*} \widetilde{H}_{q-1}(S^k \times D^l) \longrightarrow$$

Breaking this up we obtain the short exact sequences

$$0 \longrightarrow (\operatorname{coker} i_*)_q \longrightarrow \widetilde{H}_q((S^k \times D^l/S^k \times S^{l-1}) \longrightarrow (\ker i_*)_{q-1} \longrightarrow 0,$$

where we write $(\ker i_*)_q$ (resp. $(\operatorname{coker} i_*)_q$) for kernel resp. cokernel of the homomorphism

$$i_*\colon \widetilde{H}_q(S^k \times S^{l-1}) \xrightarrow{i_*} \widetilde{H}_q(S^k \times D^l).$$

We claim that $i_*$ is surjective. To show this, we pick a point $y_0 \in S^{l-1}$ and consider the map

$$s\colon S^k \times D^l \longrightarrow S^k \times S^{l-1} \qquad (x, y) \mapsto (x, y_0).$$

This map is a *right inverse to $i$ up to homotopy*, meaning that the composition $i \circ s\colon S^k \times D^l \to S^k \times D^l$ is homotopic to the identity. This follows from the fact that $i \circ s$ is the product of the identity map on $S^k$ times the constant map on $D^l$ and the constant map on the contractible space $D^l$ is homotopic to the identity map. We conclude that $i_* \circ s_* = (i \circ s)_* = \mathbb{1}$, and so $s_*$ is a right inverse of $i_*$. This implies that $i_*$ is surjective, hence the cokernel of $i_*$ is zero and we obtain an isomorphism

$$\widetilde{H}_q(S^k \times D^l / S^k \times S^{l-1}) \cong (\ker i_*)_{q-1} \tag{10}$$

for all $q$.

The space $S^k \times S^{l-1}$ with its standard CW structure has four cells of dimension $0$, $k$, $l-1$ and $k+l-1$, respectively. Each of these cells contributes a copy of $\mathbb{Z}$ to the homology group in the appropriate dimension (since the boundary maps in the cellular chain complex are zero). Passing from $H_*$ to $\widetilde{H}_*$ we loose a copy of $\mathbb{Z}$ in degree 0 and passing to the kernel of $i_*$, we loose a $\mathbb{Z}$ in degree $k$. We conclude that

$$(\ker i_*)_q \cong \begin{cases} \mathbb{Z} & q = l-1, k+l-1 \\ 0 & q \neq l-1, k+l-1 \end{cases}$$

which implies the desired statement by putting it together with the isomorphism (10). $\square$

2. Suppose $M$ is a compact manifold of dimension $n$, and suppose $f\colon S^k \times D^{n-k} \to M$ is an embedding (i.e., a homeomorphism onto its image). Let $M' := M \setminus f(S^k \times \operatorname{int}(D^{n-k}))$, and let

$$\widehat{M} := M' \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1};$$

in other words, $\widehat{M}$ is obtained from the disjoint union

$$M' \amalg D^{k+1} \times S^{n-k-1}$$

by identifying a point $(x, y) \in S^k \times S^{n-k-1} \subset D^{k+1} \times S^{n-k-1}$ with $f(x, y) \in M'$. It is not hard to show that $\widehat{M}$ is again a closed $n$-manifold. This is an important way to modify a manifold $M$ called *surgery*. More precisely we say that *$\widehat{M}$ is obtained by a $k$-surgery* from $M$. The effect on homology groups is easiest to determine for $k < m$ for $n = 2m$ or $n = 2m + 1$ ($m$ is called the 'middle dimension of $M$' and consequently this is called 'surgery below the middle dimension'). So let us assume $k < m$ for the following.

(a) Just drawing pictures, identify the surface obtained by doing a 0-surgery (resp. a 1-surgery) on the torus.

(b) Show that the inclusion $M' \to M$ induces an isomorphism on $H_q$ for $q < m$. Hint: consider the long exact sequence of the pair $(M, M')$, show that this is a good pair, and identify $M/M'$.

