CONTENTS

Homework Solutions

1. HOMEWORK ASSIGNMENT # 1

1. Show that for any pointed topological space (X, x_0), the n-homotopy group $\pi_n(X, x_0)$ is a group. More precisely, this involves showing the following:

associativity: Let f, g, and h be maps from $(I^n, \partial I^n)$ to (X, x_0) and let $f + g\colon (I^n, \partial I^n) \to (X, x_0)$ be defined by

$$(f+g)(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & 1 \le s_1 \le \frac{1}{2} \\ g(2s_1 - 1, s_2, \dots, s_n) & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

Then we need to show that $f+(g+h)$ is homotopic to $(f+g)+h$.

unit property: If $c\colon (I^n, \partial I^n) \to (X, x_0)$ is the constant map, we need to show $f + c \sim f$ and $c + f \sim f$.

inverse property: If $\bar{f}\colon (I^n, \partial I^n) \to (X, x_0)$ is given by $\bar{f}(s_1, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n)$, we need to show $\bar{f} + f \sim c$ and $f + \bar{f} \sim c$.

proof of associativity:

$$\begin{aligned}(f+(g+h))(s_1, \dots, s_n) &= \begin{cases} f(2s_1, s_2, \dots, s_n) & 1 \le s_1 \le \frac{1}{2} \\ (g+h)(2s_1 - 1, s_2, \dots, s_n) & \frac{1}{2} \le s_1 \le 1 \end{cases} \\ &= \begin{cases} f(2s_1, s_2, \dots, s_n) & 1 \le s_1 \le \frac{1}{2} \\ g(2(2s_1 - 1), s_2, \dots, s_n) & \frac{1}{2} \le s_1 \le \frac{3}{4} \\ h(2(2s_1 - 1) - 1, s_2, \dots, s_n) & \frac{3}{4} \le s_1 \le 1 \end{cases}\end{aligned}$$

$$((f+g)+h)(s_1,\dots,s_n) = \begin{cases} (f+g)(2s_1,s_2,\dots,s_n) & 1 \le s_1 \le \frac{1}{2} \\ h(2s_1-1,s_2,\dots,s_n) & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

$$= \begin{cases} f(4s_1,s_2,\dots,s_n) & 1 \le s_1 \le \frac{1}{4} \\ g(4s_1-1,s_2,\dots,s_n) & \frac{1}{4} \le s_1 \le \frac{1}{2} \\ h(2s_1-1,s_2,\dots,s_n) & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

So we see that these maps basically differ in how the interval $[0,1]$ is split up into three subintervals, each of which is mapped to X using f (resp. g resp. h). To relate these, it is convenient to compare both to the map $F\colon ([0,3],\partial[0,3]) \to (X,x_0)$ defined by 'evenly splitting up' the interval $[0,3]$ into three subintervals, and using f (resp. g resp. h) on each subinterval:

$$F(s_1,s_2,\dots,s_n) = \begin{cases} f(s_1,s_2,\dots,s_n) & 1 \le s_1 \le 2 \\ g(s_1-1,s_2,\dots,s_n) & 1 \le s_1 \le 2 \\ h(s_1-2,s_2,\dots,s_n) & 2 \le s_1 \le 3 \end{cases}$$

Then we can express $f+(g+h)$ and $(f+g)+h$ in terms of F as

$$\begin{aligned} (f+(g+h)((s_1,\dots,s_n) &= F(\phi_0(s_1),s_2,\dots,s_n) \\ ((f+g)+h)(s_1,\dots,s_n) &= F(\phi_1(s_1),s_2,\dots,s_n), \end{aligned}$$

where $\phi_0,\phi_1\colon [0,1] \to [0,3]$ are given by

$$\phi_0(s_1) = \begin{cases} 2s_1 & 0 \le s_1 \le \frac{1}{2} \\ 4s_1-1 & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

$$\phi_1(s_1) = \begin{cases} 4s_1 & 0 \le s_1 \le \frac{1}{2} \\ 2s_1+1 & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

Now we can use the 'linear homotopy' $\phi_t := (1-t)\phi_0 + t\phi_1$ between ϕ_0 and ϕ_1 to build the map $H\colon [0,1]\times I^n \to X$ defined by

$$H(t,s_1,\dots,s_n) := F(\phi_t(s_1),s_2,\dots,s_n).$$

This map is clearly continuous, and agrees with $f+(g+h)$ (resp. $(f+g)+h$ for $t=0$ (resp. $t=1$). We still need to check that we have $H(t,s)=x_0$ for $t\in[0,1]$, and $s=(s_1,\dots,s_n)\in\partial I^n$. We note that $s\in\partial I^n$ if and only if $s_i=0$ or $s_i=1$ for some $i=1,\dots,n$. If $s_1=0$ (resp. $s_1=1$), then $\phi_0(s_1)=\phi_1(s_1)=0$ (resp. $\phi_0(s_1)=\phi_1(s_1)=3$) and hence $\phi_t(s_1)=0$ (resp. $\phi_t(s_1)=3$) for all $t\in[0,1]$. It follows that $F(\phi_t(s_1),s_2,\dots,s_n)=x_0$. If $s_i=0,1$ for some $i=2,\dots,n$, then we see right away that

$F(\phi_t(s_1), s_2, \ldots, s_n) = x_0$. Summarizing, H is a homotopy between the maps $f+(g+h)$ and $(f+g)+h$ from the pair $(I^n, \partial I^n)$ to (X, x_0).

proof of the unit property: The proof is similar to the previous one. We note that

$$f(s) = f(\phi_0(s_1), s_2, \ldots, s_n) \qquad (f+c)(s) = f(\phi_1(s_1), s_2, \ldots, s_n),$$

where

$$\phi_0(s_1) = s_1 \qquad \phi_1(s_1) = \begin{cases} 2s_1 & 0 \le s_1 \frac{1}{2} \\ 1 & \frac{1}{2} \le s_1 \le 1 \end{cases}$$

Then $H\colon [0,1] \times I^n \to X$ defined by

$$H(t, s_1, \ldots, s_n) = f((1-t)\phi_0(s_1) + t\phi_1(s_1), s_2, \ldots, s_n)$$

is the desired homotopy (as above it is easy to check that $H(0,s) = f$, $H(1,s) = (f+c)(s)$, and that $H(t,s) = x_0$ for $s \in \partial I^n$). The argument for $c + f \sim f$ is analogous.

inverse property: We note that

$$\begin{aligned} (f + \bar{f})(s) &= f(\phi_0(s_1), s_2, \ldots, s_n) \\ c(s) &= f(\phi_1(s_1), s_2, \ldots, s_n), \end{aligned}$$

where

$$\phi_0(s_1) = \begin{cases} 2s_1 & 0 \le s_1 \le \frac{1}{2} \\ 2 - 2s_1 & \frac{1}{2} \le s_1 \le 1 \end{cases} \qquad \text{and} \qquad \phi_1(s_1) \equiv 0$$

As before, then $H\colon [0,1] \times I^n \to X$ given by

$$H(t, s_1, \ldots, s_n) = f((1-t)\phi_0(s_1) + t\phi_1(s_1), s_2, \ldots, s_n)$$

provides a homotopy between the maps $f+\bar{f}$ and c from $(I^n, \partial I^n)$ to (X, x_0).

2. HOMEWORK ASSIGNMENT # 2

1. Show that $\mathbb{Z}/pq$ is isomorphic to $\mathbb{Z}/p \oplus \mathbb{Z}/q$ if and only if p and q are relatively prime.

If p, q are relatively prime, we can choose $a, b \in \mathbb{Z}$ with $ap + bq = 1$. Consider the map $f\colon \mathbb{Z}/pq \to \mathbb{Z}/p \oplus \mathbb{Z}/q$ given by $[1] \mapsto ([1],[1])$. This map is surjective since

$$\begin{aligned} f([ap]) &= ([ap],[ap]) = ([0],[1]-[bq]) = ([0],[1]) \\ f([bq]) &= ([bq],[bq]) = ([1]-[ap],[0]) = ([1],[0]). \end{aligned}$$

Since domain and range of f have the same order, f is an isomorphism.

2. a) Show that if X_1 and X_2 are compact connected surfaces, then $\chi(X_1 \# X_2) = \chi(X_1) + \chi(X_2) - 2$.

Cover both surfaces by patterns of polygons, and let us write V_i (resp. E_i resp. F_i) for the number of vertices (resp. edges resp. faces) on X_i. Let us assume that we chose these pattern such that both contain a face which is a polygon with k vertices and edges for some k. The point of this is that removing that face from both surfaces and identifying the resulting surfaces along the polygon results in the connected sum $X_1 \# X_2$ and the patterns on both surfaces fit together to give a pattern of polygons on $X_1 \# X_2$. Let us determine V (= number of vertices), E (= number of edges) and F (= number of faces) of this pattern on $X_1 \# X_2$:

- **faces:** $F = F_1 + F_2 - 2$ (each face on X_1 or X_2 gives a face on $X_1 \# X_2$ except the two polygons that we removed in order to form the connected sum).
- **edges:** $E = E_1 + E_2 - k$ (each edge on X_1 or X_2 gives an edge on $X_1 \# X_2$ except there are k pairs of edges that get identified with each other when we form the connected sum).
- **vertices:** $V = V_1 + V_2 - k$ (each vertex on X_1 or X_2 gives a vertex on $X_1 \# X_2$ except there are k pairs of vertices that get identified with each other when we form the connected sum).

This implies

$$\begin{aligned}\chi(X_1 \# X_2) &= V - E + F \\ &= (V_1 + V_2 - k) - (E_1 + E_2 - k) + (F_1 + F_2 - 2) \\ &= V_1 - E_1 + F_1 + V_2 - E_2 + F_2 - 2 \\ &= \chi(X_1) + \chi(X_2) - 2\end{aligned}$$

b) Calculate the Euler characteristic of all compact connected surfaces.

According to the Classification Theorem for compact connected surfaces, every such surface is homeomorphic to S^2 or a connected sum $T \# \ldots T$ of tori or a connected sum $P \# \ldots \# P$ of projective planes. In class we've calculated:

$$\chi(S^2) = 2 \qquad \chi(T) = 0 \qquad \chi(P) = 1$$

Using the formula from part (a) plus induction we get

$$\chi(\underbrace{T \# \ldots \# T}_{g}) = 2 - 2g \qquad \chi(\underbrace{P \# \ldots \# P}_{k}) = 2 - k$$

Alternatively, we could represent these surfaces as polygons with edge identifications as in problem 3 a) below and just read off the Euler characteristic.

3. a) Calculate the homology groups of all compact connected surfaces.

Again using the Classification Theorem, it suffices to look at S^2 (homology calculated in class) and connected sums of tori and projective planes. In order to determine the homology groups, we need to choose a pattern of polygons for $T\#\dots\#T$ and $P\#\dots\#P$. To do so, it is convenient to think of these surfaces as polygons with edge identifications. I'd love to draw a picture of this, but unfortunately doing that in a TEX file is time consuming. Let's instead describe these pictures in words by reading off the edge labels going around the perimeter of the polygon in clockwise direction; let us agree that we write a^{-1} instead of a if the edge label is a, but the arrow of the edge in question is pointing in counterclockwise direction.

With these conventions, the torus T is obtained from a square by labeling the edges a, b, a^{-1}, b, b^{-1}. In class we explained how to think of the connected sum $X\#Y$ of two surfaces given by polygons with edge identification again as being obtained from a polygon. Applying this to $\underbrace{T\#\dots\#T}_{g}$, we see that this surface can be obtained from a $4g$-gon with edges labeled

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

The resulting pattern on the surface has one vertex v, one face f and $2g$ edges a_i, b_i, $1 \le i \le g$. Hence the chain complex associated to this pattern looks like

$$\mathbb{Z}v \xleftarrow{\partial_1} \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \dots \mathbb{Z}a_g \oplus \mathbb{Z}b_g \xleftarrow{\partial_2} \mathbb{Z}f$$

Since there is only one vertex involved, we have $\partial_1(e) = v - v = 0$ for any edge e and hence $\partial_1 \equiv 0$. To determine $\partial_2(f)$ we notice that every edge label occurs exactly twice, but with arrows pointing in opposite directions. Hence $\partial_2(f) = 0$ and $\partial_2 \equiv 0$ as well. It follows that the q-th homology group H_q is isomorphic to the q-th chain group C_q and we can read off the groups from the chain complex above:

$$H_q(\underbrace{T\#\dots\#T}_{g}) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ \mathbb{Z}^{2g} & q = 1 \\ 0 & q \neq 0, 1, 2 \end{cases}$$

We proceed similarly to calculate the homology groups of $P\#\dots\#P$. Since P is obtained from a bigon with edges labeled a, a, the connected sum $\underbrace{P\#\dots\#P}_{k}$ is obtained from a $2k$-gon with edges labeled

$$a_1, a_1, \dots, a_k, a_k.$$

The associated chain complex then is

$$\mathbb{Z}v \xleftarrow{\partial_1} \mathbb{Z}a_1 \oplus \dots \mathbb{Z}a_g \xleftarrow{\partial_2} \mathbb{Z}f$$

with $\partial_1(a_i) = 0$ and $\partial_2(f) = 2a_1 + \dots + 2a_k$. It follows that $H_0 = \mathbb{Z}$, $H_2 = 0$, and

$$H_1 = \mathbb{Z}a_1 \oplus \dots \oplus \mathbb{Z}a_k / \mathbb{Z}2(a_1 + \dots + a_k).$$

To identify this quotient group, we choose a different basis of the free abelian group C_1, namely $a_1, \dots, a_{k-1}, c$, with $c = a_1 + \dots + a_k$. Then we see

$$H_1 = \mathbb{Z}a_1 \oplus \dots \oplus \mathbb{Z}a_{k-1} \oplus \mathbb{Z}c/\mathbb{Z}2c \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{k-1} \oplus \mathbb{Z}/2.$$

b) Can the Euler characteristic of a compact connected surface be expressed in terms of its homology groups?

Inspection of the Euler characteristic calculation (resp. homology group calculation) for a compact connected surfaces X shows that

$$\chi(X) = \operatorname{rk} H_0(X) - \operatorname{rk} H_1(X) + \operatorname{rk} H_2(X),$$

where $\operatorname{rk} H_q(X)$ is the rank of the abelian group $H_q(X)$.

c) Looking at your calculation of $H_2(X)$ for compact connected surfaces X, what do you observe?

We note that $H_2(X) = \mathbb{Z}$ for compact connected surfaces X which are *orientable*, while $H_2(X) = 0$ if X is not orientable.

3. HOMEWORK ASSIGNMENT # 3

The solutions to this assignment were worked out and texed by Katie Grayshan (with some modifications by Stephan). Thanks, Katie!

1. Show that the singular chain complex of a topological space X is in fact a chain complex; i.e., that $\partial_q \circ \partial_{q+1} = 0$, where $\partial_q : C_q(X) \to C_{q-1}(X)$ is the boundary map. It will suffice to show that $\partial_q \circ \partial_{q+1} = 0$ on the generators of C_{q+1} since ∂_q and ∂_{q+1} are both homomorphisms.

Consider $\sigma : \Delta^{q+1} \to X$ in $C_{q+1}(X)$.

$$\begin{aligned}
&\partial_q \circ \partial_{q+1}(\sigma)\\
=&\sum_{i=0}^{q}(-1)^i \partial_{q+1}(\sigma)|_{[v_0,\ldots,\hat{v_i},\ldots,v_{q+1}]}\\
=&\sum_{i=0}^{q}(-1)^i (\sum_{j=0}^{q+1}(-1)^j \sigma|_{[v_0,\ldots,\hat{v_j},\ldots,v_{q+1}]})|_{[v_0,\ldots,\hat{v_i},\ldots,v_q]}\\
=&\sum_{0\leq i<j\leq q+1}(-1)^{i+j}\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,\hat{v_j},\ldots,v_{q+1}]} + \sum_{0\leq j<i\leq q+1}(-1)^{i+j+1}\sigma|_{[v_0,\ldots,\hat{v_j},\ldots,\hat{v_i},\ldots,v_{q+1}]}\\
=&\sum_{0\leq i<j\leq q+1}(-1)^{i+j}\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,\hat{v_j},\ldots,v_{q+1}]} - \sum_{0\leq j<i\leq q+1}(-1)^{i+j}\sigma|_{[v_0,\ldots,\hat{v_j},\ldots,\hat{v_i},\ldots,v_{q+1}]}\\
=&0.
\end{aligned}$$

Hence, since $\partial_q \circ \partial_{q+1} = 0$ on the generators of $C_{q+1}(X)$, $\partial_q \circ \partial_{q+1} = 0$, and the singular chain complex of a topological space X is in fact a chain complex. 2. Calculate the singular homology groups of the one point space. Let X=$\{x_0\}$. Notice that for $q \geq 0$, the only possible singular q-simplex is the map $\sigma_q : \Delta^q \to X$ defined by $\sigma_q(v) = x_0$ for any $v \in \Delta^q$. Hence, $C_q(X) = \{n\sigma_q \mid n \in \mathbb{Z}\} = \mathbb{Z}[\sigma_q]$ for all $q \geq 0$. For $q < 0$, by convention, $C_q(X) = 0$ since there are no q-simplices. For $q < 0$, $B_q(X) \subset Z_q(X) \subset C_q(X) = 0$, so $Z_q(X) = B_q(X) = 0$.

