

**Name:**

**Take Home Exam, March 3, 2010**

You can use any resources for this test (e.g., books, notes, internet) with the exception of fellow mathematicians: talking to anybody except me about this exam before you turn your exam in constitutes a violation of the honor code. Please return the exam to me in class on Friday, March 5. I'll be around most of the day on Thursday. Good Luck!

1. (10 points) Suppose the following diagram of abelian groups and group homomorphisms is commutative with exact rows:

$$\begin{array}{ccccccccc}
 \longrightarrow & C_{q+1} & \xrightarrow{\partial_{q+1}} & A_q & \xrightarrow{f_q} & B_q & \xrightarrow{g_q} & C_q & \xrightarrow{\partial_q} & A_{q-1} & \longrightarrow \\
 & c_{q+1} \downarrow & & a_q \downarrow & & b_q \downarrow & & c_q \downarrow & & a_{q-1} \downarrow & \\
 \longrightarrow & C'_{q+1} & \xrightarrow{\partial'_{q+1}} & A'_q & \xrightarrow{f'_q} & B'_q & \xrightarrow{g'_q} & C'_q & \xrightarrow{\partial'_q} & A'_{q-1} & \longrightarrow
 \end{array}$$

Assuming in addition that the maps  $c_q$  are isomorphisms show that there is a long exact sequence of the form

$$\longrightarrow A_q \longrightarrow A'_q \oplus B_q \longrightarrow B'_q \longrightarrow A_{q-1} \longrightarrow A'_{q-1} \oplus B_{q-1} \longrightarrow B'_{q-1} \longrightarrow$$

First define carefully the homomorphisms in the above sequence. Then prove exactness at each location.

*Proof.* We define the maps in the above sequence as follows:

$$\begin{array}{ll}
 \alpha_q: A_q \longrightarrow A'_q \oplus B_q & a \mapsto (a_q(a), f_q(a)) \\
 \beta_q: A'_q \oplus B_q \longrightarrow B'_q & (a', b) \mapsto f'_q(a') - b_q(b) \\
 \gamma_q: B'_q \longrightarrow A_{q-1} & b' \mapsto \partial_q c_q^{-1} g'_q(b')
 \end{array}$$

**Exactness at  $B'_q$ .** First we show  $\gamma_q \beta_q = 0$ . For  $(a', b) \in A'_q \oplus B_q$  we have

$$\gamma_q \beta_q(a', b) = \partial_q c_q^{-1} g'_q(f'_q a' - b_q b) = -\partial_q c_q^{-1} g'_q b_q b = \partial_q g_q b = 0$$

Here the second equality holds due to  $g'_q f'_q = 0$ , the third follows from the commutativity of the third square, and the last is due to  $\partial_q g_q = 0$ .

To show  $\ker \gamma_q \subset \text{im } \beta_q$  let  $b' \in B'_q$  with  $\gamma_q b' = \partial_q c_q^{-1} g'_q b' = 0$ . By exactness at  $C_q$  there is an element  $b \in B_q$  such that  $g_q b = c_q^{-1} g'_q b'$  or equivalently

$$g'_q b' = c_q g_q b = g'_q b_q b,$$

where the second equality follows from commutativity of the third square. It follows that  $b' - b_q b$  is in the kernel of  $g'_q$  and hence by exactness at  $B'_q$ , there is an element  $a' \in A'_q$  with  $f'_q a' = b' - b_q b$ . This implies

$$\beta_q(a', -b) = f'_q a' + b_q b = b'$$

which shows that  $b'$  is in the image of  $\beta_q$ .

**Exactness at  $A'_q \oplus B_q$ .** First we show  $\beta_q \alpha_q = 0$ . For  $a \in A_q$  we have

$$\beta_q \alpha_q a = \beta_q(a_q a, f_q a) = f'_q a_q a - b_q f_q a = 0$$

due to the commutativity of the second square.