(c) Show that the inclusion $M' \to \widehat{M}$ induces an isomorphism on $H_q$ for $q < n-1$, $q \neq k, k+1$. Hint: consider the long exact sequence of the pair $(\widehat{M}, M')$, show that this is a good pair, and identify $\widehat{M}/M'$.

(d) Let $g$ be the composition $\mathbb{Z} \cong H_k(S^k) \cong H_k(S^k \times D^{n-k}) \xrightarrow{f_*} H_k(M)$. Show that $H_k(\widehat{M}) \cong H_k(M)/g(\mathbb{Z})$ and

$$H_{k+1}(\widehat{M}) = \begin{cases} H_{k+1}(M) & g \text{ injective} \\ H_{k+1}(M) \oplus \mathbb{Z} & g \text{ not injective} \end{cases}$$

Hint: in order to identify the boundary homomorphisms in the exact sequence of the pair $(\widehat{M}, M')$, compare with the long exact sequence of the pair $(D^{k+1}, S^k)$.

*Proof.* (a) is easy, but drawing pictures in TEX is not, so I won't bother to give a solution.

To prove part (b) we note that the embedding $f$ induces a homeomorphism

$$(S^k \times D^{n-k})/(S^k \times S^{n-k-1}) \approx M/M'$$

The argument that $(M, M')$ is a good pair is completely analogous to the argument we used in problem 2: $M \setminus f(S^k \times \{0\})$ is an open neighborhood of $M'$ that has $M'$ as a deformation retract. Using these facts, the exact homology sequence of the pair $(M, M')$ takes the following form:

$$\xrightarrow{\partial} \widetilde{H}_q(M') \xrightarrow{i_*} \widetilde{H}_q(M) \longrightarrow \widetilde{H}_q(S^k \times D^{n-k}/S^k \times S^{n-k-1}) \xrightarrow{\partial}$$

By the previous problem, $\widetilde{H}_q(S^k \times D^{n-k}/S^k \times S^{n-k-1}) = 0$ except for $q = n-k, n$. Hence the exact sequence implies that $i_*$ is an isomorphism except for $q = n-k-1, n-k, n-1, n$. We note that our assumptions imply

$$n - k - 1 \geq 2m - k - 1 \geq 2m - (m-1) - 1 = m,$$

and so for $q < m$ the map $i_*$ is an isomorphism as claimed.

On to part (c): the argument that $(\widehat{M}, M')$ is a good pair is the same as that for $(M, M')$. We have a homeomorphism

$$\widehat{M}/M' \approx (D^{k+1} \times S^{n-k-1})/(S^k \times S^{n-k-1})$$

which using the result of problem (2) implies that $\widetilde{H}_q(\widehat{M}/M') = 0$ for $q \neq k+1, n$. Let $j\colon M' \to \widehat{M}$ be the inclusion map and consider the exact homology sequence of the pair $(\widehat{M}, M')$:

$$\longrightarrow \widetilde{H}_{q+1}(\widehat{M}/M') \xrightarrow{\partial} \widetilde{H}_q(M') \xrightarrow{j_*} \widetilde{H}_q(\widehat{M}) \longrightarrow \widetilde{H}_q(\widehat{M}/M) \longrightarrow$$

It follows that $j_*$ is an isomorphism for $q < n$, $q \neq k, k+1$.

To do part (d) we note that the exact sequence above implies an isomorphism

$$\widetilde{H}_k(\widehat{M}) \cong \operatorname{coker}\Big(\partial\colon \widetilde{H}_{k+1}(\widehat{M}/M') \to \widetilde{H}_k(M')\Big)$$

and a short exact sequence

$$\widetilde{H}_{k+1}(M') \longrightarrow \widetilde{H}_{k+1}(\widehat{M}) \longrightarrow \ker\Big(\widetilde{H}_{k+1}(\widehat{M}/M') \xrightarrow{\partial} \widetilde{H}_k(M')\Big) \tag{11}$$