For $q \geq 0$,

$$\begin{aligned}
\partial_q(\sigma_q) &= \sum_{i=0}^{q}(-1)^i \sigma_q|_{[v_0,\ldots,\hat{v_i},\ldots,v_q]}\\
&= \sigma_q|_{[v_1,\ldots,v_q]} - \sigma_q|_{[v_0,v_2\ldots,v_q]} + \sigma_q|_{[v_0,v_1,v_3\ldots,v_q]} - \ldots + (-1)^q\sigma_q|_{[v_0,\ldots,v_{q-1}]}\\
&= \begin{cases} \sigma_{q-1} - \sigma_{q-1} + \sigma_{q-1} - \ldots + (-1)^q\sigma_{q-1} & q > 0\\ 0 & q = 0\end{cases}\\
&= \begin{cases} 0 & q \text{ odd or } q = 0\\ \sigma_{q-1} & q \text{ even and } q > 0\end{cases}
\end{aligned}$$

Therefore, since every element of $C_q(X)$ is merely an integer multiple of σ_q, we see that for $q \geq 0$,

$$Z_q(X) = ker\ \{\partial_q : C_q(X) \to C_{q-1}(X)\} = \begin{cases} C_q(X) = \mathbb{Z}[\sigma_q] & q \text{ odd or } q = 0\\ 0 & q \text{ even and } q > 0\end{cases}$$

and

$$B_q(X) = Im\ \{\partial_{q+1} : C_{q+1}(X) \to C_q(X)\} = \begin{cases} 0 & q \text{ even} \\ \mathbb{Z}[\sigma_q] & q \text{ odd} \end{cases}$$

Hence, we have that for $q \geq 0$

$$H_q(X) = \frac{Z_q(X)}{B_q(X)} = \begin{cases} \mathbb{Z}[\sigma_0] & q = 0 \\ 0 & q > 0 \end{cases}$$

and $H_q(X) = 0$ for $q < 0$

3. Let X be a topological space with path components X_α, $\alpha \in A$. Show that $H_q(X)$ is isomorphic to $\bigoplus_{\alpha \in A} H_q(X_\alpha)$.

Since the image of a path connected set is also path connected, we have for each $\sigma : \Delta^q \to X \in C_q(X)$, $\sigma(\Delta^q) \subset X_\alpha$ for some $\alpha \in A$, so $\sigma : \Delta^q \to X \in C_q(X_\alpha)$. Hence, given an arbitrary element $\sum\limits_{i=0}^{m} n_i \sigma_i \in C_q(X)$, where $\sigma_i : \Delta^q \to X$, $\sum\limits_{i=0}^{m} n_i \sigma_i \in \left\langle \bigcup_{\alpha \in A} C_q(X_\alpha) \right\rangle$. Clearly, we also have $C_q(X) \supset \left\langle \bigcup_{\alpha \in A} C_q(X_\alpha) \right\rangle$, so $C_q(X) = \left\langle \bigcup_{\alpha \in A} C_q(X_\alpha) \right\rangle$. Moreover, each $X_{\alpha_i} \cap X_{\alpha_j} = \emptyset$ for $i \neq j$ since path components are disjoint, whence $C(X_{\alpha_i}) \cap \left\langle \bigcup\limits_{\substack{k \neq i \\ \alpha_k \in A}} C(X_{\alpha_k}) \right\rangle = 0$. Any subgroup of an abelian group is normal, so we then have that $C_q(X) \cong \bigoplus_{\alpha \in A} C_q(X_\alpha)$. Similarly, for $\alpha \in A$, $\partial_q(C_q(X_\alpha)) \subset C_{q-1}(X_\alpha)$ since the boundary map doesn't affect the range of a singular q-simplex. As a result, $\partial_q(C_q(X)) \cong \bigoplus_{\alpha \in A} \partial_q(C_q(X_\alpha))$; i.e., $B_{q-1}(X) \cong \bigoplus_{\alpha \in A} B_{q-1}(X_\alpha)$. Moreover, from the decomposition of X into the (external) direct sum of X_α's, we have $Z_q(X) \cong \bigoplus_{\alpha \in A} Z_q(X_\alpha)$. Thus,

$$H_q(X) = \frac{Z_q(X)}{B_q(X)} \cong \frac{\bigoplus_{\alpha \in A} Z_q(X_\alpha)}{\bigoplus_{\alpha \in A} B_q(X_\alpha)} \cong \bigoplus_{\alpha \in A} \frac{Z_q(X_\alpha)}{B_q(X_\alpha)} = \bigoplus_{\alpha \in A} H_q(X_\alpha).$$

4. a) Show that the Hurewicz map $h : \pi_1(X, x_0) \to H_1(X)$ given by $[\gamma] \to [[\gamma]]$ is a homomorphism. Let $[\gamma], [\gamma'] \in \pi_1(X, x_0)$, and let $\gamma \cdot \gamma'$ denote the product of paths γ and γ' (previously denoted $\gamma + \gamma'$). To show that h is a homomorphism, we need to verify that $h([\gamma][\gamma']) = h([\gamma]) + h([\gamma'])$; i.e., $h([\gamma \cdot \gamma']) = [[\gamma \cdot \gamma']] = [[\gamma]] + [[\gamma']]$. We will do so by showing that $\gamma + \gamma' - \gamma \cdot \gamma' \in B_1(X)$ which will then imply that $[[\gamma\gamma']] = [[\gamma + \gamma']] = [[\gamma]] + [[\gamma']]$, as desired. Consider the singular 2-simplex, $\sigma : \Delta^2 \to X$, defined as the composition of the affine linear

map

$$\Delta^2 = [v_0, v_1, v_2] \longrightarrow [v_0, v_2]$$

determined by $v_0 \mapsto v_0$, $v_2 \mapsto v_2$, $v_1 \mapsto \frac{1}{2}(v_0 + v_2)$ followed by $\gamma \cdot \gamma' : [v_0, v_2] \to X$. Notice that $\partial_2(\sigma) = \gamma' - \gamma \cdot \gamma' + \gamma$, so $\gamma' - \gamma \cdot \gamma' + \gamma \in B_1(X)$ as desired. Thus, h is indeed a homomorphism. b) Let $\Psi : C_1(X)/B_1(X) \to \pi_1(X, x_0)^{ab}$ be the map defined by $[[\gamma]] \mapsto [\lambda_{\gamma(0)}\gamma\overline{\lambda_{\gamma(1)}}]$ for any singular 1-simplex γ (as in class we have chosen for every point $x \in X$ a path λ_x from the basepoint x_0 to x). Show that the restriction of Ψ to $H_1(X) \subset C_1(X)/B_1(X)$ provides an inverse to the map $\bar{h} : \pi_1^{ab}(X, x_0) \to H_1(X)$.

We need to show that $\Psi|_{H_1(X)} \circ \bar{h} = id_{\pi_1^{ab}(X,x_0)}$ and $\bar{h} \circ \Psi|_{H_1(X)} = id_{H_1(X)}$. Let $[\gamma : (I, \partial I) \to (X, x_0)] \in \pi_1^{ab}(X, x_0)$. Then

$$\begin{aligned}\Psi|_{H_1(X)} \circ \bar{h}([\gamma]) &= \Psi|_{H_1(X)}([[\gamma]]) \\ &= [\lambda_{\gamma(0)}\gamma\overline{\lambda_{\gamma(1)}}] = [\lambda_{x_0}\gamma\overline{\lambda_{x_0}}] \\ &= [\lambda_{x_0}][\gamma][\overline{\lambda_{x_0}}] = [\lambda_{x_0}][\overline{\lambda_{x_0}}][\gamma] = [\gamma],\end{aligned}$$

which shows $\Psi|_{H_1(X)} \circ \bar{h} = id_{\pi_1^{ab}(X,x_0)}$.

To prove that $\bar{h} \circ \Psi|_{H_1(X)} = id_{H_1(X)}$, it is useful to first calculate the left hand side not just for elements in $H_1(X)$, but for elements of $C_1(X)/B_1(X)$; the advantage is that $C_1(X)/B_1(X)$ is *generated by 1-simplices.* If γ is a 1-simplex, i.e., a path (or more generally a 1-chain, i.e., linear combination of paths), let us denote by $[[\gamma]]$ the element represented by γ in the quotient group $C_1(X)/B_1(X)$. We note that the considerations in part (a) show that if γ, γ' are paths with $\gamma(1) = \gamma(0)$, then $[[\gamma \cdot \gamma']] = [[\gamma]] + [[\gamma']]$; moreover, $[[\bar{\gamma}]] = -[[\gamma]]$.

With these preliminaries, we can calculate for any path γ:

$$\begin{aligned}\bar{h} \circ \Psi([[\gamma]]) &= \bar{h}([\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}] = [[\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}]] \\ &= [[\lambda_{\gamma(0)}]] + [[\gamma]] + [[\bar{\lambda}_{\gamma(1)}]] = [[\lambda_{\gamma(0)}]] + [[\gamma]] - [[\lambda_{\gamma(1)}]] \qquad (1) \\ &= [[\gamma]] + [[\lambda(\partial(\gamma))]],\end{aligned}$$

where $\lambda \colon C_0(X) \to C_1(X)$ is given on generators by $x \mapsto \lambda_x$ (and hence λ maps $\partial\gamma = \gamma(1) - \gamma(0)$ to $\lambda_{\gamma(0)} - \lambda_{\gamma(1)}$). Since formula (10) holds for the generators $\gamma \in C_1(X)$, it hold for every element $z \in C_1(X)$; in particular, if z is a *cycle*, we have $\bar{h} \circ \Psi([[z]]) = [[z]]$, which is what we wanted to prove.

4. HOMEWORK ASSIGNMENT # 4

The solutions to this assignment were worked out and texed by Katie Grayshan (with some modifications by Stephan). Thanks, Katie!

1. Let $A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$ be a short exact sequence of chain complexes. Show that the sequence

$$... \longrightarrow H_q(A) \xrightarrow{f_*} H_q(B) \xrightarrow{g_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow ...$$

is exact at $H_q(C)$ and $H_{q-1}(B)$.

Proof. In order to show that the sequence is exact at $H_q(C)$ and $H_{q-1}(B)$, we need to verify the following: a) $Im\ g_* = ker\ \partial$ b) $Im\ f_* = ker\ g_*$

a) $Im\ g_* \subset ker\ \partial$ Recall that for $q \in \mathbb{Z}$, the chain map g_* induces a map $g_* : H_q(B) \longrightarrow H_q(C)$ where $g_*([b]) = [g(b)]$. Let $g_*([b]) \in Im\ g_* \subset H_q(C)$, where $[b] \in H_q(B)$ (so $b \in Z_q(B)$), then $\partial g_*([b]) = \partial([g(b)])$. Recall the construction of ∂: ∂ assigns c, which is $g(b)$ in our case, to an $a \in A_{q-1}$ for which $f(a) = \partial(b')$, with b' chosen so that $c = g(b) = g(b')$ for $b \in B_q$. Since we showed that the choice of such a b' is irrelevant, we can then choose b' to be b. Hence, $f(a) = \partial(b) = 0$ since $b \in Z_q(B)$. f_q is injective (see the exactness of the given sequence), so we have that $a = 0$, whence $\partial g_*([b]) = \partial([g(b)]) = [0]$. Therefore, $Im\ g_* \subset ker\ \partial$ at $H_q(C)$. $Im\ g_* \supset ker\ \partial$ Let $[c] \in ker\ \partial \subset H_q(C)$. Using the notation from the construction of ∂, there exists a $b \in B_q$ such that $g(b) = c$ and an $a \in A_{q-1}$ for which $f(a) = \partial(b)$, and since $[c] \in ker\ \partial$, $\partial a' = a$ for some $a' \in A_q$. We notice that the element $b - f(a') \in B_q$ is a q-cycle since

$$\partial(b - f(a')) = \partial b - \partial f(a') = f(a) - f\partial a' = 0,$$

whence $[b - f(a')] \in H_q(B)$. Furthermore, by the exactness of the original sequence, $gf = 0$, so

$$g(b - f(a')) = g(b) - gf(a') = g(b) - 0 = g(b) = c.$$

Thus, $[c] = [g(b - f(a'))] = g_*([b - f(a')]) \in Im\ g_*$, and $Im\ g_* \supset ker\ \partial$.

b) $Im\ f_* \subset ker\ g_*$ Let $f_*([a]) = [f(a)] \in Im\ f_* \subset H_{q-1}(B)$, where $a \in Z_{q-1}(A)$. Then, by the exactness of the original sequence, $gf = 0$, so $g_*([f(a)]) = [gf(a)] = [0]$. Hence, $Im\ f_* \subset ker\ g_*$.

$Im\ f_* \supset ker\ g_*$ Let $[b] \in H_{q-1}(B)$ such that $g_*([b]) = [0]$; i.e., $g(b) \in Im(\partial : C_q \to C_{q-1})$. Therefore, $\exists c \in C_q$ such that $\partial(c) = g(b)$. Since g is onto (by the exactness of the given sequence), $\exists b' \in B_q$ such that $g(b') = c$. The commutativity of the diagram (of the chain complexes, chain maps, and boundary maps) yields:

$$g\partial(b') = \partial g(b') = \partial(c) = g(b).$$

Since each g is a homomorphism, we then have $g(b - \partial(b')) = 0$, so $b - \partial(b') \in ker\ g = Im\ f$. Thus, $\exists a \in A_{q-1}$ such that $f(a) = b - \partial(b')$. Again, by the commutativity of the diagram, $f(\partial a) = \partial f(a) = \partial b - \partial\partial b' = \partial b = 0$, so $a = 0$ since f is injective ($ker\ f = 0$). Thus,

$a \in Z_{q-1}(A)$ and $[a] \in H_{q-1}(A)$. Moreover, since $b - f(a) = \partial(b')$, $f_*([a]) = [f(a)] = [b] \in H_{q-1}(B)$. Thus, *Im* $f_* \supset$ *ker* g_*. To recap, we now have that:

$$...\longrightarrow H_q(A) \xrightarrow{f_*} H_q(B) \xrightarrow{g_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow ...$$

is exact at $H_q(C)$ and $H_{q-1}(B)$. □

2. Recall that the *reduced homology groups* $\tilde{H}_q(X)$ of a space X are the homology groups of the *augmented chain complex*

$$...0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_3} ...$$

For a pair (X, A) with $A \neq \emptyset$, we define $\tilde{H}_q(X, A) = H_q(X, A)$. a) Show that $\tilde{H}_q(X) \cong H_q(X)$ for $q \neq 0$, and that there is a short exact sequence

: (1)
$$0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0.$$

Hint: one way of doing this is to construct a chain map from the augmented chain complex to the chain complex of X, try to fit it in a short exact sequence of chain complexes, and use the exact homology sequence of problem 1.

We note that equation (1) implies in particular that there is an isomorphism $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$, but this is not canonical.

Proof. Consider the chain complex D_*:

$$...0 \longleftarrow \mathbb{Z} = D_{-1} \longleftarrow 0 \longleftarrow 0 \longleftarrow 0....$$

Notice that the augmented chain complex, the original, and this new chain complex together form a commutative diagram for which each column is exact:

$$\begin{tikzcd}
... \arrow[d] & 0 \arrow[l] \arrow[d, "id_0"] & \mathbb{Z} \arrow[l] \arrow[d, "id_{\mathbb{Z}}"] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & ... \arrow[l] \\
... \arrow[d] & 0 \arrow[l] \arrow[d, "id_0"] & \mathbb{Z} \arrow[l] \arrow[d] & C_0(X) \arrow[l] \arrow[d, "id_{C_0(X)}"] & C_1(X) \arrow[l] \arrow[d, "id_{C_1(X)}"] & ... \arrow[l] \\
... & 0 \arrow[l] & 0 \arrow[l] & C_0(X) \arrow[l] & C_1(X) \arrow[l] & ... \arrow[l]
\end{tikzcd} \tag{2}$$

In other words, we have a short exact sequence of chain complexes:

$$D_* \longrightarrow \tilde{C}_*(X) \longrightarrow C_*(X)$$

where we denote the augmented chain complex $\tilde{C}_*$. Hence, problem 1. gives us a long exact sequence of homology groups. Substituting

$H_q(\tilde{C}_*(X)) = \tilde{H}_q(X)$ and $H_q(C_*(X)) = H_q(X)$, we obtain the long exact sequence
(3)

$$\longrightarrow H_q(D_*) \longrightarrow \tilde{H}_q(X) \longrightarrow H_q(X) \xrightarrow{\partial} H_{q-1}(D_*) \longrightarrow \tilde{H}_{q-1}(X) \longrightarrow .$$

We note that the homology groups of D_* are given by

$$H_q(D_*) = D_q = \begin{cases} \mathbb{Z} & q = -1 \\ 0 & q \neq -1 \end{cases}$$

It follows from the exact sequence (10) that the map $\tilde{H}_q(X) \to H_q(X)$ is an isomorphism for $q > 0$, since the groups in the sequence to the left and right of this map are zero. We also note that this is obviously true for $q < 0$, since both $\tilde{H}_q(X)$ and $H_q(X)$ are zero for $q < 0$. For $q = 0$ the terms $H_0(D_*)$ and $\tilde{H}_{q-1}(X)$ in the long exact sequence (10) are both zero, and hence we obtain the desired short exact sequence

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

□

b) Show that there is an exact sequence of reduced homology groups

$$(4) \qquad \dots \longrightarrow \tilde{H}_q(A) \xrightarrow{i_*} \tilde{H}_q(X) \xrightarrow{j_*} \tilde{H}_q(X, A) \xrightarrow{\partial} \tilde{H}_{q-1}(A) \xrightarrow{i_*} \dots$$

Proof. Suppose $A \neq \emptyset$. Consider the following commutative diagram
(5)

$$\begin{tikzcd}
{} \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "id_{\mathbb{Z}}"] & C_0(A) \arrow[l] \arrow[d] & C_1(A) \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & C_0(X) \arrow[l] \arrow[d] & C_1(X) \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots & 0 \arrow[l] & 0 \arrow[l] & C_0(X, A) \arrow[l] & C_1(X, A) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

The three rows are the reduced singular chain complex of A resp. the reduced singular chain complex of X resp. the singular chain complex of the pair (X, A). Each column is exact and hence we have the following short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{C}_*(A) \xrightarrow{i_*} \tilde{C}_*(X) \xrightarrow{j_*} C_*(X, A) \longrightarrow 0$$

The associated long exact homology sequence is the desired sequence (4). □

3. Prove the following statement which is known as the 5-*lemma.* Suppose we have a commutative diagram of abelian groups and group homomorphisms

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd} \tag{6}$$

such that the rows are exact sequences. Show that if the outer two vertical maps h_1, h_2, h_4, h_5 are all isomorphisms are slightly stronger than needed for the proof. What weaker assumptions will do?