To show  $\ker \beta_q \subset \text{im } \alpha_q$ , let  $(a', b) \in A'_q \oplus B_q$  with

$$\beta_q(a', b) = f'_q a' - b_q b = 0.$$

Then we have

$$c_q g_q b = g'_q b_q b = g'_q f'_q a' = 0,$$

where the first equality is due to the commutativity of the third square, and the last is due to exactness at  $B'_q$ . Since  $c_q$  is an isomorphism, this implies  $g_q b = 0$  and hence by exactness at  $B_q$ , there is an element  $a \in A_q$  with  $f_q a = b$ . If we could show  $a_q a = a'$ , we would be done. However we can only say the following:

$$f'_q(a_q a - a') = f'_q a_q a - f'_q a' = b_q f_q a - b_q b = 0$$

where the second equality follows from the commutativity of the second square and our assumption  $f'_q a' = b_q b$ . Since  $f'_q$  is not necessarily injective, we can't conclude that  $a_q a = a'$ , but thanks to exactness at  $A'_q$ , it implies that there is an element  $c' \in C'_{q+1}$  with  $\partial_{q+1} c' = a_q a - a'$ . Moreover, since  $c_{q+1}$  is an isomorphism, there is a  $c \in C_{q+1}$  with  $c_{q+1} c = c'$ . Now we modify the element  $a \in A_q$  by defining  $\bar{a} \stackrel{\text{def}}{=} a - \partial_{q+1} c$ . We calculate

$$\begin{aligned} f_q \bar{a} &= f_q(a - \partial_{q+1} c) = f_q a = b \\ a_q \bar{a} &= a_q(a - \partial_{q+1} c) = a_q a - \partial'_{q+1} c_{q+1} c = a_q a - (a_q a - a') = a' \end{aligned}$$

This shows that  $\alpha_q(\bar{a}) = (a', b)$  as desired.

**Exactness at  $A_q$ .** First let us show  $\alpha_q \circ \gamma_{q+1} = 0$ . For  $b' \in B'_{q+1}$  we have

$$\begin{aligned}\alpha_q \gamma_{q+1} b' &= \alpha_q(\partial_{q+1} c_{q+1}^{-1} g'_{q+1} b') \\ &= (a_q \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b', f_q \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b') \\ &= (\partial'_{q+1} g'_{q+1} b', 0) = (0, 0)\end{aligned}$$

since the compositions  $f_q \partial_{q+1}$  and  $\partial'_{q+1} g'_{q+1}$  are zero due to the exactness at  $A_q$  resp.  $A'_q$ .

To show  $\ker \alpha_q \subset \text{im } \gamma_{q+1}$ , let  $a \in A_q$  with  $\alpha_q a = (a_q a, f_q a) = (0, 0)$ . By exactness at  $A_q$  there is an element  $c \in C_{q+1}$  with  $\partial_{q+1} c = a$ . Then

$$\partial'_{q+1} c_{q+1} c = a_q \partial_{q+1} c = a_q a = 0$$

and hence by exactness at  $C'_{q+1}$ , there is an element  $b' \in B'_{q+1}$  with  $g'_{q+1} b' = c_{q+1} c$ . This implies

$$\gamma_{q+1} b' = \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b' = \partial_{q+1} c = a,$$

which shows that  $a$  is in the image of  $\gamma_{q+1}$ .

□

2. (10 points) Let  $X$  be a topological space which is the union of two open subsets  $U, V$ . Let  $h_n(\ )$  be a generalized homology theory (i.e., a family of functors satisfying all the Eilenberg-Steenrod axioms except possibly the dimension axiom). Show that the homology groups of  $X, U, V$  and  $U \cap V$  fit into a long exact sequence (called the Mayer-Vietoris sequence)

$$\rightarrow h_n(U \cap V) \rightarrow h_n(U) \oplus h_n(V) \rightarrow h_n(X) \rightarrow h_{n-1}(U \cap V) \rightarrow .$$

Hint: apply Problem 1 to the long exact sequences of the pairs  $(V, U \cap V)$  and  $(X, U)$ .