So we need to analyze the connecting homomorphism $\partial$. As usual in algebraic topology, we don't do that by going back to the definition of the connecting homomorphism (coming from the short exact sequence of chain complexes), but rather by comparison with the exact sequence of a different pairs. We pick a point $y_0 \in S^{n-k-1}$, and consider the following maps of pairs:

$$(D^{k+1}, S^k) \xrightarrow{r} (D^{k+1} \times S^{n-k-1}, S^k \times S^{n-k-1}) \xrightarrow{s} (\widehat{M}, M') \,.$$

Here $r$ maps $x \in D^{k+1}$ to $(x, y_0) \in D^{k+1} \times S^{n-k-1}$ and $s$ is the obvious inclusion map which induces the homeomorphism between quotient spaces we used above. In particular,

$$s_*\colon \widetilde{H}_q(D^{k+1} \times S^{n-k-1}/S^k \times S^{n-k-1}) \longrightarrow \widetilde{H}_q(\widehat{M}/M')$$

is an isomorphism. We note that the map $r$ has a left-inverse given by the projection onto the first factor. In other words, $p \circ r = \mathbb{1}$ and hence $r_* \circ h_*$ is the identity on $\widetilde{H}_{k+1}(D^{k+1}/S^k)$. Since

$$\widetilde{H}_{k+1}(D^{k+1} \times S^{n-k-1}/S^k \times S^{n-k-1}) \cong \mathbb{Z}$$

by problem # 2, it follows that

$$r_*\colon H_{k+1}(D^{k+1}/S^k) \longrightarrow \widetilde{H}_{k+1}(D^{k+1} \times S^{n-k-1}/S^k \times S^{n-k-1})$$

is an isomorphism. Putting these statements together, we see that the composition $t := s \circ r\colon (D^{k+1}, S^k) \to (\widehat{M}, M')$ induces an isomorphism $t_*\colon \widetilde{H}_{k+1}(D^{k+1}/S^k) \to \widetilde{H}_{k+1}(\widehat{M}/M')$.

Now consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{H}_{k+1}(D^{k+1}/S^k) \arrow[r, "\partial", "\cong"'] \arrow[d, "t_*"', "\cong"] & \widetilde{H}_k(S^k) \arrow[d, "t_*"] \\
\widetilde{H}_{k+1}(\widehat{M}/M') \arrow[r, "\partial"] & \widetilde{H}_k(M') \arrow[d, "i_*"', "\cong"] \\
 & \widetilde{H}_k(M)
\end{tikzcd}$$

We note that the map $i \circ t\colon S^n \to M$ is equal to the restriction of $f\colon S^n \times D^{n-k}$ to $S^n \times \{y_0\}$, and hence $i_* \circ t_*$ can be identified with $g\colon \mathbb{Z} \to H_k(M)$. The diagram above then implies that the kernel resp. cokernel of $\partial$ can be identified with the kernel resp. cokernel of $g$. This implies the statement about $H_k(\widehat{M})$ and the statement about $H_{k+1}(\widehat{M})$ if $g$ is injective. If $g$ is not injective, the kernel of $g$ is a non-trivial subgroup of $\widehat{H}_{k+1}(\widehat{M}/M') \cong \mathbb{Z}$ and hence isomorphic to $\mathbb{Z}$. This implies that the exact sequence (11) splits and hence we have the isomorphisms

$$\widetilde{H}_{k+1}(\widehat{M}) \cong \widetilde{H}_{k+1}(M') \oplus \mathbb{Z} \cong \widetilde{H}_{k+1}(M) \oplus \mathbb{Z}$$

□

## 10. HOMEWORK ASSIGNMENT # 10

1. Show $\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$ for cochains $\varphi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$.