Proof. We first prove that h_3 is injective. Let $a_3 \in A_3$ such that $h_3a_3 = 0 \in B_3$. Then by the commutativity of the diagram, we have $h_4f_3a_3 = g_3h_3a_3 = g_3(0) = 0$, so $f_3(a_3) \in \ker\ h_4 = 0$ since h_4 is injective. Hence, $a_3 \in \ker\ f_3 = \operatorname{im} f_2$, so there is an $a_2 \in A_2$ such that $f_2(a_2) = a_3$. Again, by the commutativity of the diagram, we have $g_2h_2(a_2) = h_3f_2(a_2) = h_3(a_3) = 0$, so $h_2(a_2) \in \ker\ g_2 = \operatorname{im}\ g_1$. Thus, there is an element $b_1 \in B_1$ such that $g_1(b_1) = h_2(a_2)$. Moreover, since h_1 is surjective, there is some $a_1 \in A_1$ with $h_1a_1 = b_1$. It follows that $h_2f_1a_1 = g_1h_1a_1 = g_1b_1 = h_2a_2$. Thus, since h_2 is injective, $a_2 = f_1a_1$, so $a_3 = f_2a_2 = f_2f_1a_1 = 0$. Therefore, h_3 is injectiv.

To show surjectivity of h_3, let $b_3 \in B_3$. Since h_4 is surjective, there is some $a_4 \in A_4$ with $h_4a_4 = g_3b_3 \in A_4$ and by commutativity, $h_5f_4a_4 = g_4h_4a_4 = g_4g_3b_3 = 0$. Since h_5 is injective, this implies $f_4a_4 = 0$. By exactness of the top row at A_4, there is an $a_3 \in A_3$ such that $f_3(a_3) = a_4$. Hence,

$$g_3(h_3(a_3) - b_3) = h_4f_3a_3 - g_3b_3 = h_4a_4 - g_3b_3 = 0.$$

By the exactness of the lower row at B_3, this implies that there exists $b_2 \in B_2$ such that $g_2(b_2) = h_3(a_3) - b_3$. Since h_2 is surjective, there is some $a_2 \in A_2$ with $h_2a_2 = b_2$ and hence

$$h_3(a_3 - f_2a_2) = h_3a_3 - h_3f_2a_2 = h_3a_3 - g_2h_2a_2 = b_3,$$

which shows that b_3 is in the image of h_3. Since b_3 was arbitrary, this shows that h_3 is surjective.

We see that we've used the assumptions that h_4, h_2 are injective, and that h_1 is surjective to show injectivity of h_3. Our proof that h_3 is surjective required the assumptions that h_2, h_4 are surjective, and that h_5 is injective. So it is sufficient to assume that h_2 and h_4 are isomorphisms, that h_1 is an epimorphism, and that h_5 is a monomorphism. □

5. HOMEWORK ASSIGNMENT # 5

1. Let X be a topological space and let ΣX be the *suspension of* X which is defined as the quotient space $X \times [0,1]/\sim$, where the equivalence relation is generated by $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. Show that $\widetilde{H}_q(X) \cong \widetilde{H}_{q+1}(\Sigma X)$ (this is called the *suspension isomorphism*).

Hint: note that the suspension of S^n is homeomorphic to S^{n+1}, and think of the suspension isomorphism as a generalization of the isomorphism $\widetilde{H}_q(S^n) \cong \widetilde{H}_{q+1}(S^{n+1})$ proved in class. For that proof the decomposition of S^{n+1} into upper and lower hemisphere was important. Here the role of the upper/lower hemispheres is played by subspaces of ΣX consisting of those points $[x,t] \in \Sigma X$ with $t \geq 1/2$ resp. $t \leq 1/2$.

Proof. Let $x^+ = [x,1] \in \Sigma X$, and $x_- = [x,0] \in \Sigma X$ (they are analogous to the north pole resp. south pole of S^{n+1}), and let $C^+X := \Sigma X \setminus \{x^-\}$ and $C^-X := \Sigma X \setminus \{x^+\}$.

Claim 1: The spaces $C^{\pm}X$ are contractible (i.e., homotopy equivalent to the one point space).

To prove that C^+X is contractible, we will show that the inclusion map $i\colon \{x^+\} \to C^+X$ is the homotopy inverse to the constant map $r\colon C^+X \to \{x^+\}$ (i.e., that the compositions $r \circ i$ and $i \circ r$ are homotopic to the identities on $\{x^+\}$ resp. C^+X). We note that $r \circ i$ is *equal* to the identity on $\{x^+\}$; a homotopy between $i \circ r$ and the identity on C^+X is provided by the map

$$H\colon [0,1] \times C^+X \longrightarrow C^+X \qquad (s,[x,t]) \mapsto [x, st + (1-s)].$$

The proof that C^-X is contractible is similar.

Step 1. Consider the long exact homology sequence of the pair (Σ, Σ^+X):

$$\longrightarrow \widetilde{H}_q(C^+X) \longrightarrow \widetilde{H}_q(\Sigma X) \xrightarrow{j_*} H_q(\Sigma X, C^+X) \xrightarrow{\partial} \widetilde{H}_{q-1}(C^+X) \longrightarrow$$

The contractibility of the space C^+ implies the vanishing of its reduced homology groups and hence it follows that the map j_* in the above sequence is an isomorphism.

Step 2. Removing the point x^+ from the pair $(\Sigma X, C^+X)$ we obtain the pair

$$(\Sigma X \setminus \{x^+\}, C^+X \setminus \{x^+\}) = (C^-X, C^+X \cap C^-X)$$

The excision axiom then implies that the inclusion map

$$(C^-X, C^+X \cap C^-X) \longrightarrow (\Sigma X, C^+X)$$

induces an isomorphism on homology groups.

Step 3. Consider the long exact homology sequence of the pair (C^-X, C), $C = C^+X \cap C^-X$:

$$\longrightarrow \widetilde{H}_{q+1}(C^-X) \longrightarrow H_{q+1}(C^-X, C) \xrightarrow{\partial} \widetilde{H}_q(C) \longrightarrow \widetilde{H}_q(C^-X) \longrightarrow$$

The contractibility of C^-X implies the vanishing of its reduced homology groups. Hence the connecting homomorphism ∂ in the above exact sequence is an isomorphism.

Step 4. We claim that the map $i\colon X \to C^+X \cap C^-X$ given by $i(x) = [x, \frac{1}{2}]$ is a homotopy equivalence. In particular, i induces isomorphisms on homology groups.

Let $r\colon C^+X \cap C^-X \to X$ be defined by $[x,t] \mapsto x$ (note that this is well-defined and continuous). We note that $r \circ i$ is the identity on X; a homotopy between $i \circ r$ and the identity on $C^+X \cap C^-X$ is given by

$$H\colon [0,1] \times C^+X \cap C^-X \longrightarrow C^+X \cap C^-X \qquad (s, [x,t]) \mapsto [x, st + (1-s)\frac{1}{2}]$$

Composing the isomorphisms constructed in steps (1)-(4) we obtain the desired isomorphism

$$\widetilde{H}_{q+1}(\Sigma X) \xrightarrow[\cong]{j_*} H_{q+1}(\Sigma X, C^+) \xleftarrow[\cong]{k_*} H_{q+1}(C^-, C) \xrightarrow[\cong]{\partial} \widetilde{H}_q(C) \xleftarrow[\cong]{i_*} \widetilde{H}_q(X)$$

□

2. Let $f\colon S^n \to S^n$ be a map and

$$\Sigma f\colon S^{n+1} \approx \Sigma S^n \to \Sigma S^n \approx S^{n+1}$$

its suspension. Show that $\deg(\Sigma f) = \deg(f)$. Hint: f induces an endomorphism on all the homology groups you used in your proof of the suspension isomorphism $\widetilde{H}_q(S^n) \cong \widetilde{H}_{q+1}(\Sigma S^n)$.

Proof. We note that the map $\Sigma f\colon \Sigma S^n \to \Sigma S^n$ leaves the subspaces $C^\pm \subset \Sigma S^n$ used in problem #1 invariant. In particular, Σf induces selfmaps of the pairs $(\Sigma S^n, C^+)$ and (C^-, C). Abusing notation we will write Σf for all these selfmaps. These maps are compatible with the maps j, k and i in the sense that the following diagrams are commutative:

$$\begin{tikzcd}
\Sigma S^n \arrow[r, "j"] \arrow[d, "\Sigma f"'] & (\Sigma S^n, C^+) \arrow[d, "\Sigma f"'] & (C^-, C) \arrow[l, "k"'] \arrow[d, "\Sigma f"] \\
\Sigma S^n \arrow[r, "j"] & (\Sigma S^n, C^+) & (C^-, C) \arrow[l, "k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d, "\Sigma f"'] & S^n \arrow[l, "i"'] \arrow[d, "f"'] \\
C & S^n \arrow[l, "i"]
\end{tikzcd}$$

Now we consider the following diagram of homology groups

$$\begin{tikzcd}
\widetilde{H}_{n+1}(\Sigma S^n) \arrow[r, "j_*"] \arrow[d, "\Sigma f_*"'] & H_{n+1}(\Sigma S^n, C^+) \arrow[d, "\Sigma f_*"'] & H_{n+1}(C^-, C) \arrow[l, "k_*"'] \arrow[r, "\partial"] \arrow[d, "\Sigma f_*"'] & \widetilde{H}_n(C) \arrow[d, "\Sigma f_*"'] & \widetilde{H}_n(S^n) \arrow[l, "i_*"'] \arrow[d, "f_*"'] \\
\widetilde{H}_{n+1}(\Sigma S^n) \arrow[r, "j_*"] & H_{n+1}(\Sigma S^n, C^+) & H_{n+1}(C^-, C) \arrow[l, "k_*"'] \arrow[r, "\partial"] & \widetilde{H}_n(C) & \widetilde{H}_n(S^n) \arrow[l, "i_*"']
\end{tikzcd}$$

We note that the first, second and fourth square of this diagram are commutative, since the corresponding square of maps between topological spaces commute. The third square is commutative due to the 'naturality of the connecting homomorphism' (see next week's homework problems). We pick a generator $\alpha_n \in \widetilde{H}_n(S^n)$ and denote by $\alpha_{n+1} \in \widetilde{H}_{n+1}(\Sigma S^n)$ the generator corresponding to α_n via the isomorphism $\Psi\colon \widetilde{H}_{n+1}(\Sigma S^n) \to \widetilde{H}_n(S^n)$ given by the rows of the above diagram. We have

$$\begin{aligned}
f_* \circ \Psi(\alpha_{n+1}) &= f_*(\alpha_n) = \deg(f)\alpha_n \\
\Psi \circ \Sigma f_*(\alpha_{n+1}) &= \Psi(\deg(\Sigma f)\alpha_{n+1}) = \deg(\Sigma f)\alpha_n
\end{aligned}$$

The commutativity of the homology diagram above implies $f_* \circ \Psi = \Psi \circ \Sigma f_*$ and hence $\deg(f) = \deg(\Sigma f)$. $\square$

3. Prove the following statements for the local degree.

a) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry (i.e., f belongs to the orthogonal group), then $\deg(f, 0) = \deg(f_{|S^{n-1}})$.

b) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then

$$\deg(f, 0) = \begin{cases} +1 & \det(f) > 0 \\ -1 & \det(f) < 0 \end{cases}$$

c) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map which is differentiable at the point $x_0 \in \mathbb{R}^n$. Let Df_{x_0} be the derivative at x_0 (which is a linear map $Df_{x_0}\colon \mathbb{R}^n \to \mathbb{R}^n$; the corresponding matrix is the Jacobian of f at the point x_0). Show that if Df_{x_0} is invertible, then

$$\deg(f, x_0) = \begin{cases} +1 & \det(Df_{x_0}) > 0 \\ -1 & \det(Df_{x_0}) < 0 \end{cases}$$

Hint: Try to compare $\deg(f, x_0)$ with $\deg(g, x_0)$, where g is the best affine linear approximation to f which is given by $g(x) = f(x) + Df_{x_0}(x)$ by showing that restricted to a small enough ball around x there is a homotopy f_t, $0 \leq t \leq 1$ between these two maps such that $f_t^{-1}(f(x))$ consists of the point x_0 only.

Proof. Let $g\colon S^{n-1} \to S^{n-1}$ be the restriction of f to S^{n-1}. By the previous problem we know that $\deg(g) = \deg(\Sigma g)$. We note that $(\Sigma g)^{-1}(x_-) = \{x_-\}$, where $x_- = [x, 0] \in \Sigma S^{n-1}$. Hence by the theorem from class, we have $\deg(\Sigma g) = \deg(\Sigma g, x_-)$. To calculate that local degree, we can restrict Σg to a neighborhood of x_-, which we take to be $C^- = \{[x,t] \in \Sigma X \mid 0 \le t < 1\}$. We note that C^- is homeomorphic to $\mathbb{R}^n$ via the homeomorphism

$$h\colon C^- \longrightarrow \mathbb{R}^n \qquad [x,t] \mapsto \left(\tan t\frac{\pi}{2}\right) x$$

Moreover, via this homeomorphism the map Σg corresponds to the original map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ in the sense that the diagram

$$\begin{tikzcd}
C^- \arrow[r, "h"] \arrow[d, "\Sigma g"'] & \mathbb{R}^n \arrow[d, "f"] \\
C^- \arrow[r, "h"] & \mathbb{R}^n
\end{tikzcd}$$

is commutative:

$$h(\Sigma g([x,t])) = h([f(x),t]) = \left(\tan t\frac{\pi}{2}\right) f(x) = f(\left(\tan t\frac{\pi}{2}\right) x) = f(h([x,t]))$$

Since $h(x_-) = 0$, it follows that $\deg(\Sigma g, x_-) = \deg(f, 0)$. This proves part (a).

To prove part (b), we note that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ belongs to the orthogonal group $O(n)$ of linear isometries, then f restricts to a map $f_|\colon S^{n-1} \to S^{n-1}$. We've proved in class that $\deg(f_|) = \det(f)$, and together with part (a) this proves the desired statement if f is an *isometry*. To prove the result for a general isomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, it suffices to show that there is a path $f(t)$ in the group $GL_n(\mathbb{R})$ of isomorphism of $\mathbb{R}^n$ which connects $f = f(0)$ with an element $f(1) \in O(n) \subset GL_n(\mathbb{R})$. We note that the sign of $\det(f(t))$ is independent of t, since it depends continuously on t, and $\det(f(t)) \neq 0$ for all $t \in [0,1]$.

To construct the path, we identify linear maps $\mathbb{R}^n \to \mathbb{R}^n$ with $n \times n$ matrices in the usual way. We will write a matrix A in the form $A = (a_1, a_2, \dots, a_n)$, where $a_i \in \mathbb{R}^n$ are the column vectors of the matrix. We recall that A belongs to $GL_n(\mathbb{R})$ if and only if the vectors $a_1, \dots, a_n$ are linearly independent; $A \in O(n)$ if and only if the vectors a_i are unit vectors which are mutually perpendicular. Let $p_i\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the subspace spanned by $a_1, \dots, a_i$, and let $p_i^\perp\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the orthogonal complement of that subspace; in particular, we have $v = p_i^\perp v + p_i v$ for all $v \in \mathbb{R}^n$. Given a matrix A, let us define

$$A_t = (v_1, p_1^\perp v_2 + t p_1 v_2, \dots, p_{i-1}^\perp v_i + t p_{i-1} v_i, \dots, p_{n-1}^\perp v_n + t p_{n-1} v_n)$$

We note that $\det(A_t)$ is independent of t, since $\det(v_1,\dots,v_n)=0$ depends linearly on each column vector and this determinant is zero if the vectors are linearly dependent. In particular, if $A=A_1$ invertible, then so is A_t. The column vectors of $B=A_0$ are mutually perpendicular, but not necessarily of unit length. Now for $B=(w_1,\dots,w_n)$ we define

$$B_t=((1-t)w_1+t\frac{w_1}{||w_1||},\dots,(1-t)w_n+t\frac{w_n}{||w_n||}),$$

which is a path connecting $B=B_0$ with $B_1\in O(n)$. Concatenating these paths, we obtain the desired path from A to $B_0\in O(n)$.

To prove part (c), we note that the assumption that f is differentiable at x_0 means that $f(x_0+h)$ can be written in the form

$$f(x_0+h)=f(x_0)+Df_{x_0}(h)+e(h),$$

where the 'error term' $e(h)$ is $o(h)$ for $h\to 0$, which means that

$$\lim_{h\to 0}\frac{e(h)}{|h|}=0. \tag{7}$$

We define

$$f_t(x_0+h)=f(x_0)+Df_{x_0}(h)+te(h)$$

and want to argue that f_t is a map of pairs

$$f_t\colon (B_\epsilon(x_0),B_\epsilon(x_0)\setminus\{x_0\})\longrightarrow(\mathbb{R}^n,\mathbb{R}^n\setminus f(x_0)),$$

for sufficiently small $\epsilon>0$, where $B_\epsilon(x_0)$ is the ball of radius ϵ around x_0. In other words, we want to argue that $f_t^{-1}(f(x_0))\cap B_\epsilon(x_0)=\{x_0\}$, or equivalently, that $Df_{x_0}(h)+te(h)\neq 0$ for all h with $0<h<\epsilon$. The idea is to show that for $0<h<\epsilon$ the norm of $Df_{x_0}(h)$ is large compared to the norm of $te(h)$.