*Proof.* Consider the following diagram

$$\begin{array}{ccccccccc} \rightarrow & H_{q+1}(V, U \cap V) & \xrightarrow{\partial} & H_q(U \cap V) & \longrightarrow & H_q(V) & \longrightarrow & H_q(V, U \cap V) & \xrightarrow{\partial} & H_{q-1}(U \cap V) & \rightarrow \\ & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & \\ \rightarrow & H_{q+1}(X, U) & \xrightarrow{\partial} & H_q(U) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X, U) & \xrightarrow{\partial} & H_{q-1}(U) & \rightarrow \end{array}$$

The rows of this diagram are the long exact homology sequences of the pairs  $(V, U \cap V)$  and  $(X, U)$ , respectively. The vertical maps are induced by the inclusion map  $i: (V, U \cap V) \rightarrow (X, U)$ . The Second and third square commute by functoriality of the homology groups. The first and fourth square commute since the boundary homomorphism in the long exact homology sequence of a pair is a natural transformation. We note that the pair  $(V, U \cap V)$  is obtained from  $(X, U)$  by excising  $Z = X \setminus V$ . We have

$$\bar{Z} = Z \hookrightarrow U = \int(U)$$

since  $U$  is open, and  $Z$  is closed (as the complement of the open subset  $V$ ), and hence excision implies that the indicated vertical maps are isomorphisms. Hence we can apply the statement of problem 1 to deduce the exact Mayer-Vietoris sequence.  $\square$

3. (10 points) We say that the homology groups of a space  $X$  are *finitely generated and bounded* if  $H_q(X)$  is a finitely generated  $\mathbb{Z}$ -modules for every  $q$  and if there is some  $N$  such that  $H_q(X) = 0$  for  $q > N$ . This condition guarantees that the Euler characteristic  $\chi(X) := \sum_q (-1)^q \text{rk } H_q(X)$  can be defined.

- (a) Assume that  $X$  is the union of two open subsets  $U, V$  such that  $U, V$  and  $U \cap V$  have finitely generated bounded homology groups. Show that then also  $X$  has finitely generated bounded homology groups.
- (b) Assuming the conditions of part (a) show that  $\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$ .

Hint: Use the Mayer-Vietoris sequence.

*Proof.* Consider the Mayer-Vietoris sequence

$$\longrightarrow H_q(U \cap V) \longrightarrow H_q(U) \oplus H_q(V) \longrightarrow H_q(X) \longrightarrow H_{q-1}(U \cap V) \longrightarrow (1)$$

First we observe that since  $U, V,$  and  $U \cap V$  are of bounded finite type, there is some  $N$  such that for  $q \geq N$  the homology groups  $H_q(U), H_q(V), H_{q-1}(U \cap V)$  are all zero. By the exactness of the sequence above, this implies  $H_q(X) = 0$  for  $q \geq N$ .

To prove that all homology groups  $H_q(X)$  are finitely generated, let us write  $f_q$  for the map from  $H_q(U \cap V)$  to  $H_q(U) \oplus H_q(V)$ . Then the exact sequence (1) gives rise to short exact sequences of the form

$$0 \longrightarrow \operatorname{coker}(f_q) \longrightarrow H_q(X) \longrightarrow \ker(f_{q-1}) \longrightarrow 0.$$

We note that  $\ker(f_{q-1})$  is finitely generated (as a subgroup of the finitely generated abelian group  $H_{q-1}(U \cap V)$ ) and  $\operatorname{coker}(f_q)$  is finitely generated (as quotient of the finitely generated group  $H_q(U) \oplus H_q(V)$ ). This implies by the short exact sequence above that  $H_q(X)$  is finitely generated.

To prove part (b) we recall that in class we showed that if  $C_*$  is a chain complex of bounded finite type (i.e., all the groups  $U_q$  are finitely generated, and only finitely many of them are non-zero), then

$$\sum_{q \in \mathbb{Z}} (-1)^q \operatorname{rk} C_q = \sum_{q \in \mathbb{Z}} (-1)^q \operatorname{rk} H_q(C_*).$$

In particular, we see that if  $C_*$  is an *exact sequence*, then all homology of  $C_*$  vanish and hence the alternating sum  $\sum_{q \in \mathbb{Z}} (-1)^q \operatorname{rk} C_q$  is zero.