*Proof.* For $\sigma\colon \Delta^{k+l+1} \to X$ we have

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{k+1}])\psi(\sigma \circ [e_k, \dots, v_{k+\ell+1}])$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma \circ [e_0, \dots, e_k])\psi(\sigma \circ [e_k, \dots, \widehat{e_i}, \dots, v_{k+\ell+1}])$$

Adding these two expressions, the last term of the first sum cancels the first term of the second sum, and the remaining terms are exactly

$$\delta(\varphi \cup \psi) = (\varphi \cup \psi)(\partial\sigma)$$

since

$$\partial\sigma = \sum_{i=0}^{k+\ell+1} (-1)^i \sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{k+\ell+1}].$$

□

2. Show that the cup product is compatible with pull-back of cohomology classes in the sense that for a map $f\colon X \to Y$ and cohomology classes $\alpha \in H^k(Y; R)$, $\beta \in H^l(Y; R)$ we have

$$f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta).$$

Hint: Show first the analogous statement for cochains.

*Proof.* Let us write $f^{\#}\colon C^*(Y; R) \to C^*(X; R)$ for the cochain map induced by $f$; i.e.,

$$(f^{\#}\varphi)(\sigma) = \varphi(f \circ \sigma) \qquad \text{for } \varphi \in C^q(X; R),\ \sigma\colon \Delta^q \to X$$

Then for $\varphi \in C^k(Y; R)$, $\psi \in C^\ell(Y; R)$ and $\sigma\colon \Delta^{k+\ell} \to X$ we have

$$\begin{aligned}(f^{\#}(\varphi \cup \psi))(\sigma) &= (\varphi \cup \psi)(f \circ \sigma)\\ &= \varphi(f \circ \sigma \circ [e_0 \dots, e_k])\psi(f \circ \sigma \circ [e_k, \dots, e_{k+\ell}])\\ &= (f^{\#}\varphi)(\sigma \circ [e_0 \dots, e_k])(f^{\#}\psi)(\sigma \circ [e_k, \dots, e_{k+\ell}])\\ &= (f^{\#}\varphi \cup f^{\#}\psi)(\sigma)\end{aligned}$$

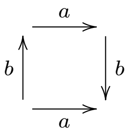
If the cohomology classes $\alpha$, $\beta$ are represented by the cocycles $\varphi$ resp. $\psi$, we obtain

$$\begin{aligned}f^*(\alpha \cup \beta) = f^*([\varphi \cup \psi)]) &= [f^{\#}(\varphi \cup \psi)]\\ &= [f^{\#}\varphi \cup f^{\#}\psi] = [f^{\#}\varphi] \cup [f^{\#}\psi]\\ &= f^*\alpha \cup f^*\beta\end{aligned}$$

□

3. (a) Use the cellular chain complex of the Klein bottle $K$ to determine its cohomology groups $H^q(K; \mathbb{Z}/2)$.
(b) Determine the cup products on cohomology with $\mathbb{Z}/2$ coefficients. Hint: proceed similarly to what we did in class for determining the cup products for the cohomology of the connected sum of two tori.

*Proof.* Our standard picture for the Kleinbottle



shows that $K$ has a CW structure with one 0-cell (given by the common vertex $v$), two 1-cells (given by the edges $a$ and $b$) and one 2-cell (given by the square $f$). Hence the cellular chain complex of $K$ has the form

$$C_0^{CW}(K) = \mathbb{Z}v \xleftarrow{\partial} C_1^{CW}(K) = \mathbb{Z}a \oplus \mathbb{Z}b \xleftarrow{\partial} C_2^{CW}(K) = \mathbb{Z}f$$

Looking at the degrees of the attaching maps we see that $\partial f = 2b$ and $\partial a = \partial b = 0$. Hence the cellular cochain complex $C^*_{CW}(K;\mathbb{Z}/2) = \mathrm{Hom}(C_*^{CW}(K);\mathbb{Z}/2)$ has the form

$$C^0_{CW}(K) = \mathbb{Z}/2v^* \xrightarrow{\delta} C^1_{CW}(K) = \mathbb{Z}/2a^* \oplus \mathbb{Z}/2b^* \xrightarrow{\delta} C^2_{CW}(K) = \mathbb{Z}/2f^*$$