To make this precise, let $m:=\min_{h\in S^{n-1}}||Df_{x_0}(h)||$. We note that $m>0$, since $m=||Df_{x_0}(h_0)||$ for some $h_0\in S^{n-1}$, and $Df_{x_0}(h_0)\neq 0$ due to our assumption that Df_{x_0} is invertible. Now the statement (7) allows us to choose $\epsilon>0$ such that $\frac{||e(h)||}{|h|}<m$ for $||h||<\epsilon$. This implies that for $0<||h||<\epsilon$ we have

$$||te(h)||\le||e(t)||<m||h||\le||Df_{x_0}(\frac{h}{||h||})||\,||h||=||Df_{x_0}(h)||$$

and hence $Df_{x_0}(h)+te(h)\neq 0$ as desired. We conclude that $f=f_1$ is homotopic to $g=f_0$ as maps from $(B_\epsilon(x_0),B_\epsilon(x_0)\setminus\{x_0\})$ to $(\mathbb{R}^n,\mathbb{R}^n\setminus f(x_0))$ and hence $\deg(f,x_0)=\deg(g,x_0)$.

Finally, we want to compare $\deg(g,x_0)$ and $\deg(D,0)$, where $D=Df_{x_0}$. Since $g(x)=f(x_0)+D(x-x_0)$, the map g can be written in the form $g=T_{f(x_0)}\circ D\circ T_{-x_0}$, where $T_v\colon\mathbb{R}^n\to\mathbb{R}^n$ is the *translation map* given by $x\mapsto x+v$ for $v\in\mathbb{R}^n$. To relate the local degrees of g and

D we note that the maps D, T_v and g extend to continuous selfmaps $\widehat{D}$, $\widehat{T}_v$, $\widehat{g}$ of the one-point compactification $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$. The space $\widehat{\mathbb{R}}^n$ is homeomorphic to S^n via the stereographic projection, and hence we can consider the degree of these maps. We note that

$$\deg(\widehat{D}) = \deg(D, 0) \qquad \deg(\widehat{g}) = \deg(g, x_0)$$

by our theorem from class, since $\widehat{D}^{-1}(0) = D^{-1}(0) = \{0\}$ and $\widehat{g}^{-1}(f(x_0)) = g^{-1}(f(x_0)) = \{x_0\}$. Moreover, $\deg(\widehat{T}_v) = 1$, since the map

$$\widehat{R}^n \times [0,1] \to \widehat{\mathbb{R}}^n \qquad (x,t) \mapsto \widehat{T}_{tv}(x)$$

provides a homotopy between $\widehat{T}_v$ and $\widehat{T}_0 = 1_{\widehat{R}^n}$. It follows that

$$\deg(\widehat{g}) = \deg(\widehat{T}_{f(x_0)} \circ \widehat{D} \circ \widehat{T}_{-x_0}) = \deg(T_{f(x_0)}) \cdot \deg(\widehat{D}) \cdot \deg(\widehat{T}_{-x_0}) = \deg(\widehat{D})$$

which finishes the proof. □

6. Homework Assignment # 6

1. Let $x_1, \dots, x_l$ be points in S^n, and let $U_1, \dots, U_l$ be mutually disjoint neighborhoods of these points. Fix a generator $\alpha \in H_n(S^n) \cong \mathbb{Z}$ and let $\alpha_k \in H_n(U_k, U_k \setminus \{x_k\})$ be the image of α via the isomorphism

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "(j_k)_*", "\cong"'] & H_n(S^n, S^n \setminus \{x_k\}) & H_n(U_k, U_k \setminus \{x_k\}) \arrow[l, "(i_k)_*"', "\cong"]
\end{tikzcd}$$

Show that the composition

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "j_*"] & H_n(S^n, S^n \setminus \{x_1, \dots, x_l\}) & H_n(\amalg U_m, \amalg\,(U_m \setminus \{x_m\})) \arrow[l, "i_*"', "\cong"] \\
 & & \bigoplus_{m=1}^{l} H_n(U_m, U_m \setminus \{x_m\}) \arrow[u, "\cong", "\bigoplus(\iota_m)_*"']
\end{tikzcd}$$

maps α to $\alpha_1 + \dots + \alpha_l$.

Proof. Consider the following diagram

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "j_*"] \arrow[ddr, "\cong", "(j_k)_*"'] & H_n(S^n, S^n \setminus \{x_1, \dots, x_l\}) \arrow[dd, "(g_k)_*"] & H_n(\amalg U_m, \amalg\,(U_m \setminus \{x_m\})) \arrow[l, "i_*"', "\cong"] \\
 & & \bigoplus_{m=1}^{l} H_n(U_m, U_m \setminus \{x_m\}) \arrow[u, "\cong", "\bigoplus(\iota_m)_*"'] \arrow[d, "p_k"] \\
 & H_n(S^n, S^n \setminus \{x_k\}) & H_n(U_k, U_k \setminus \{x_k\}) \arrow[l, "(i_k)_*"', "\cong"]
\end{tikzcd}$$

Here p_k is the projection onto the k-th summand; all other maps are induced by the obvious maps of pairs of spaces. In particular the triangle on the left is commutative, since the corresponding diagram of pairs of spaces is commutative. We claim that the right hand side of the diagram is commutative as well. To prove this, let

$$\beta = (\beta_1, \ldots, \beta_l) \in \bigoplus_{k=1}^{l} H_n(U_k, U_k \setminus \{x_k\}).$$

Then

$$\begin{aligned}(g_k)_* i_* (\bigoplus (\iota_m)_*)(\beta) &= \sum_{m=1}^{l} (g_k)_* i_* (\iota_m)_* (\beta_m) = \sum_{m=1}^{l} (g_k i \iota_m)_* (\beta_m) \\ &= (g_k i \iota_k)_* (\beta_k) = (i_k)_* p_k (\beta)\end{aligned}$$

We note that the second but last equality holds since for $m \neq k$ the natural map

$$g_k i \iota_m \colon (U_m, U_m \setminus \{x_m\}) \longrightarrow (S^n, S^n \setminus \{x_k\})$$

factors through the pair (U_m, U_m) whose homology vanishes.

The commutativity of the diagram implies that the k-th component of the image of α in $\bigoplus_{k=1}^{l} H_n(U_k, U_k \setminus \{x_k\})$ is α_k as desired. □

2. Let

$$\begin{tikzcd}
A_* \arrow[r, "f_*"] \arrow[d, "a_*"] & B_* \arrow[r, "g_*"] \arrow[d, "b_*"] & C_* \arrow[d, "c_*"] \\
A'_* \arrow[r, "f_*"] & B'_* \arrow[r, "g_*"] & C'_*
\end{tikzcd}$$

be a commutative diagram of chain complexes and chain maps whose rows are short exact. Show that the following diagram is commutative:

$$\begin{tikzcd}
H_q(C_*) \arrow[r, "\partial"] \arrow[d, "c_*"] & H_q(A_*) \arrow[d, "a_*"] \\
H_q(C'_*) \arrow[r, "\partial"] & H_{q-1}(A'_*)
\end{tikzcd}$$

This statement is referred to as the *naturality of the connecting homomorphism.* In particular, if $f \colon (X, A) \to (X', A')$ is a map between pairs of topological spaces, then the singular chain complexes of A, X, (X, A), A', X', (X', A') fit together in a diagram as above and we conclude that the diagram

$$\begin{tikzcd}
H_q(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{q-1}(A) \arrow[d, "f_*"] \\
H_q(X',A') \arrow[r, "\partial"] & H_{q-1}(A')
\end{tikzcd}$$

is commutative.

Proof. Let c be a q-cycle in the chain complex C_* and let us denote as usual by $[c] \in H_q(C_*)$ the homology class it represents. We recall that $\partial([c]) = [a]$, if there is some $b \in B_q$ with $g_q b = c$ and $\partial_q^A b = f_{q-1} a$ (where we write ∂^A for the boundary map in the chain complex A_*). We note that the commutativity of the diagram above implies that

$$g'_q(b_q b) = c_q c \quad \text{and} \quad f'_{q-1} a_{q-1} a = b_{q-1} f_{q-1} a = b_{q-1} \partial_q^A b = \partial_q^{A'} b_q b.$$

Here the last equation follows from the fact that b_* is a chain map. It follows that $\partial\colon H_q(C'_*) \to H_{q-1}(A'_*)$ maps $[c_q c]$ to $[a_q a]$. ☐

3. Let $p(z)$ be a non-constant polynomial with roots $z_1, \dots, z_l$. Show that the local degree $\deg(p, z_k)$ is equal to the multiplicity of the root z_k.

Proof. If m is the multiplicity of the root z_k, we can factor $p(z)$ in the form $p(z) = (z - z_k)^m q(z)$, where $q(z)$ is a polynomial with $q(z_k) \neq 0$. By continuity of $q(z)$, there is some neighborhood U of z_k such that for all $z \in U$ we have $||q(z) - q(z_k)|| < ||q(z_k)||$. This guarantees that the function

$$p_t(z) = (z - z_k)^m((1-t)q(z) + tq(z_k))$$

is non-zero for $z \in U$, $z \neq z_k$, $t \in [0,1]$. This in turn implies that $p_t(z)$ is a homotopy between $p(z) = p_0(z)$ and $p_1(z) = (z - z_k)^m q(z_k)$ considered as maps $(U, U \setminus \{z_k\}) \to (\mathbb{C}, \mathbb{C} \setminus \{0\})$. It follows that the local degrees $\deg(p, z_k)$, $\deg(p_1, z_k)$ are equal. Moreover, since z_k is the *only* root of the polynomial $p_1(z)$, it follows that

$$\deg(p_1, z_k) = \deg(\widehat{p_1}) = m.$$

Here $\hat{p}_1\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the extension of p_1 to a continuous map of the Riemann sphere $\hat{C} = \mathbb{C} \cup \{\infty\} \approx S^2$; the degree of $\hat{p}_1$ is equal to m, the usual degree of the polynomial p_1 by a result from class. ☐

4. Show that the complex projective space $\mathbb{CP}^n$ is a CW complex with one cell of dimension $2i$ for $0 \leq i \leq n$.

Proof. It suffices to show that $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell of dimension $2n$. Define

$$\begin{aligned} \Phi\colon D^{2n} &\longrightarrow \mathbb{C}P^n \\ z = (z_0, \dots, z_{n-1}) &\mapsto [z_0, z_1, \dots, z_{n-1}, \sqrt{1 - ||z||^2}], \end{aligned}$$

and let $\varphi\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ be the natural projection map given by $(z_0, \dots, z_{n-1}) \mapsto [z_0, z_1, \dots, z_{n-1}]$. We note that these maps are compatible in the sense that the following diagram is commutative

$$\begin{tikzcd} S^{2n-1} \arrow[r, "\varphi"] \arrow[d, "j"] & \mathbb{C}P^{n-1} \arrow[d, "i"] \\ D^{2n} \arrow[r, "\Phi"] & \mathbb{C}P^n, \end{tikzcd}$$

where i, j are the obvious inclusion maps. It follows that the map

$$\mathbb{C}P^{n-1} \cup_\varphi D^{2n} \xrightarrow{i \cup \Phi} \mathbb{C}P^n$$

is well-defined and continuous. We note that this map is surjective, since if $[z_0, \dots, z_n] \in \mathbb{C}P^n$ with $z_n \neq 0$, then multiplying all components by the unit complex number $z_n^{-1}||z_n||$, we can assume w.l.o.g. that z_n is a positive real number. Injectivity is obvious and it follows that $f = i \cup \Phi$ is a continuous bijection. It follows that f is a homeomorphism since the domain of f is compact and its range is Hausdorff. $\square$

7. HOMEWORK ASSIGNMENT # 7

1. The *Moore space* $M(\mathbb{Z}/p, n)$ for $n \geq 1$ is defined to by $M(\mathbb{Z}/p, n) = S^n \cup e^{n+1}$, where the attaching map $\varphi\colon S^n \to S^n$ of the $n+1$-cell has degree p. Calculate the reduced homology groups of the Moore space $M(\mathbb{Z}/p, n)$.

Proof. The Moore space $M(\mathbb{Z}/p, n)$ has a cell-decomposition of the form $M(\mathbb{Z}/p, n) = e^0 \cup e^n \cup e^{n+1}$ and the cellular chain complex has the following form

$$\leftarrow \mathbb{Z}e^0 \leftarrow 0 \leftarrow \dots \leftarrow 0 \leftarrow \mathbb{Z}e^n \leftarrow \mathbb{Z}e^{n+1} \leftarrow 0 \leftarrow \dots$$

According to the cellular boundary formula, $d(e^{n+1}) = ae^n$, where the integer a is the degree of the map

$$S^n \xrightarrow{\varphi} X^n \xrightarrow{p} X^n/X^{n-1} = S^n;$$

Here we write X for the Moore space $M(\mathbb{Z}/p, n)$. We note that the projection map p is just the identity here since X^{n-1} consists just of one point. Hence $a = \deg(\varphi) = p$ and it follows that

$$\widetilde{H}_q(M(\mathbb{Z}/p, n)) = \begin{cases} \mathbb{Z}/p & q = n \\ 0 & q \neq n \end{cases}$$

□

2. If X is a topological space, we can define its *Euler characteristic* as

$$\chi(X) = \sum_i (-1)^i \operatorname{rk} H_i(X),$$

provided each homology group $H_i(X)$ is finitely generated and all but finitely many homology groups are zero (in this case we say that X is of *bounded finite type*). Let X be a topological space and $A, B \subset X$ subspaces with $X = \operatorname{int}(A) \cup \operatorname{int}(B)$.
a) Show that if A, B and $A \cup B$ are of bounded finite type, then so is X.
b) Show that in this case

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

We remark that the Euler characteristic of a topological space X of bounded finite type is a generalization of the cardinality of a finite set, since if X is a finite set, then its Euler characteristic equals its cardinality. We note that the formula to be proved in (b) obviously holds if X is a finite set.

Proof. Consider the Mayer-Vietoris sequence
(8)

$$\longrightarrow H_q(A \cap B) \longrightarrow H_q(A) \oplus H_q(B) \longrightarrow H_q(X) \longrightarrow H_{q-1}(A \cap B) \longrightarrow$$

First we observe that since A, B, and $A \cap B$ are of bounded finite type, there is some N such that for $q \geq N$ the homology groups $H_q(A)$, $H_q(B)$, $H_{q-1}(A \cap B)$ are all zero. By the exactness of the sequence above, this implies $H_q(X) = 0$ for $q \geq N$.

To prove that all homology groups $H_q(X)$ are finitely generated, let us write f_q for the map from $H_q(A \cap B)$ to $H_q(A) \oplus H_q(B)$. Then the exact sequence (8) gives rise to short exact sequences of the form

$$0 \longrightarrow \operatorname{coker}(f_q) \longrightarrow H_q(X) \longrightarrow \ker(f_{q-1}) \longrightarrow 0.$$

We note that $\ker(f_{q-1})$ is finitely generated (as a subgroup of the finitely generated abelian group $H_{q-1}(A \cap B)$) and $\operatorname{coker}(f_q)$ is finitely generated (as quotient of the finitely generated group $H_q(A) \oplus H_q(B)$). This implies by the short exact sequence above that $H_q(X)$ is finitely generated.

To prove part (b) we recall that in class we showed that if C_* is a chain complex of bounded finite type (i.e., all the groups A_q are finitely generated, and only finitely many of them are non-zero), then

$$\sum_{q\in\mathbb{Z}}(-1)^q \operatorname{rk} C_q = \sum_{q\in\mathbb{Z}}(-1)^q \operatorname{rk} H_q(C_*).$$

In particular, we see that if C_* is an *exact sequence*, then all homology of C_* vanish and hence the alternating sum $\sum_{q\in\mathbb{Z}}(-1)^q \operatorname{rk} C_q$ is zero.

Now we apply this to the Mayer-Vietoris sequence (8). The alternating sum here is equal to

$$\begin{aligned}&\sum_{q\in\mathbb{Z}}(-1)^q \left(\operatorname{rk} H_q(A\cap B) - \operatorname{rk} H_q(A) - \operatorname{rk} H_q(B) + \operatorname{rk} H_q(X)\right)\\ =&\chi(A\cap B) - \chi(A) - \chi(B) + \chi(X)\end{aligned}$$

Since this alternating sum is zero due to the observation above, the desired formula for $\chi(X)$ follows. $\square$

3. If X, Y are finite CW complexes, then the product $X\times Y$ again has a CW structure whose cells correspond to products of cells of X and Y. More precisely, if e^m_α is an m-cell of X, and e^n_β is an n-cell of Y, then these determine a $m+n$-cell of $X\times Y$ denoted $e^m_\alpha \times e^n_\beta$. In particular, $C^{CW}_q(X\times Y)$ is the free abelian group generated by products cells $e^m_\alpha\times e^n_\beta$ with $m+n=q$. The cellular boundary map is determined by the formula

$$d(e^m_\alpha \times e^n_\beta) = (de^m_\alpha)\times e^n_\beta + (-1)^m e^m_\alpha \times d(e^n_\beta). \tag{9}$$

a) Calculate $H_*(S^m\times S^n)$.
b) Calculate the homology groups of $\underbrace{S^n\times\cdots\times S^n}_{k}$.

Proof. To calculate the homology of $S^m\times S^n$, let us furnish the sphere S^m with the CW structure consisting of one 0-cell e^0 and one m-cell e^m. Then the product CW structure on $S^m\times S^n$ has four cells $e^0\times e^0$, $e^m\times e^0$, $e^0\times e^n$, $e^m\times e^n$ of dimension 0, m, n and $m+n$. The cellular boundary map in the cellular chain complex of S^m and S^n is zero, and hence the 'product rule' (9) implies that the cellular boundary map for $S^m\times S^n$ is trivial. Hence $H_q(S^m\times S^n)\cong C^{CW}_q(S^m\times S^n)$ is a direct sum of as many copies of $\mathbb{Z}$ as there are cells of dimension q. In particular, for $m, n > 0$ and $m \neq n$ we have

$$H_q(S^m \times S^n) \cong \begin{cases} \mathbb{Z} & q = 0, m, n, m+n \\ 0 & \text{otherwise} \end{cases}$$

If $n > 0$ we have:

$$H_q(S^n \times S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & q = n \\ \mathbb{Z} & q = 0, 2n \\ 0 & \text{otherwise} \end{cases}$$

The case $m = 0$ (or $n = 0$) is not so interesting, since we see immediately that $S^0 \times S^n$ is the disjoint union of two copies of S^n, and hence its homology groups are just the direct sum of two copies of the homology groups of S^n.