Now we apply this to the Mayer-Vietoris sequence (1). The alternating sum here is equal to

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} (-1)^q (\operatorname{rk} H_q(U \cap V) - \operatorname{rk} H_q(U) - \operatorname{rk} H_q(V) + \operatorname{rk} H_q(X)) \\ &= \chi(U \cap V) - \chi(U) - \chi(V) + \chi(X) \end{aligned}$$

Since this alternating sum is zero due to the observation above, the desired formula for  $\chi(X)$  follows.  $\square$

4. (10 points) Compute  $H_q(\mathbb{R}P^n/\mathbb{R}P^m)$  for  $m < n$  by cellular homology (Hint: note that the standard CW structure on  $\mathbb{R}P^n$  induces a CW structure on  $\mathbb{R}P^n/\mathbb{R}P^m$ ).

Since  $\mathbb{R}P^k$  is obtained from  $\mathbb{R}P^{k-1}$  by attaching a  $k$ -cell, for  $m < k$  the quotient space  $\mathbb{R}P^k/\mathbb{R}P^m$  is obtained from  $\mathbb{R}P^{k-1}/\mathbb{R}P^m$  by attaching a  $k$ -cell. Hence  $\mathbb{R}P^n/\mathbb{R}P^m$  is a CW complex with one 0-cell and one  $k$ -cell for every  $k$  satisfying  $m < k \leq n$ . Moreover the projection map  $p: \mathbb{R}P^n \rightarrow \mathbb{R}P^n/\mathbb{R}P^m$  preserves the skeletons (i.e., it maps each  $k$ -skeleton of  $\mathbb{R}P^n$  to the  $k$ -skeleton of  $\mathbb{R}P^n/\mathbb{R}P^m$ ). Hence  $p$  induces a chain map

$$p_*: C_*^{CW}(\mathbb{R}P^n) \longrightarrow C_*^{CW}(\mathbb{R}P^n/\mathbb{R}P^m)$$

which is an isomorphism on  $C_k^{CW}(-)$  for  $k = 0$  and  $m < k \leq n$ . It follows that the cellular chain complex for  $\mathbb{R}P^n/\mathbb{R}P^m$  is obtained from the cellular chain complex by replacing the terms  $C_k^{CW}(\mathbb{R}P^n)$  for  $0 < k \leq m$  by 0 (the analogous statement holds for *any* connected CW complex  $X$ ). We recall that the cellular chain complex of  $\mathbb{R}P^n$  looks as follows

$$C_0 = \mathbb{Z} \xleftarrow{0} C_1 = \mathbb{Z} \xleftarrow{2} C_2 = \mathbb{Z} \xleftarrow{0} C_3 = \mathbb{Z} \xleftarrow{2} \dots$$

Hence the first interesting portion of  $C_*^{CW}(\mathbb{R}P^n/\mathbb{R}P^m)$  has the form

$$\leftarrow C_m = 0 \leftarrow C_{m+1} = \mathbb{Z} \xleftarrow{\partial_{m+2}} C_{m+2} = \mathbb{Z} \leftarrow \dots$$

where  $\partial_{m+2}$  is multiplication by 2 if  $m$  is even and 0 if  $m$  is odd. It follows that

$$H_q(\mathbb{R}P^n/\mathbb{R}P^m) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & 0 < q \leq m \\ \mathbb{Z}/2 & q = m + 1, m \text{ even} \\ \mathbb{Z} & q = m + 1, m \text{ odd} \\ H_q(\mathbb{R}P^n) & q > m + 1 \end{cases}$$

$$= \begin{cases} \mathbb{Z} & q = 0 \text{ or } q = m + 1, m \text{ odd, or } q = n \text{ and } n \text{ odd} \\ 0 & 0 < q \leq m, \text{ or } q > n, \text{ or } m + 1 \leq q < n, q \text{ even} \\ \mathbb{Z}/2 & m + 1 \leq q < n \text{ and } q \text{ odd} \end{cases}$$