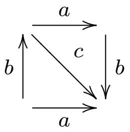
Here $v^*$, $a^*$, $b^*$, $f^*$ are the elements dual to $v, a, b, f$ (i.e., $a^*(a) = 1 \in \mathbb{Z}/2$, $a^*(b) = 0$, etc). We note that the coboudary maps $\delta$ are trivial. This is clear for $\delta\colon C^0_{CW} \to C^1_{CW}$ since it is dual to $\partial\colon C_1^{CW} \to C_0^{CW}$ which is zero. Concerning $\delta\colon C^1_{CW} \to C^2_{CW}$ we have

$$\begin{aligned}\delta a^*(f) &= a^*(\partial f) = a^*(2b) = 0\\ \delta b^*(f) &= b^*(\partial f) = b^*(2b) = 2 = 0 \in \mathbb{Z}/2\end{aligned}$$

It follows that

$$H^q(K;\mathbb{Z}/2) = H^q(C^*_{CW}(K;\mathbb{Z}/2) = \begin{cases}\mathbb{Z}/2 & q = 0,2\\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & q = 1\\ 0 & \text{otherwise}\end{cases}$$

To calculate the cup products for part (b), we choose a sub chain complex $C_*(K)'$ of the singular chain comples $C_*(K)$ to consist of the 0-simplices, 1-simplices and 2-simplices indicated in the picture



Here we use the same convention as in class: the arrows allow us to identify the edges with affine 1-simplices in the square; projection from the square to the Klein bottle then gives us singular simplices $a$, $b$ in $K$. The arrows allow us to identify triangles with affine 2-simplices (the 0-th vertex of a triangle is the vertex from which two edges are emanating, the 1-th vertex has one edge coming in and one edge going out and the 2-th vertex has two incoming edges). So we get two singular 2-simplices in $K$, say $\sigma_1$ (corresponding to the top right triangle) and $\sigma_2$ (corresponding to the bottom left triangle). In particular:

| | |
|---|---|
| 1-front face of $\sigma_1 = a$ | 1-back face of $\sigma_1 = b$ |
| 1-front face of $\sigma_2 = b$ | 1-back face of $\sigma_2 = c$ |

We note that $\sigma_1 + \sigma_2$ is a cycle, and that given cochains $\varphi, \psi \in C^1(K;\mathbb{Z}/2)' := \mathrm{Hom}(C_1(K)', \mathbb{Z}/2)$ we have

$$(\varphi \cup \psi)(\sigma_1 + \sigma_2) = \varphi(a)\psi(b) + \varphi(b)\psi(c)$$

Now we define cochains $\alpha, \beta \in C^1(K; \mathbb{Z}/2)'$ by

$$\begin{aligned} \alpha(a) &= 1 & \alpha(b) &= 0 & \alpha(c) &= 1 \\ \beta(a) &= 0 & \beta(b) &= 1 & \beta(c) &= 1 \end{aligned}$$

We note that $\alpha$ and $\beta$ are cocycles, since

$$(\delta\alpha)(\sigma_1) = \alpha(\partial\sigma_i) = \alpha(a + b + c) = 0 \in \mathbb{Z}/2$$

and similarly for $\beta$. Moreover, $\alpha$, $\beta$ form a basis for $H^1(K; \mathbb{Z}/2)$ since they are dual to the basis $\{a, b\}$ of $H_2(K; \mathbb{Z})$. We calculate:

$$\begin{aligned} (\alpha \cup \alpha)(\sigma_1 + \sigma_2) &= 1 \cdot 0 + 0 \cdot 1 = 0 \\ (\beta \cup \beta)(\sigma_1 + \sigma_2) &= 0 \cdot 1 + 1 \cdot 1 = 1 \\ (\alpha \cup \beta)(\sigma_1 + \sigma_2) &= 1 \cdot 1 + 0 \cdot 1 = 1 \\ (\beta \cup \alpha)(\sigma_1 + \sigma_2) &= 0 \cdot 0 + 1 \cdot 1 = 1 \end{aligned}$$

This implies that $\beta \cup \beta = \alpha \cup \beta = \beta \cup \alpha$ is the non-zero element of $H^2(K; \mathbb{Z}/2) \cong \mathbb{Z}/2$. In particular, the cycle $\sigma_1 + \sigma_2$ represents a generator of $H_2(K)$ and hence the calculation above shows that $\alpha \cup \alpha = 0$. ☐