As in part (a) we see that the boundary maps in the cellular chain complex for $S^n \times \cdots \times S^n$ are zero, and hence the q-th homology group is a direct sum of as many copies of $\mathbb{Z}$ as there are q-cells in this product. Let me use the following useful gadget to keep track of the number of cells. Suppose X is a finite CW complex then define the *Poincaré series* $P(X)$ to be the power series

$$P(X) = a_0 + a_1 z + a_2 z^2 + \ldots$$

where the coefficient a_q is the number of q-cells of X. For example, $P(S^n) = 1 + z^n$ and from part (a) we know

$$P(S^m \times S^n) = 1 + z^m + z^n + z^{m+n} = (1 + z^m)(1 + z^n) = P(S^m) \cdot P(S^n).$$

This suggests the following result:

Lemma 1. *If X, Y are finite CW complexes, then*

$$P(X \times Y) = P(X) \cdot P(Y)$$

Let a_q be the number of q-cells of X, and let b_q be the number of q-cells of Y. We recall that the product cell $e^m_\alpha \times e^n_\beta$ has dimension $m + n$. This implies that the number of q-cells of $X \times Y$ is

$$a_0 b_q + a_1 b_{q-1} + \cdots + a_{q-1} b_1 + a_q b_0$$

We observe that this sum is also the coefficient of z^q in

$$P(X) \cdot P(Y) = (a_0 + a_1 z + a_2 z^2 + \ldots)(b_0 + b_1 z + b_2 z^2 + \ldots)$$

which proves the lemma.

In particular,

$$P(\underbrace{S^n \times \cdots \times S^n}_{k}) = P(S^n)^k = (1 + z^n)^k = \sum_{i=0}^{k} \binom{k}{i} z^{ni}$$

We conclude that the q-th homology group of $\underbrace{S^n \times \cdots \times S^n}_{k}$ is trivial for q not divisible by n. If $q = ni$, then the q-th homology group is a direct sum of $\binom{k}{i}$ copies of $\mathbb{Z}$. ☐

4. a) Calculate the homology groups of $\mathbb{R}P^2 \times \mathbb{R}P^2$.
b) Generalizing part a), calculate the homology groups of a product of two Moore spaces $M(\mathbb{Z}/p, m) \times M(\mathbb{Z}/q, n)$.

Proof. Part a) is good as a 'warm-up' for doing the general case of part b), but here let us go straight for the general case. The Moore space $M(\mathbb{Z}/p, m)$ has three cells e^0, e^m, e^{m+1} whose dimensions are indicated by the superscripts. Hence the product $M(\mathbb{Z}/p, m) \times M(\mathbb{Z}/q, n)$ has nine cells. To simplify matters, we note that the cell e^0 does not 'interact' with the cells e^m, e^{m+1} in the cellular chain complex of $M(\mathbb{Z}/p, m)$ in the sense that it is a direct sum of chain complexes; the first summand is generated by e^0, the other is generated by e^m and e^{m+1}. It follows that the chain complex of the product can be written as a direct sum of *four* chain complexes:

$$\begin{aligned}
A_* &= \langle e^0 \times e^0 \rangle \\
B_* &= \langle e^0 \times e^n, e^0 \times e^{n+1} \rangle \\
C_* &= \langle e^m \times e^0, e^{m+1} \times e^0 \rangle \\
D_* &= \langle e^m \times e^n, e^m \times e^{n+1}, e^{m+1} \times e^n, e^{m+1} \times e^{n+1} \rangle
\end{aligned}$$

The boundary maps of these chain complexes are readily determined by the product formula (9). Also the homology groups of A_*, B_* and C_* are straightforward to determine and so we only state the result:

$$H_i(A_*) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H_i(B_*) = \begin{cases} \mathbb{Z}/q & i = n \\ 0 & i \neq n \end{cases}$$

$$H_i(C_*) = \begin{cases} \mathbb{Z}/p & i = m \\ 0 & i \neq m \end{cases}$$

To calculate the homology groups of the chain complex D_*, we first determine the boundary maps of D_*:

$$\begin{aligned}
d(e^m \times e^n) &= 0 \\
d(e^{m+1} \times e^n) &= p(e^m \times e^n) \\
d(e^m \times e^{n+1}) &= (-1)^m q(e^m \times e^n) \\
d(e^{m+1} \times e^{n+1}) &= p(e^m \times e^{n+1}) + (-1)^{m+1} q(e^{m+1} \times e^n)
\end{aligned}$$

To determine the homology groups it is convenient to use a different basis for the chain group D_{m+n+1}. Let $g = \gcd pq$, and write $p = gp'$, $q = gq'$. Then p' and q' are relatively prime and hence there are integers a, b such that $ap' + bq' = 1$. We note that the element $d(e^{m+1} \times e^{n+1})$ is divisible by g. Let $e_1 \in D_{m+n+1}$ be the element obtained by that division, i.e.,

$$e_1 := p'(e^m \times e^{n+1}) + (-1)^{m+1} q'(e^{m+1} \times e^n)$$

and define

$$e_2 := (-1)^m b(e^m \times e^{n+1}) + a(e^{m+1} \times e^n)$$

This is taylor made so that the base change matrix

$$\begin{pmatrix} p' & (-1)^m b \\ (-1)^{m+1} q' & a \end{pmatrix}$$

has determinant one which implies that $\{e_1, e_2\}$ is in fact a new basis for D_{m+n+1}. The advantage of this new basis is that the boundary map of the chain complex D_* has the following simple form

$$\begin{aligned}
d(e^{m+1} \times e^{n+1}) &= ge_1 \\
d(e_1) &= 0 \\
d(e_2) &= (-1)^m b d(e^m \times e^{n+1}) + a d(e^{m+1} \times e^n) \\
&= (-1)^m b (-1)^m q(e^m \times e^n) + ap(e^m \times e^n) \\
&= (bq + ap)(e^m \times e^n) = g(e^m \times e^n)
\end{aligned}$$

It follows that

$$H_q(D_*) = \begin{cases} \mathbb{Z}/g & q = m+n, m+n+1 \\ 0 & \text{otherwise} \end{cases}$$

□

8. HOMEWORK ASSIGNMENT # 8

1. Let $H\colon X\times I \to Y$ be a homotopy between the maps $f\colon X\to Y$ and $g\colon X\to Y$. Let $f_\#, g_\#\colon C_*(X)\to C_*(Y)$ be the chain maps induced by f, g on the singular chain complexes. Show that the *prism operator* $P\colon C_n(X)\to C_{n+1}(Y)$ defined by

$$P(\sigma) := \sum_{i=0}^{n}(-1)^i H\circ(\sigma\times \mathbb{1})\circ[v_0,\dots,v_i,w_i,\dots,w_n]$$

is a chain homotopy from $f_\#$ to $g_\#$, i.e.,

$$\partial P + P\partial = g_\# - f_\#$$

Proof.

$$\begin{aligned}\partial P(\sigma) = &\sum_{0\le j\le i\le q}(-1)^i(-1)^j H\circ(\sigma\times \mathbb{1})\circ[v_0,\dots,\widehat{v}_j,\dots,v_i,w_i,\dots,w_q]\\ &+\sum_{0\le i\le j\le q}(-1)^i(-1)^{j+1} H\circ(\sigma\times \mathbb{1})\circ[v_0,\dots,v_i,w_i,\dots,\widehat{w}_j,\dots,w_q].\end{aligned}$$

We note that

$$[v_0,\dots,v_{i-1},\widehat{v}_i,w_i,\dots,w_q] = [v_0,\dots,v_{i-1},\widehat{w}_{i-1},w_i,\dots,w_q]$$

and hence for any k with $0<k\le q$ the $i=j=k$ summand of the first sum and the $i=j=k-1$ of the second sum cancel each other. So the only surviving contribution with $i=j$ to the first sum is

$$H\circ(\sigma\times \mathbb{1})\circ[\widehat{v}_0,w_0,\dots,w_q] = g\circ\sigma = g_\#\sigma.$$

The only surviving contribution with $i=j$ to the second sum is

$$-H\circ(\sigma\times \mathbb{1})\circ v_0,\dots,v_q,\widehat{w}_q] = -f\circ\sigma = -f_\#\sigma.$$

The terms with $i\neq j$ are exactly $-P\partial(\sigma)$, since

$$\partial\sigma = \sum_{j=0}^{q}(-1)^j\sigma\circ[e_0,\dots,\widehat{e}_j,\dots,e_q]$$

and hence

$$\begin{aligned}P\partial(\sigma) = &\sum_{0\le i<j\le q}(-1)^i(-1)^j H\circ(\sigma\times \mathbb{1})\circ[v_0,\dots,v_i,w_i,\dots,\widehat{w}_j,\dots,w_q]\\ &+\sum_{0\le j<i\le q}(-1)^{i-1}(-1)^j H\circ(\sigma\times \mathbb{1})\circ[v_0,\dots,\widehat{v}_j,\dots,v_i,w_i,\dots,w_q].\end{aligned}$$

It follows that $\partial P + P\partial = g_\# - f_\#$ as claimed. $\square$

2. Suppose that

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow D \xrightarrow{g} E$$

is an exact sequence of abelian groups. Show that it gives rise to a short exact sequence

$$0 \longrightarrow \operatorname{coker} f \longrightarrow C \longrightarrow \ker g \longrightarrow 0$$

Proof. Let us write $h\colon B \to C$ and $k\colon C \to D$ for the given maps. We note that by exactness at B the image of f is equal to the kernel of h, and hence h induces a well-defined monomorphism

$$\bar{h}\colon \operatorname{coker} f = B/\operatorname{im} f \longrightarrow C.$$

Similarly, the image of k is equal to the kernel of g by exactness at D, and hence k is a surjective map from C to $\ker g$. Furthermore,

$$\operatorname{im} \bar{h} = \operatorname{im} h = \ker k,$$

by exactness of the original sequence at C and hence we have shown that the sequence

$$0 \longrightarrow \operatorname{coker} f \xrightarrow{\bar{h}} C \xrightarrow{k} \ker g \longrightarrow 0$$

is exact. □

3. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of abelian groups. We say that it is *split exact* (or that *the sequence splits*) if there is a homomorphism $s\colon C \to B$ which is a right-inverse to g in the sense that $gs = \mathbb{1}$. Show that if the sequence splits, then B is isomorphic to $A \oplus C$.

Proof. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "1"'] & A \oplus C \arrow[r, "p"] \arrow[d, "f\oplus s"'] & C \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

where i and p are the obvious inclusion resp. projection maps. The left square evidently commutes and the right square commutes since $g \circ f = 0$ and $g \circ s = \mathbb{1}$. Both rows are exact and hence the 5-Lemma implies that $f \oplus s$ is an isomorphism. □

4. Show that if X is a finite CW complex, then its suspension ΣX has the structure of a finite CW complex (this is true without the finiteness assumption, but this avoids pointset topology issues). If C_* is a chain complex, define its *suspension* ΣC_* in such a way that the cellular chain complex $C_*^{CW}(\Sigma X)$ is isomorphic to the suspension of the celluar chain complex $C_*^{CW}(X)$.

Proof. We recall that $\Sigma X = X \times I/\sim$, $I = [0,1]$, where the equivalence relation $\sim$ is generated by $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for $x, x' \in X$. We claim that ΣX has a CW structure with two 0-cells (= points), namely $\bar{e}_0^0 := [x,0]$ and $\bar{e}_1^0 = [x,1]$ and one $(q+1)$-cell $\bar{e}_\alpha^{q+1}$ for each q-cell e_α^q of X. If $\Phi_\alpha \colon I_\alpha^q \to X$ is the characteristic map of the q-cell e_α^q of X, we want the map

$$\bar{\Phi}_\alpha \colon I^{q+1} = I^q \times I \xrightarrow{\Phi_\alpha^q \times 1} X \times I \xrightarrow{p} \Sigma X$$

to be the characteristic map of a $(q+1)$-cell $\bar{e}_\alpha^{q+1}$ of ΣX, where p is the projection map. This suggests to define the q-skeleton of ΣX for $q > 0$ to be

$$(\Sigma X)^q := \Sigma(X^{q-1}) \subset \Sigma X.$$

We note that with this definition the map $\bar{\Phi}_\alpha$ maps the boundary

$$\partial I^{q+1} = \partial I^q \times I \cup I^p \times \partial I$$

of the $(q+1)$-cube I^{q+1} indeed to the q-skeleton, since for $x \in \partial I^q$ the point $\Phi_\alpha(x)$ belongs to X^{q-1}, and hence for any $t \in I$,

$$\bar{\Phi}_\alpha(x,t) = [\Phi_\alpha(x), t] \in (\Sigma X)^q.$$

For $(x,t) \in I^q \times \partial I$, we have $t = 0$ or $t = 1$, and hence $\bar{\Phi}_\alpha(x,t)$ is in the 0-skeleton of ΣX consisting of the two points $[x,0]$ and $[x,1]$.

It remains to show that $(\Sigma X)^{q+1}$ is obtained from $(\Sigma X)^q$ by attaching $(q+1)$-cells via the attaching maps

$$\bar{\varphi}_\alpha := (\bar{\Phi}_\alpha)_{|\partial I^{q+1}} \colon \partial I^{q+1} \longrightarrow (\Sigma X)^q.$$

To prove this, we note that the inclusion map $(\Sigma X)^q \hookrightarrow (\Sigma X)^{q+1}$ and the maps $\bar{\Phi}_\alpha$ fit together to give a well-defined continuous map

$$\left((\Sigma X)^q \amalg \coprod_\alpha I_\alpha^{q+1} \right) / \sim \quad \longrightarrow \quad (\Sigma X)^{q+1}.$$

It is not hard to check that this is a bijection; since the spaces involved are compact and Hausdorff, this is in fact a homeomorphism.

To determine the cellular chain complex of ΣX, it is convenient to compare it with the cellular chain complex of $X \times I$ via the projection map $p \colon X \times I \to \Sigma X$. Let us equip I with its standard CW structure (consisting of two 0-cells e_0^0 and e_1^0 – the points $0, 1 \in I$ – and one

1-cell e^1), and give $X \times I$ its CW structure as a product of two CW complexes. We note that the projection map $p\colon X \times I \to \Sigma X$ is a map of CW complexes in the sense that for every $q \geq 0$ the image of the q-skeleton of $X \times I$ under p is contained in the q-skeleton of ΣX. This implies that p induces a chain map

$$p_{\#}\colon C_*^{CW}(X \times I) \longrightarrow C_*^{CW}(\Sigma X)$$

of cellular chain complexes. We also note that the characteristic map of the cell $e^1 \times e^q_\alpha$ composed with p is the characteristic map $\bar{\Phi}_\alpha$. This implies that the chain map $p_{\#}$ sends the generator $e^1 \times e^q_\alpha \in C^{CW}_{q+1}(I \times X)$ to the generator $\bar{e}^{q+1}_\alpha \in C^{CW}_{q+1}$. We further observe that the images of the characteristic maps of the cells $e^0_i \times e^q_\alpha$, $i = 0, 1$ composed with p are contained in the 0-skeleton of ΣX. This implies that $p_{\#} e^0_i \times e^q_\alpha = 0$ for $q > 0$. We also note that for any 0-cell e^0_α, the projection map p sends the 0-cell $e^0_i \times e^0_\alpha$ of $I \times X$ to the 0-cell e^0_i of ΣX for $i = 0, 1$.

This information suffices to determine the boundary map in $C_*^{CW}(\Sigma X)$ in terms of the boundary map of $C_*^{CW}(X)$. Assume that

$$\partial e^q_\alpha = \sum_\beta n_{\alpha\beta} e^{q-1}_\beta.$$

Then

$$\begin{aligned}
\partial \bar{e}^{q+1}_\alpha &= \partial p_{\#}(e^q_\alpha \times e^1) = p_{\#}\partial(e^q_\alpha \times e^1) \\
=& p_{\#}\left((\sum_\beta n_{\alpha\beta} e^{q-1}_\beta) \times e^1 + e^q_\alpha \times \left((-1)^q (e^0_1 - e^0_0)\right) \right) \\
=& \sum_\beta n_{\alpha\beta} p_{\#}(e^{q-1}_\beta \times e^1) = \sum_\beta n_{\alpha\beta} \bar{e}^q_\beta
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial \bar{e}^1_\alpha &= \partial p_{\#}(e^0_\alpha \times e^1) = p_{\#}\partial(e^0_\alpha \times e^1) = p_{\#}(e^0_\alpha \times (\partial e^1)) \\
=& p_{\#}(e^0_\alpha \times (e^0_1 - e^0_0)) = \bar{e}^0_1 - \bar{e}^0_0
\end{aligned}$$

□

9. HOMEWORK ASSIGNMENT # 9

1. Let Y be a convex subset of a euclidean space, and let $AC_*(Y) \subset C_*(Y)$ be the subcomplex generated by affine linear simplices. Let

$$S\colon AC_q(Y) \longrightarrow AC_q(Y) \qquad T\colon AC_q(Y) \longrightarrow AC_{q+1}(Y) \tag{10}$$

be the linear maps constructed in class. We recall that the 'barycentric subdivision map' is a chain map $AC_*(Y) \to AC_*(Y)$, and that T is a chain homotopy from S to $\mathbb{1}$, i.e.,

$$T\partial + \partial T = \mathbb{1} - S.$$

We extend S and T to linear maps

$$S\colon C_q(X) \longrightarrow C_q(X) \qquad T\colon C_q(X) \longrightarrow C_{q+1}(X)$$

on chains on any space X by defining for a simplex $\sigma\colon \Delta^q \to X$

$$S(\sigma) = \sigma_\# S[e_0, \dots, e_q] \qquad T(\sigma) = \sigma_\# T[e_0, \dots, e_q].$$

Here $[e_0, \dots, e_q]\colon \Delta^n \to \Delta^n$ is the affine linear simplex given by the identity map; in particular, we can apply the maps (10) to it (for $Y = \Delta^q$) and obtain affine linear chains $S[e_0, \dots, e_q] \in AC_q(\Delta^q) \subset C_q(\Delta^q)$ (resp. $T[e_0, \dots, e_q] \in AC_{q+1}(\Delta^q) \subset C_{q+1}(\Delta^q)$). Then we can apply the chain map

$$\sigma_\#\colon C_*(\Delta^q) \longrightarrow C_*(X)$$

induced by $\sigma\colon \Delta^q \to X$ to obtain a chain in $C_*(X)$.
(a) Show that $S\colon C_*(X) \longrightarrow C_*(X)$ is a chain map.
(b) Show that T is a chain homotopy from S to the identity of $C_*(X)$.

Proof. This problem is more involved than I thought. Hatcher's argument is not complete. There is an important property of S and T that is needed here:

Let Y, Y' be convex subsets of some euclidean vector spaces and let $f\colon Y \to Y'$ be an affine linear map, i.e.,

$$f(\sum t_i w_i) = \sum t_i f(w_i) \qquad w_i \in Y,\ t_i \in \mathbb{R}_+,\ \sum t_i = 1.$$

In particular, if $\lambda\colon \Delta^q \to Y$ is an affine linear simplex, then $f \circ \lambda$ is an affine linear simplex in Y', and hence f induces a chain map

$$f_\#\colon \widetilde{AC}_*(Y) \longrightarrow \widetilde{AC}_*(Y') \qquad \lambda \mapsto f \circ \lambda.$$

Lemma 2. *$f_\# S = S f_\#$ and $f_\# T = T f_\#$.*

The inductive construction of S and T involves the homomorphism

$$b\colon \widetilde{AC}_q(Y) \longrightarrow \widetilde{AC}_{q+1}(Y) \qquad [w_0, \dots, w_q] \mapsto [b, w_0, \dots, w_q].$$

We calculate:

$$\begin{aligned} f_\# b[w_0, \dots, w_q] &= f \circ [b, w_0, \dots, w_q] = [f(b), f(w_0), \dots, f(w_q)] \\ &= f(b)[f(w_0), \dots, f(w_q)] = f(b) f_\#[w_0, \dots, w_q]. \end{aligned}$$

We recall that S is defined inductively by $S(\lambda) = b_\lambda S\partial\lambda$, where λ is an affine q-simplex, b_λ is its barycenter, and $S\partial\lambda$ is already defined since $\partial\lambda$ is a linear combination of affine $(q-1)$-simplices. Then using induction over q we have

$$f_\# S(\lambda) = f(b_\lambda) f_\# S\partial\lambda = b_{f_\#\lambda} S\partial f_\#\lambda = S f_\#\lambda$$

Here the second equality holds since the affine linear map sends the barycenter of λ to the barycenter of $f \circ \lambda = f_\#\lambda$, and since S and $f_\#$ commute by inductive hypothesis when applied to $(q-1)$-chains.

The argument for T is entirely similar. We recall that T is defined inductively by $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ and calculate using induction over q:

$$\begin{aligned} f_\# T\lambda = f_\# b_\lambda(\lambda - T\partial\lambda) &= f(b_\lambda) f_\#(\lambda - T\partial\lambda) \\ &= b_{f_\#\lambda}(f_\#\lambda - T\partial f_\#\lambda) = T f_\#\lambda \end{aligned}$$

Now we prove that $S\colon C_*(X) \to C_*(X)$ is a chain map

$$\begin{aligned} S(\partial\sigma) =& S\left(\sum(-1)^i \sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_q]\right) \\ =& \sum(-1)^i S(\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_q]) \\ =& \sum(-1)^i (\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_q])_\# S[e_0 \dots, e_{q-1}] \\ =& \sum(-1)^i \sigma_\# \circ [e_0, \dots, \widehat{e_i}, \dots, e_q]_\# S[e_0 \dots, e_{q-1}] \\ =& \sum(-1)^i \sigma_\# S[e_0, \dots, \widehat{e_i}, \dots, e_q]_\# [e_0 \dots, e_{q-1}] \\ =& \sum(-1)^i \sigma_\# S[e_0, \dots, \widehat{e_i}, \dots, e_q] \\ =& \sigma_\# S\partial[e_0, \dots, e_q] \end{aligned}$$

Furthermore, we have

$$\sigma_\# S\partial[e_0, \dots, e_q] = \sigma_\# \partial S[e_0, \dots, e_q] = \partial\sigma_\# S[e_0, \dots, e_q] = \partial S\sigma$$

which shows that $S\colon C_*(X) \to C_*(X)$ is a chain map.

The reason for splitting the calculation above in two pieces is that the the first calculation holds with S replaced by T. This allows us to compute

$$\begin{aligned} T(\partial\sigma) =& \sigma_\# T\partial[e_0, \dots, e_q] \\ =& \sigma_\#(-\partial T[e_0, \dots, e_q] + [e_0, \dots, e_q] - S[e_0, \dots, e_q]) \\ =& -\partial\sigma_\# T[e_0, \dots, e_q] + \sigma - \sigma\# S[e_0, \dots, e_q] \\ =& -\partial T\sigma + \sigma - S\sigma \end{aligned}$$

□

2. Show that for $k, l \geq 1$

$$\widetilde{H}_q((S^k \times D^l)/(S^k \times S^{l-1})) = \begin{cases} \mathbb{Z} & q = l, k+l \\ 0 & q \neq l, k+l \end{cases}$$

Proof. We claim that $(S^k \times D^l, S^k \times S^{l-1})$ is a good pair. This follows by crossing with S^k if we can show that (D^l, S^{l-1}) is a good pair, i.e., S^{l-1} is a closed subset of D^l (this is clear) and that S^{l-1} is a deformation retract of some neighborhood of S^{l-1}. To prove the latter, we note that $D^l \setminus \{0\}$ is an open neighborhood of S^{l-1}, and $r \colon D^l \setminus \{0\} \to S^{l-1}$ given by $x \mapsto x/||x||$ is a retraction. The composition of r with the inclusion map $i \colon S^{l-1} \to D^l \setminus \{0\}$ is homotopic to the identity map relative S^{l-1} via the homotopy

$$H \colon D^l \setminus \{0\} \times [0,1] \longrightarrow D^l \setminus \{0\} \qquad (x,t) \mapsto (1-t)\frac{x}{||x||} + tx.$$

We recall that the homology groups of good pairs (X, A) can be identified with the homology groups of the quotient X/A. Hence the long exact homology sequence of the good pair $(S^k \times D^l, S^k \times S^{l-1})$ leads to the following exact sequence:

$$\widetilde{H}_q(S^k \times S^{l-1}) \xrightarrow{i_*} \widetilde{H}_q(S^k \times D^l) \longrightarrow \widetilde{H}_q((S^k \times D^l/S^k \times S^{l-1})$$
$$\xrightarrow{\partial} \widetilde{H}_{q-1}(S^k \times S^{l-1}) \xrightarrow{i_*} \widetilde{H}_{q-1}(S^k \times D^l) \longrightarrow$$

Breaking this up we obtain the short exact sequences

$$0 \longrightarrow (\operatorname{coker} i_*)_q \longrightarrow \widetilde{H}_q((S^k \times D^l/S^k \times S^{l-1}) \longrightarrow (\ker i_*)_{q-1} \longrightarrow 0,$$

where we write $(\ker i_*)_q$ (resp. $(\operatorname{coker} i_*)_q$) for kernel resp. cokernel of the homomorphism

$$i_* \colon \widetilde{H}_q(S^k \times S^{l-1}) \xrightarrow{i_*} \widetilde{H}_q(S^k \times D^l).$$

We claim that i_* is surjective. To show this, we pick a point $y_0 \in S^{l-1}$ and consider the map

$$s \colon S^k \times D^l \longrightarrow S^k \times S^{l-1} \qquad (x,y) \mapsto (x, y_0).$$

This map is a *right inverse to i up to homotopy*, meaning that the composition $i \circ s \colon S^k \times D^l \to S^k \times D^l$ is homotopic to the identity. This follows from the fact that $i \circ s$ is the product of the identity map on S^k times the constant map on D^l and the constant map on the contractible space D^l is homotopic to the identity map. We conclude that $i_* \circ s_* = (i \circ s)_* = \mathbb{1}$, and so s_* is a right inverse of i_*. This implies that i_* is surjective, hence the cokernel of i_* is zero and we obtain an isomorphism

$$\widetilde{H}_q(S^k \times D^l / S^k \times S^{l-1}) \cong (\ker i_*)_{q-1} \tag{11}$$

for all q.

The space $S^k \times S^{l-1}$ with its standard CW structure has four cells of dimension 0, k, $l-1$ and $k+l-1$, respectively. Each of these cells contributes a copy of $\mathbb{Z}$ to the homology group in the appropriate dimension (since the boundary maps in the cellular chain complex are zero). Passing from H_* to $\widetilde{H}_*$ we loose a copy of $\mathbb{Z}$ in degree 0 and passing to the kernel of i_*, we loose a $\mathbb{Z}$ in degree k. We conclude that

$$(\ker i_*)_q \cong \begin{cases} \mathbb{Z} & q = l-1, k+l-1 \\ 0 & q \neq l-1, k+l-1 \end{cases}$$

which implies the desired statement by putting it together with the isomorphism (11). □

3. Suppose M is a compact manifold of dimension n, and suppose $f\colon S^k \times D^{n-k} \to M$ is an embedding (i.e., a homeomorphism onto its image). Let $M' := M \setminus f(S^k \times \operatorname{int}(D^{n-k}))$, and let

$$\widehat{M} := M' \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1};$$

in other words, $\widehat{M}$ is obtained from the disjoint union

$$M' \amalg D^{k+1} \times S^{n-k-1}$$

by identifying a point $(x, y) \in S^k \times S^{n-k-1} \subset D^{k+1} \times S^{n-k-1}$ with $f(x, y) \in M'$. It is not hard to show that $\widehat{M}$ is again a closed n-manifold. This is an important way to modify a manifold M called *surgery*. More precisely we say that *$\widehat{M}$ is obtained by a k-surgery* from M. The effect on homology groups is easiest to determine for $k < m$ for $n = 2m$ or $n = 2m+1$ (m is called the 'middle dimension of M' and consequently this is called 'surgery below the middle dimension'). So let us assume $k < m$ for the following.

(a) Show that the inclusion $M' \to M$ induces an isomorphism on H_q for $q < m$. Hint: consider the long exact sequence of the pair (M, M'), show that this is a good pair, and identify M/M'.

(b) Show that the inclusion $M' \to \widehat{M}$ induces an isomorphism on H_q for $q < n-1$, $q \neq k, k+1$. Hint: consider the long exact sequence of the pair $(\widehat{M}, M')$, show that this is a good pair, and identify $\widehat{M}/M'$.

(c) Let g be the composition $\mathbb{Z} \cong H_k(S^k) \cong H_k(S^k \times D^{n-k}) \xrightarrow{f_*} H_k(M)$. Show that $H_k(\widehat{M}) \cong H_k(M)/g(\mathbb{Z})$ and

$$H_{k+1}(\widehat{M}) = \begin{cases} H_{k+1}(M) & g \text{ injective} \\ H_{k+1}(M) \oplus \mathbb{Z} & g \text{ not injective} \end{cases}$$

Hint: in order to identify the boundary homomorphisms in the exact sequence of the pair $(\widehat{M}, M')$, compare with the long exact sequence of the pair (D^{k+1}, S^k).

Proof. To prove part (a) we note that the embedding f induces a homeomorphism

$$(S^k \times D^{n-k})/(S^k \times S^{n-k-1}) \approx M/M'$$

The argument that (M, M') is a good pair is completely analogous to the argument we used in problem 2: $M \setminus f(S^k \times \{0\})$ is an open neighborhood of M' that has M' as a deformation retract. Using these facts, the exact homology sequence of the pair (M, M') takes the following form:

$$\xrightarrow{\partial} \widetilde{H}_q(M') \xrightarrow{i_*} \widetilde{H}_q(M) \longrightarrow \widetilde{H}_q(S^k \times D^{n-k}/S^k \times S^{n-k-1}) \xrightarrow{\partial}$$

By the previous problem, $\widetilde{H}_q(S^k \times D^{n-k}/S^k \times S^{n-k-1}) = 0$ except for $q = n-k, n$. Hence the exact sequence implies that i_* is an isomorphism except for $q = n-k-1, n-k, n-1, n$. We note that our assumptions imply

$$n-k-1 \geq 2m-k-1 \geq 2m-(m-1)-1 = m,$$

and so for $q < m$ the map i_* is an isomorphism as claimed.

On to part (b): the argument that $(\widehat{M}, M')$ is a good pair is the same as that for (M, M'). We have a homeomorphism

$$\widehat{M}/M' \approx (D^{k+1} \times S^{n-k-1})/(S^k \times S^{n-k-1})$$

which using the result of problem (2) implies that $\widetilde{H}_q(\widehat{M}/M') = 0$ for $q \neq k+1, n$. Let $j\colon M' \to \widehat{M}$ be the inclusion map and consider the exact homology sequence of the pair $(\widehat{M}, M')$:

$$\longrightarrow \widetilde{H}_{q+1}(\widehat{M}/M') \xrightarrow{\partial} \widetilde{H}_q(M') \xrightarrow{j_*} \widetilde{H}_q(\widehat{M}) \longrightarrow \widetilde{H}_q(\widehat{M}/M) \longrightarrow$$

It follows that j_* is an isomorphism for $q < n$, $q \neq k, k+1$.

To do part (c) we note that the exact sequence above implies an isomorphism

$$\widetilde{H}_k(\widehat{M}) \cong \operatorname{coker}\Big(\partial\colon \widetilde{H}_{k+1}(\widehat{M}/M') \to \widetilde{H}_k(M')\Big)$$

and a short exact sequence

$$(12)\quad \widetilde{H}_{k+1}(M') \longrightarrow \widetilde{H}_{k+1}(\widehat{M}) \longrightarrow \ker\Big(\widetilde{H}_{k+1}(\widehat{M}/M') \xrightarrow{\partial} \widetilde{H}_k(M')\Big)$$

So we need to analyze the connecting homomorphism ∂. As usual in algebraic topology, we don't do that by going back to the definition of the connecting homomorphism (coming from the short exact sequence of chain complexes), but rather by comparison with the exact sequence of a different pairs. We pick a point $y_0 \in S^{n-k-1}$, and consider the following maps of pairs:

$$(D^{k+1}, S^k) \xrightarrow{r} (D^{k+1} \times S^{n-k-1}, S^k \times S^{n-k-1}) \xrightarrow{s} (\widehat{M}, M') \ .$$

Here r maps $x \in D^{k+1}$ to $(x, y_0) \in D^{k+1} \times S^{n-k-1}$ and s is the obvious inclusion map which induces the homeomorphism between quotient spaces we used above. In particular,

$$s_*\colon \widetilde{H}_q(D^{k+1} \times S^{n-k-1}/S^k \times S^{n-k-1}) \longrightarrow \widetilde{H}_q(\widehat{M}/M')$$

is an isomorphism. We note that the map r has a left-inverse given by the projection onto the first factor. In other words, $p \circ r = \mathbb{1}$ and hence $r_* \circ h_*$ is the identity on $\widetilde{H}_{k+1}(D^{k+1}/S^k)$. Since

$$\widetilde{H}_{k+1}(D^{k+1} \times S^{n-k-1}/S^k \times S^{n-k-1}) \cong \mathbb{Z}$$

by problem # 2, it follows that

$$r_*\colon H_{k+1}(D^{k+1}/S^k) \longrightarrow \widetilde{H}_{k+1}(D^{k+1} \times S^{n-k-1}/S^k \times S^{n-k-1})$$

is an isomorphism. Putting these statements together, we see that the composition $t := s \circ r\colon (D^{k+1}, S^k) \to (\widehat{M}, M')$ induces an isomorphism $t_*\colon \widetilde{H}_{k+1}(D^{k+1}/S^k) \to \widetilde{H}_{k+1}(\widehat{M}/M')$.

Now consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{H}_{k+1}(D^{k+1}/S^k) \arrow[r, "\partial", "\cong"'] \arrow[d, "t_*"', "\cong"] & \widetilde{H}_k(S^k) \arrow[d, "t_*"] \\
\widetilde{H}_{k+1}(\widehat{M}/M') \arrow[r, "\partial"] & \widetilde{H}_k(M') \arrow[d, "i_*"', "\cong"] \\
 & \widetilde{H}_k(M)
\end{tikzcd}$$

We note that the map $i \circ t\colon S^n \to M$ is equal to the restriction of $f\colon S^n \times D^{n-k}$ to $S^n \times \{y_0\}$, and hence $i_* \circ t_*$ can be identified with $g\colon \mathbb{Z} \to H_k(M)$. The diagram above then implies that the kernel resp. cokernel of ∂ can be identified with the kernel resp. cokernel of g.

This implies the statement about $H_k(\widehat{M})$ and the statement about $H_{k+1}(\widehat{M})$ if g is injective. If g is not injective, the kernel of g is a non-trivial subgroup of $\widehat{H}_{k+1}(\widehat{M}/M') \cong \mathbb{Z}$ and hence isomorphic to $\mathbb{Z}$. This implies that the exact sequence (12) splits and hence we have the isomorphisms

$$\widetilde{H}_{k+1}(\widehat{M}) \cong \widetilde{H}_{k+1}(M') \oplus \mathbb{Z} \cong \widetilde{H}_{k+1}(M) \oplus \mathbb{Z}$$

□

10. HOMEWORK ASSIGNMENT # 10

1. Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence of left modules over a ring R. Let M be a left R-module and N be a right R-module.