4. The real projective space $\mathbb{RP}^n$, $n = k + l$ contains the disjoint subspaces

$$\begin{aligned} \mathbb{RP}^{k-1} &:= \{[x_0, \dots, x_{k-1}, 0, \dots, 0] \in \mathbb{RP}^n\} \\ \mathbb{RP}^l &:= \{[0, \dots, 0, x_k, \dots, x_n] \in \mathbb{RP}^n\} \end{aligned}$$

(a) Show that $\mathbb{RP}^{k-1}$ is a deformation retract of $\mathbb{RP}^n \setminus \mathbb{RP}^l$.
(b) Show that the inclusion map $(\mathbb{RP}^n, \mathbb{RP}^{k-1}) \to (\mathbb{RP}^n, \mathbb{RP}^n \setminus \mathbb{RP}^l)$ induces an isomorphism on cohomology with any coefficients $R$.

*Proof.* To prove part (a), it will be convenient to think of $\mathbb{RP}^n$ as the quotient $\mathbb{R}^{n+1} \setminus \{0\}/\sim$ where vectors $\vec{x}, \vec{y} \in \mathbb{R}^{n+1} \setminus \{0\}$ are equivalent if and only ifthere is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $y = \lambda x$. We'll write $[x_0, \dots, x_n] \in \mathbb{RP}^n$ for the equivalence class of $\vec{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. Consider the map

$$\begin{aligned} H\colon (\mathbb{RP}^n \setminus \mathbb{RP}^\ell) \times [0,1] &\longrightarrow \mathbb{RP}^n \setminus \mathbb{RP}^\ell \\ ([x_0, \dots, x_n], t) &\mapsto [x_0, \dots, x_{k-1}, tx_k, \dots, tx_n] \end{aligned}$$

It is clear that this map is well-defined and continuous (since the corresponding map before passing to the quotient spaces is continuous). Moreover, restricting to $t = 0$, the image is contained in $\mathbb{RP}^{k-1}$; in other words, we obtain a retraction map $r\colon \mathbb{RP}^n \setminus \mathbb{RP}^\ell \to \mathbb{RP}^{k-1}$ whose restriction to $\mathbb{RP}^{k-1}$ is the identity. The map $H$ above then provides a homotopy between the identity map on $\mathbb{RP}^n \setminus \mathbb{RP}^\ell$ (for $t = 1$) and the composition $i \circ r$ of the retraction map and the inclusion $i\colon \mathbb{RP}^{k-1} \to \mathbb{RP}^n \setminus \mathbb{RP}^\ell$ (for $t = 0$).

To prove part (b) we note that the inclusion map of pairs

$$i\colon (P^n, P^{k-1}) \longrightarrow (P^n, P^n \setminus P^\ell)$$

induces a commutative diagram whose rows are the long exact cohomology sequences (with coefficients in $R$) of the pairs involved:

$$\begin{tikzcd}
H^q(P^n \setminus P^\ell) \arrow[d, "i^*"] & H^q(P^n) \arrow[l] \arrow[d, equal] & H^q(P^n, P^n \setminus P^\ell) \arrow[l] \arrow[d, "i^*"] & H^{q-1}(P^n \setminus P^\ell) \arrow[l, "\delta"'] \arrow[d, "i^*"] & H^{q-1}(P^n) \arrow[l] \arrow[d, equal] \\
H^q(P^{k-1}) & H^q(P^n) \arrow[l] & H^q(P^n, P^{k-1}) \arrow[l] & H^{q-1}(P^{k-1}) \arrow[l, "\delta"'] & H^{q-1}(P^n) \arrow[l]
\end{tikzcd}$$

Part (a) implies that the inclusion map $i\colon P^{k-1} \hookrightarrow P^n \setminus P^{k-1}$ induces an isomorphism on cohomology. The the 5-lemma then implies that $i^*\colon H^q(P^n, P^n \setminus P^\ell) \to H^q(P^n, P^{k-1})$ is an isomorphism. $\square$