(a) Show that

$$\begin{tikzcd} \mathrm{Hom}_R(A,M) & \mathrm{Hom}_R(B,M) \arrow[l, "f^*"'] & \mathrm{Hom}_R(C,M) \arrow[l, "g^*"'] & 0 \arrow[l] \end{tikzcd}$$

is an exact sequence of $\mathbb{Z}$-modules. Hint: the exactness of the above sequence is equivalent to saying that g^* induces an isomorphism between $\mathrm{Hom}_R(C,M)$ and the kernel of f^*. Prove this by constructing an inverse map.

(b) Show that

$$\begin{tikzcd} N \otimes_R A \arrow[r, "1\otimes f"] & N \otimes_R B \arrow[r, "1\otimes g"] & N \otimes_R C \arrow[r] & 0 \end{tikzcd}$$

is an exact sequence of $\mathbb{Z}$-modules. Hint: the exactness of this sequence is equivalent to saying that $\mathbb{1} \otimes g$ induces an isomorphism between $N \otimes_R C$ and the cokernel of $\mathbb{1} \otimes f$. Prove this by constructing a map $N \otimes_R C \to \mathrm{coker}(\mathbb{1} \otimes f)$ inverse to $\mathbb{1} \otimes g$. We note that constructing a homomorphism out of $N \otimes_R C$ amounts to constructing a *bilinear map* by using the universal property of this tensor product (see e.g. the definition of the tensor product of modules in wikipedia).

Proof. We note that exactness of the sequence in part (a) is equivalent to the statement that g^* maps $\mathrm{Hom}_R(C,M)$ isomorphically onto the kernel of f^*. To prove this, we will construct an inverse map

$$\psi\colon \ker f^* \longrightarrow \mathrm{Hom}_R(C,M).$$

For $h \in \ker f^*$ we define $\psi(h)\colon C \to M$ by $c \mapsto h(b)$, where $b \in B$ is any element with $g(b) = c$. This is well-defined, since if $b' \in B$ with $g(b') = c$, then $b - b' \in \ker g$ and hence by exactness at B, there is some

$a \in A$ with $f(a) = b - b'$. Then $g(b) - g(b') = g(b - b') = g(f(a)) = (f^*g)(a) = 0$. It is straightforward to check that ψ is R-linear and that it is an inverse to f^*.

The argument for part (b) is quite analogous. The claimed exactness is equivalent to the statement that $1 \otimes g$ maps to cokernel of $1 \otimes f$ isomorphically to $N \otimes_R C$. Again, we will prove this by constructing an inverse

$$\psi \colon N \otimes_R C \longrightarrow \operatorname{coker}(q \otimes f).$$

We define ψ by setting $\psi(n \otimes c) := n \otimes b$, where $b \in B$ is any element with $g(b) = c$. This is well-defined since if $b' \in B$ with $g(b') = c$, then $b - b' \in \ker g$ and by exactness at B, there is some $a \in A$ with $f(a) = b - b'$. Hence $n \otimes b - n \otimes b' = (\mathbb{1} \otimes f)(n \otimes a)$ and so $[n \otimes b] = [n \otimes b'] \in \operatorname{coker}(\mathbb{1} \otimes f)$. It is easy to check that ψ is a map of $\mathbb{Z}$-modules, and that it is inverse to $\mathbb{1} \otimes g$. $\square$

2. Let M be a left module over a ring R. We recall that a *free resolution of M* is an exact sequence of left R-modules and R-module homomorphisms

$$\begin{tikzcd}
0 & M \arrow[l] & M_0 \arrow[l, "\epsilon"'] & M_1 \arrow[l, "d_1"'] & M_2 \arrow[l, "d_2"'] & \dots \arrow[l]
\end{tikzcd}$$

such that all the M_q's are *free* R-modules.
(a) Let (M_*, ϵ^M) be a free resolution of a left R-module M and let (N_*, ϵ^N) be a free resolution of a left R-module N. Show that if $f \colon M \to N$ is an R-linear map, then there is a R-linear maps $f_q \colon M_q \to N_q$ such that the diagram

$$\begin{tikzcd}
M \arrow[d, "f"] & M_0 \arrow[l, "\epsilon^M"'] \arrow[d, "f_0"] & M_1 \arrow[l] \arrow[d, "f_1"] & M_2 \arrow[l] \arrow[d, "f_2"] & \cdots \arrow[l] \\
N & N_0 \arrow[l, "\epsilon^N"'] & N_1 \arrow[l] & N_2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

is commutative. Hint: construct the R-linear maps f_q inductively using the following property of a free module: if $g \colon A \to B$ is an R-module map whose domain A is a *free* module, then g factors through any R-linear surjection $h \colon C \to B$; i.e., there is an R-linear map $\widehat{g} \colon A \to C$ making the following diagram commutative:

$$\begin{tikzcd}
 & C \arrow[d, two heads, "h"] \\
A \arrow[ur, "\widehat{g}"] \arrow[r, "g"] & B
\end{tikzcd}$$

(b) Show that the R-linear chain map $f_*\colon M_* \to N_*$ constructed in (a) is unique up to R-linear chain homotopies, i.e., if $f'_*\colon M_* \to N_*$ is another solution to (a), show that there is a chain homotopy T between them.

Proof. Since M_0 is a free R-module, the module map $f \circ \epsilon^M\colon M_0 \to N$ factors through the surjective map $\epsilon^N\colon N_0 \to N$; i.e., there is an R-linear map $f_0\colon M_0\colon N_0$ making the first square commutative. We will construct the f_q's by induction. Let us assume that we already constructed R-linear maps $f_0, \ldots, f_q$ making all diagrams to the left of f_q commutative. We note that this implies in particular that f_q maps $\ker(M_q \to M_{q-1})$ to $\ker(N_q \to N_{q-1})$. Now we want to construct f_{q+1} such that the following diagram commutes:

$$\begin{tikzcd}
\ker(M_q \to M_{q-1}) \arrow[d, "f_q"] & M_{q+1} \arrow[l, "d^M_{q+1}"'] \arrow[d, "f_{q+1}"] \\
\ker(N_q \to N_{q-1}) & N_{q+1} \arrow[l, "d^N_{q+1}"']
\end{tikzcd}$$

This map exists since M_{q+1} is free and the map d^N_{q+1} is surjective by exactness of the resolution N_* at N_q.

To prove part (b), assume that $f'_*\colon M_* \to N_*$ is another chain map lifting the map f. Our goal is to construct a chain homotopy T between them; i.e., we want R-linear maps $T_q\colon M_q \to N_{q+1}$ with

$$d^N_{q+1} T_q + T_{q-1} d^M_q = f_q - f'_q \tag{13}$$

where the modules M_q, N_q are interpreted as the trivial modules for $q < 0$. We will construct the T_q's inductively. To construct T_0, we note that

$$\epsilon^N \circ f_0 = f \circ \epsilon^M = \epsilon^N \circ f'_0$$

implies that the range of $f_0 - f'_0$ is contained in $\ker \epsilon^N$, and hence there is a map T_0 making the diagram

$$\begin{tikzcd}
M_0 \arrow[d, "f_0-f'_0"'] \arrow[dr, "T_0"] & \\
\ker \epsilon^N & N_1 \arrow[l, "d^N_1"]
\end{tikzcd}$$

commutative since M_0 is free and the horizontal map is surjective.

Now let us assume that we have constructed $T_0, \ldots, T_{k-1}$ satisfying equation (13) for $q < k$. To construct T_k, we consider this equation for

$q = k$ and put the term $T_{k-1}d_k^M$ on the right side of the above equation and try to solve for T_k. We note that the image of

$$g := f_k - f_k' - T_{k-1}d_k^M \colon M_k \to N_k$$

is contained in the kernel of d_k^N since

$$\begin{aligned} d_k^N(f_k - f_k' - T_{k-1}d_k^M) &= f_{k-1}d_k^M - f_{k-1}'d_k^M - d_k^N T_{k-1}d_k^M \\ =& f_{k-1}d_k^M - f_{k-1}'d_k^M - (T_{k-2}d_{k-1}^M d_k^M - f_{k-1}d_k^M - f_{k-1}'d_k^M) = 0 \end{aligned}$$

Here the first equation holds since f_*, f_*' are chain maps, and the second equation follows from the inductive assumption.

Now we can construct T_k making the diagram

$$\begin{tikzcd} M_k \arrow[d, "g"'] \arrow[dr, "T_k"] & \\ \ker d_k^N & N_{k+1} \arrow[l, "d_{k+1}^N"] \end{tikzcd}$$

commutative, since M_k is free and d_{k+1}^N is surjective onto the kernel of d_k^N by the exactness of N_*.

□

3. Let $R = \mathbb{Z}$, and let $M = \mathbb{Z}/s$.

(a) Construct a free resolution of the R-module M. Hint: you can find a resolution M_* with $M_q = 0$ for $q \geq 2$.

(b) Use the free resolution M_* to compute the groups $\mathrm{Tor}_q^R(N, M)$ for $N = \mathbb{Z}$ and $N = \mathbb{Z}/t$. The torsion group $\mathrm{Tor}_q^R(N, M)$ for a right R-module N is defined as the q-th homology group of the chain complex

$$N \otimes_R M_0 \xleftarrow{1\otimes d_1} N \otimes_R M_1 \xleftarrow{1\otimes d_2} N \otimes_R M_2 \longleftarrow \dots$$

(c) Use the free resolution M_* to compute the groups $\mathrm{Ext}_R^q(M, P)$ for $P = \mathbb{Z}$ and $P = \mathbb{Z}/t$. The Ext group $\mathrm{Ext}_R^q(M, P)$ for a left R-module P is defined as the q-th homology group of

$$\mathrm{Hom}_R(M_0, P) \xrightarrow{d_1^*} \mathrm{Hom}_R(M_1, P) \xrightarrow{d_2^*} \mathrm{Hom}_R(M_2, P) \xrightarrow{d_3^*} \dots$$

Proof. The sequence

$$0 \longleftarrow M = \mathbb{Z}/s \longleftarrow M_0 = \mathbb{Z} \xleftarrow{s} M_1 = \mathbb{Z} \longleftarrow M_2 = 0 \longleftarrow$$

is a free resolution of $\mathbb{Z}/s$, where the map $s \colon \mathbb{Z} \to \mathbb{Z}$ stands for multiplication by s.

Part (b): tensoring M_* with $N = \mathbb{Z}$ over $\mathbb{Z}$, we obtain the chain complex

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xleftarrow{1\otimes s} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$$

We note that $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ is isomorphic to $\mathbb{Z}$, where the isomorphism sends $a \otimes b \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ to $ab \in \mathbb{Z}$. Via these isomorphism, the above chain complex is isomorphic to

$$\mathbb{Z} \xleftarrow{s} \mathbb{Z}$$

and hence

$$\mathrm{Tor}_q^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/s) = H_q(N \otimes_{\mathbb{Z}} M_*) = \begin{cases} \mathbb{Z}/s & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Tensoring M_* with $N = \mathbb{Z}/t$ over $\mathbb{Z}$, we obtain the chain complex

$$\mathbb{Z}/t \otimes_{\mathbb{Z}} \mathbb{Z} \xleftarrow{1\otimes s} \mathbb{Z}/t \otimes_{\mathbb{Z}} \mathbb{Z}$$

We note that $\mathbb{Z}/t \otimes_{\mathbb{Z}} \mathbb{Z}$ is isomorphic to $\mathbb{Z}/t$, where the isomorphism sends $a \otimes b \in \mathbb{Z}/t \otimes_{\mathbb{Z}} \mathbb{Z}$ to $ab \in \mathbb{Z}/t$. Via these isomorphism, the above chain complex is isomorphic to

$$\mathbb{Z}/t \xleftarrow{s} \mathbb{Z}/t$$

We note that the cokernel of the map s is $\mathbb{Z}$ modulo the ideal generated by t and s. This is equal to the ideal generated by $g = \gcd(s, t)$ and hence the cokernel of the map $s\colon \mathbb{Z}/t \to \mathbb{Z}/t$ is isomorphic to $\mathbb{Z}/g$. Considering the order of the image, the domain and the kernel of the map s, we see that $\ker s$ has order g; as a subgroup of the cyclic group $\mathbb{Z}/t$ it is isomorphic to $\mathbb{Z}/g$. We conclude:

$$\mathrm{Tor}_q^{\mathbb{Z}}(\mathbb{Z}/t, \mathbb{Z}/s) = H_q(\mathbb{Z}/t \otimes_{\mathbb{Z}} M_*) = \begin{cases} \mathbb{Z}/g & q = 0, 1 \\ 0 & q \neq 0, 1 \end{cases}$$

For part (c) and $N = \mathbb{Z}$ we consider the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{s^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$$

We note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, where the isomorphism is given by $f \mapsto f(1)$. This implies that the above cochain complex is isomorphic to

$$\mathbb{Z} \xrightarrow{s} \mathbb{Z}$$

and hence

$$\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z}) = H^q(\mathrm{Hom}_{\mathbb{Z}}(M_*, \mathbb{Z})) = \begin{cases} \mathbb{Z}/s & q = 1 \\ 0 & q \neq 1 \end{cases}$$

For $N = \mathbb{Z}/t$ we obtain the cochain complex

$$\begin{tikzcd} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/t) & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/t) \arrow[l, "s^*"'] \end{tikzcd}$$

We note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/t)$ is isomorphic to $\mathbb{Z}/t$, where the isomorphism is given by $f \mapsto f(1)$. This implies that the above cochain complex is isomorphic to

$$\mathbb{Z}/t \xrightarrow{s} \mathbb{Z}/t$$

and hence arguing as in part (b) we obtain

$$\operatorname{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z}/t) = H^q(\operatorname{Hom}_{\mathbb{Z}}(M_*, \mathbb{Z}/t)) = \begin{cases} \mathbb{Z}/g & q = 0,1 \\ 0 & q \neq 0,1 \end{cases}$$

□

11. HOMEWORK ASSIGNMENT # 11

1. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of left modules over a ring R. Let $\epsilon^A \colon A_* \to A$ and $\epsilon^C \colon C_* \to C$ be free resolutions. Show that there exists a free resolution $\epsilon^B \colon B_* \to B$ and chain maps $f* \colon A_* \to B_*$, $g_* \colon B_* \to C_*$ such that the diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & A_2 \arrow[r, "f_2"] \arrow[d, "d_2^A"'] & B_2 \arrow[r, "g_2"] \arrow[d, "d_2^B"'] & C_2 \arrow[r] \arrow[d, "d_2^C"'] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "d_1^A"'] & B_1 \arrow[r, "g_1"] \arrow[d, "d_1^B"'] & C_1 \arrow[r] \arrow[d, "d_1^C"'] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "f_0"] \arrow[d, "\epsilon^A"'] & B_0 \arrow[r, "g_0"] \arrow[d, "\epsilon^B"'] & C_0 \arrow[r] \arrow[d, "\epsilon^C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative and that the first row is an exact sequence of chain complexes. Hint: make the following Ansatz for B_* (this German word doesn't have an english translation and is used in english; if you don't know it, look it up in wikipedia): let $B_q = A_q \oplus C_q$, let $f_q \colon A_q \to A_q \oplus C_q$ be the inclusion, and let $g_q \colon A_q \oplus C_q \to C_q$ be the projection map. Then construct the homomorphism ϵ^B and then inductively the boundary maps $d^B_q\colon B_q \to B_{q-1}$ in such a way that $d^B_{q-1} \circ d^B_q = 0$ (here we interpret d^B_0 as ϵ^B) and the above diagram is commutative. Show that the exactness of the middle column of the above diagram is a consequence of the exactness of the left and right column.

Proof. We note that the desired homomorphism

$$d^B_n\colon A_n \oplus C_n \to A_{n-1} \oplus C_{n-1}$$

can be written as a matrix

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

whose entries are homomorphisms $a\colon A_n \to A_{n-1}$, $b\colon C_n \to A_{n-1}$, $c\colon A_n \to C_{n-1}$ and $d\colon C_n \to C_{n-1}$. For $n = 0$, this matrix reduces to a row vector $(a_0\, b_0)$ with $a_0\colon A_1 \to B$ and $b_0\colon C_0 \to B$. The commutativity of the diagram implies conditions on the matrix entries. The commutativity of the left squares are equivalent to the conditions $a_n = d^A_n$ and $c_n = 0$. The commutativity of the right squares is equivalent to $d_n = d^C_n$ for $n > 0$ and $gb_0 = d^C_0$.

Next let us analyze the conditions coming from the requirement $d^B_{n-1} \circ d^B_n = 0$ for $n \leq 2$:

$$d^B_{n-1} \circ d^B_n = \begin{pmatrix} d^A_{n-1} & b_{n-1} \\ 0 & d^C_{n-1} \end{pmatrix} \begin{pmatrix} d^A_n & b_n \\ 0 & d^C_n \end{pmatrix} = \begin{pmatrix} d^A_{n-1} d^A_n & d^A_{n-1} b_n + b_{n-1} d^C_n \\ 0 & d^C_{n-1} d^C_n \end{pmatrix}$$

This shows that $d^B_{n-1} \circ d^B_n = 0$ is equivalent to

$$d^A_{n-1} b_n + b_{n-1} d^C_n = 0 \tag{14}$$

For $n = 1$ the situation is slightly different in that the matrices representing d^B_0 and $d^b_0 \circ d^B_1$ are 1×2 matrices instead of 2×2 matrices. To obtain the relevant calculation in that case, we just ignore the bottom of the matrices representing d^B_{n-1} resp. $d^B_{n-1} \circ d^B_n$ and still arrive at equation (14).

Now we will construct the homomorphisms $b_0, b_1, b_2, \ldots$ such that $gb_0 = d^C_0$ and the condition (14) are satisfied. We note that there is a homomorphism b_0 making the diagram

$$\begin{tikzcd}
 & B \arrow[d, "g"] \\
C_0 \arrow[ur, "b"] \arrow[r, "\epsilon^C"'] & C
\end{tikzcd}$$

commutative since g is surjective and C_0 is a free module.

Now assume inductively that we've constructed $b_0, \dots, b_{n-1}$ satisfying condition (14) and we want to produce the homomorphism $b_n \colon C_n \to A_{n-1}$ such that (14) holds. We observe that the image of $d^A_{n-1} \colon A_n \to A_{n-1}$ and $b_{n-1}d^C_n$ is contained in the kernel of d^A_{n-2}. For d^A_{n-1} this follows from the exactness of A_*, for $b_{n-1}d^C_n$, it follows from

$$d^A_{n-2}b_{n-1}d^C_n = -b_{n-2}d^C_{n-1}d^C_n = 0,$$

where the first equation follows from equation (14) and our inductive assumption. This implies that there is a homomorphism b_n making the diagram

$$\begin{tikzcd}
 & A_{n-1} \arrow[d, "d^A_{n-1}"] \\
C_n \arrow[ur, "b_n"] \arrow[r, "-b_{n-1}d^C_n"'] & \ker d^A_{n-2}
\end{tikzcd}$$

commutative since d^A_{n-1} maps onto the kernel of d^A_{n-1} and C_n is a free module. The homomorphism b_n thus constructed satisfies our condition (14).

By our construction, the middle column of diagram (**??**) is a chain complex. To show that it is in fact a resolution, it suffices to show that its homology groups in fact vanish. We note that the diagram (**??**) can be interpreted as a short exact sequence of chain complexes (defining $A_{-1} = A$, $B_{-1} = B$ and $C_{-1} = C$). The associated long exact seqence of homology groups then shows that the homology groups of the middle column are zero, since the homology groups of the left and right column are zero, since they are free resolutions and hence in particular exact sequences. □

2. Prove the Universal Coefficient Theorem for cohomology groups; i.e., show that if C_* is a chain complex of free left modules over a PID R, and P is a left R-module, then there is a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_R(H_{q-1}(C_*), P) \longrightarrow H^q(C_*, P) \longrightarrow \operatorname{Hom}_R(H_q(C_*), P) \longrightarrow 0$$

Hint: Adapt the proof of the homology Universal Coefficient Theorem we did in class to the cohomology case.

Proof. We first assume that all boundary maps in the chain complex C_* are zero. Then the coboundary maps δ in the cochain complex $\operatorname{Hom}_{\mathbb{R}}(C_*, P)$ are trivial as well and hence we have isomorphisms

$$H^q(C_*; P) \cong \operatorname{Hom}_{\mathbb{R}}(C_q, P) \cong \operatorname{Hom}_{\mathbb{R}}(H_q(C_*), P).$$

This proves the exact sequence for chain complexes C_* with vanishing boundary maps, since in this case the Ext group is trivial due to $H_{q-1}(C_*) \cong C_{q-1}$ being free.

The idea to deal with the general case is to fit C_* in an exact sequence of chain complexes where the two other complexes are of the special kind discussed above:

$$\begin{tikzcd} 0 \arrow[r] & Z_* \arrow[r, "i"] & C_* \arrow[r, "\partial"] & \Sigma B_* \arrow[r] & 0 . \end{tikzcd}$$

Here $i_q \colon Z_q \subset C_q$ is the inclusion of q-cycles in q-chains, and ΣB_* is the suspension of the chain complex B_* defined by $(\Sigma B_*)_q = B_{q-1}$ where B_{q-1} is the group of $(q-1)$-boundaries. The boundary maps in the chain complexes Z_* and B_* are trivial. Now we apply the functor $\mathrm{Hom}_R(\quad, P)$ and obtain the following half exact sequence of cochain complexes

$$\begin{tikzcd} \mathrm{Hom}_R(Z_*, P) & \mathrm{Hom}_R(C_*, P) \arrow[l, "i^*"'] & \mathrm{Hom}_R(\Sigma B_*, P) \arrow[l, "\partial^*"'] & 0 . \arrow[l] \end{tikzcd}$$

We observe that our assumption that R is a PID guarantees that i^* is surjective and hence the above is a short exact sequence of chain complexes: the assumption that C_{q-1} is free implies that the submodule $(\Sigma B)_q = B_{q-1}$ is free and hence the group $\mathrm{Ext}_1^{\mathbb{Z}}((\Sigma B)_q, P)$ which extends the above half exact sequence (for a fixed dimension q) to a longer exact sequence is zero.

The short exact sequence of cochain complexes induces a long exact sequence of cohomology groups

$$\begin{aligned} H^{q+1}(\mathrm{Hom}_R(\Sigma B_*, P)) \overset{\delta_q}{\leftarrow} H^q(\mathrm{Hom}_R(Z_*, P)) \leftarrow H^q(\mathrm{Hom}_R(C_*, P)) \\ \leftarrow H^q(\mathrm{Hom}_R(\Sigma B_*, P)) \overset{\delta_{q-1}}{\leftarrow} H^{q-1}(\mathrm{Hom}_R(Z_*, P)) \end{aligned}$$

Splitting the long exact sequence, we obtain short exact sequences

$$0 \longleftarrow \ker \delta_q \longleftarrow H^q(\mathrm{Hom}_R(C_*, P)) \longleftarrow \operatorname{coker} \delta_{q-1} \longleftarrow 0 \tag{15}$$

In order to identify the kernel and cokernel of δ, we note that since the boundary maps in Z_* and B_* are trivial, we have

$$\begin{aligned} H^q(\mathrm{Hom}_R(Z_*, P)) &= \mathrm{Hom}_R(Z_q, P) \\ H^{q+1}(\mathrm{Hom}_R(\Sigma B_*, P)) &= \mathrm{Hom}_R((\Sigma B)_{q+1}, P) = \mathrm{Hom}_R(B_q, P) \end{aligned}$$

Moreover, the map δ can be identified with the map

$$j_q^* \colon \mathrm{Hom}_R(B_q, P) \longrightarrow \mathrm{Hom}_R(Z_q, P)$$

induced by the inclusion map $j_q\colon B_q \to Z_q$. We note that by definition of homology groups, we have a short exact sequence

$$\begin{tikzcd} 0 & H_q(C_*) \arrow[l] & Z_q \arrow[l] & B_q \arrow[l, "j_q"'] & 0 \arrow[l] \end{tikzcd}$$

This can be interpreted as a free resolution of $H_q(C_*)$, since the submodules $Z_q, B_q \subset C_q$ are both free due to our assumption that C_q is free and R is a PID. This implies that the kernel (resp. cokernel) of j_q^* is just $\mathrm{Hom}_R(H_q(C_*), P)$ (resp. $\mathrm{Ext}^1_R(H_q(C_*), P)$). This shows that the sequence (15) gives the desired exact sequence. $\square$

3. Use our knowledge of the homology groups of the real projective space $\mathbb{RP}^n$ to calculate
   (a) the homology groups $H_q(\mathbb{RP}^n; \mathbb{Z}/2)$;
   (b) the cohomology groups $H^q(\mathbb{RP}^n; \mathbb{Z})$;
   (c) the cohomology groups $H^q(\mathbb{RP}^n; \mathbb{Z}/2)$.
   (d) Calculate the cohomology groups $H^q(\mathbb{RP}^n; \mathbb{Z}/2)$ using only the result you obtained in part (a), but no knowledge of the homology groups $H_q(\mathbb{RP}^n)$.

Proof. We recall that

$$H_q(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & q = 0 \text{ or } q = n = 2k+1 \\ \mathbb{Z} & q = 2k+1,\ 1 \le q < n \\ 0 & \text{otherwise} \end{cases}$$

For part (a) we use the Universal Coefficient Theorem for homology and obtain an isomorphism

$$H_q(\mathbb{RP}^n; \mathbb{Z}/2) \cong H_q(\mathbb{RP}^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \oplus \mathrm{Tor}_1^Z(H_{q-1}(\mathbb{RP}^n), \mathbb{Z}/2).$$

For $q = 0$ and $q = n = 2k+1$, the tensor term gives $\mathbb{Z} \otimes_{\mathbb{Z}} /2 \cong \mathbb{Z}/2$, while the tor term is zero and hence $H_q(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$. For $q = 2k+1$, $1 \le q < n$, the tensor term gives $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2$, the tor-term is zero and hence $H_q(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$. For $q = 2k$, $1 < q < n$, the tensor term is zero, the tor term gives $\mathrm{Tor}_1^Z(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$ and hence again $H_q(\mathbb{RP}^n; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$. For all other q, both terms are clearly zero. Summarizing we have

$$H^q(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le q \le n \\ 0 & \text{otherwise} \end{cases}$$

For part (b) using the UCT for cohomology we obtain

$$H^q(\mathbb{RP}^n; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_q(\mathbb{RP}^n), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(\mathbb{RP}^n), \mathbb{Z}).$$

For $q = 0$ or $q = 2k+1 = n$, the hom-term gives $\mathbb{Z}$, the ext term is 0 and hence $H^q(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$ in these cases. For $q = 2k+1$, $1 \leq q < n$, the hom-term as well as the ext-term are zero. For $q = 2k$, $q < 1 < n$, the hom-term is zero while the ext-term is $\mathbb{Z}/2$. For all other q, both terms are zero. Collecting these data we have

$$H^q(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2 & q = 2k \text{ and } 0 < q < n \\ \mathbb{Z} & q = 0 \text{ or } q = n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

For part (c) using the UCT for cohomology we obtain

$$H^q(\mathbb{RP}^n;\mathbb{Z}/2) \cong \operatorname{Hom}_{\mathbb{Z}}(H_q(\mathbb{RP}^n),\mathbb{Z}/2) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(\mathbb{RP}^n),\mathbb{Z}/2).$$

For $q = 0$ or $q = 2k+1 = n$, the hom-term gives $\mathbb{Z}/2$, the ext term is 0 and hence $H^q(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}/2$. For $q = 2k+1$, $1 \leq q < n$, the hom-term gives $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2$ and the ext term is zero. For $q = 2k$, $q < 1 < n$, the hom-term is zero while the ext-term is $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2$. For all other q, both terms are zero. Collecting these data we have

$$H^q(\mathbb{RP}^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \leq q \leq n \\ 0 & \text{otherwise} \end{cases}$$

For part (d) we note that for any $\mathbb{Z}$-module M there is a canonical isomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z}/2) \cong \operatorname{Hom}_{\mathbb{Z}/2}(M \otimes_{\mathbb{Z}} \mathbb{Z}/2,\mathbb{Z}/2).$$

In particular, for any space X, the cochain complex $C^*(X;\mathbb{Z}/2) = \operatorname{Hom}_{\mathbb{Z}}(C_*(X),\mathbb{Z}/2)$ is canonically isomorphic to the cochain complex

$$\operatorname{Hom}_{\mathbb{Z}/2}(C_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/2,\mathbb{Z}/2) = \operatorname{Hom}_{\mathbb{Z}/2}(C_*(X;\mathbb{Z}/2),\mathbb{Z}/2)$$

Then we can use the Universal Coefficient Theorem for the ring $R = \mathbb{Z}/2$ to conclude

$$H^q(X;\mathbb{Z}/2) \cong \operatorname{Hom}_{\mathbb{Z}/2}(H_q(X;\mathbb{Z}/2),\mathbb{Z}/2).$$

In particular, the $\mathbb{Z}/2$ cohomology group $H^q(X;\mathbb{Z}/2)$ is (non-canonically) isomorphic to the homology group $H_q(X;\mathbb{Z}/2)$, provided the latter is a finite dimensional vector space over $\mathbb{Z}/2$. Applying this observation to $X = \mathbb{RP}^n$ gives part (d). $\square$

4. Let X be a topological space whose homology groups $H_q(X)$ are all finitely generated. Then the Universal Coefficient Theorem implies that for any field $\mathbb{F}$ the vector spaces $H_q(X;\mathbb{F})$ are finite dimensional.

A convenient way to encode *all* the $\mathbb{F}$-homology groups is the *Poincaré series*

$$P(X;\mathbb{F})(z) := \sum_{q=0}^{\infty} \dim_{\mathbb{F}} H_q(X;\mathbb{F})\, z^q$$

Show that

$$\begin{aligned} P(X \amalg Y;\mathbb{F}) &= P(X;\mathbb{F}) + P(Y;\mathbb{F}) \\ P(X \times Y;\mathbb{F}) &= P(X;\mathbb{F}) \cdot P(Y;\mathbb{F}) \end{aligned}$$

Proof. We've seen that $H_q(X \amalg Y)$ is isomorphic to $H_q(X) \oplus H_q(Y)$ and the same argument shows that that is true for homology groups with coefficients. This implies

$$P(X \amalg Y;\mathbb{F}) = \sum_{q=0}^{\infty} \dim_{\mathbb{F}}(H_q(X;\mathbb{F}) \oplus H_q(Y;\mathbb{F}))\, z^q = P(X;\mathbb{F}) + P(X;\mathbb{F})$$

For the second statement we need to determine the dimension of the vector space $H_q(X \times Y;\mathbb{F})$. We want to use the Künneth Theorem for the field $\mathbb{F}$ in order to avoid torsion groups. To do so we note that for $\mathbb{Z}$-modules M, N we have a canonical isomorphism

$$(M \otimes_{\mathbb{Z}} \mathbb{F}) \otimes_{\mathbb{F}} (N \otimes_{\mathbb{Z}} \mathbb{F}) \cong (M \otimes_{\mathbb{Z}} N) \otimes_{\mathbb{Z}} \mathbb{F}$$

This implies that we have an isomorphism of chain complexes

$$(C_*(X) \otimes_{\mathbb{Z}} \mathbb{F}) \otimes_{\mathbb{F}} (C_*(Y) \otimes_{\mathbb{Z}} \mathbb{F}) \cong (C_*(X) \otimes_{\mathbb{Z}} C_*(Y)) \otimes_{\mathbb{Z}} \mathbb{F}.$$

Combining the fact that $C_*(X) \otimes_{\mathbb{Z}} C_*(Y)$ is chain homotopy equivalent to $C_*(X \times Y)$ and the Künneth theorem we obtain the isomorphism

$$H_q(X \times Y;\mathbb{F}) \cong \bigoplus_{k+l=q} H_k(X;\mathbb{F}) \otimes H_l(Y;\mathbb{F}).$$

Taking dimensions of these vector spaces this implies

$$\dim_{\mathbb{F}} H_q(X \times Y;\mathbb{F}) = \sum_{k+l=q} (\dim_{\mathbb{F}} H_k(X;\mathbb{F}))(\dim_{\mathbb{F}} H_l(Y;\mathbb{F}))$$

Now we are ready to calculate:

$$\begin{aligned} P(X;\mathbb{F})P(Y;\mathbb{F}) = &\sum_{q=0}^{\infty} \left(\sum_{k+l=q} (\dim_{\mathbb{F}} H_k(X;\mathbb{F}))(\dim_{\mathbb{F}} H_l(Y;\mathbb{F})) \right) z^q \\ &\sum_{q=0}^{\infty} \dim_{\mathbb{F}} H_q(X \times Y;\mathbb{F}) z^q = P(X \times Y;\mathbb{F}) \end{aligned}$$

□

12. Homework Assignment # 12

1. Show $\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$ for cochains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.

For the proof see Hatcher, Lemma 3.6 on page 206.

2. Show that the cup product is compatible with pull-back of cohomology classes in the sense that for a map $f\colon X \to Y$ and cohomology classes $\alpha \in H^k(Y;R)$, $\beta \in H^l(Y;R)$ we have

$$f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta).$$

Hint: Show first the analogous statement for cochains.

For the proof see Hatcher, Proposition 3.10 on page 210.

3. We recall that the homology of the Klein bottle K is given by

$$H_q(K) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & q = 1 \end{cases}$$

(a) Use the UCT to determine the cohomology groups $H^q(K;\mathbb{Z}/2)$.
(b) Determine the cup products on cohomology with $\mathbb{Z}/2$ coefficients. Hint: proceed similarly to what we did in class for determining the cup products for the cohomology of the torus.

Proof. We recall that $H_q(K)$ is $\mathbb{Z}$ for $q = 0$, $\mathbb{Z} \oplus \mathbb{Z}/2$ for $q = 1$ and that it is trivial for other q's. We use the UCT

$$H^q(K;\mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}}(H_q(K), \mathbb{Z}/2) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(K), \mathbb{Z}/2)$$

For $q = 0$, the hom term gives $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2$, while the ext term is trivial. For $q = 1$, the hom term gives $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, while the ext term is trivial. For $q = 2$, the hom term is trivial, while the ext term gives $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$. For all other q's both terms are trivial. Summarizing we have

$$H^q(K;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & q = 0, 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & q = 1 \\ 0 & q \neq 0, 1, 2 \end{cases}$$

For part (b) we do a cochain level calculation of cup products on a small subcomplex of the singular chain complex of K. To identify this subcomplex, we use our standard picture of the Klein bottle being obtained from the square by suitably gluing opposite edges, and subdivide the square into two triangles as shown in the following picture.

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, "c"] & \bullet \\
\bullet \arrow[r, "a"'] & \bullet \arrow[u, "b"']
\end{tikzcd}$$

Using our standard convention of how the orientation of edges determines a labeling of the vertices of a triangle (the 0-vertex has two tails, the 1-vertex has a tip and a tail, the 2-vertex has two tips), we obtain an affine linear map $\Delta^2 \to I \times I$ for each triangle which maps the i-th vertex of Δ to the $i$nb-th vertex of the triangle in question. Composing with the quotient map $I \times I \to K$ we obtain two simplices $\sigma_1 \colon \Delta^2 \to K$ (corresponding to the upper right triangle) and $\sigma_2 \colon \Delta^2 \to K$ (corresponding to the lower left triangle). Interpreting as usual the edges as 1-simplices (using an affine linear map plus the convention that the tail is the 0-th vertex) we have:

$$\partial\sigma_1 = b - a + c \qquad \partial\sigma_2 = a - c + b$$

Let $\widehat{C}_*(K) \subset C_*(K)$ be the subcomplex generated by the 1-simplices a, b, c, the 2-simplices σ_1, σ_2 and the 0-simplex v corresponding to the vertex. We note that $\widehat{C}_*(K)$ can be interpreted as the cellular chain complex of the CW structure of K given by the above picture and hence the inclusion map $\widehat{C}_*(K) \to C_*(K)$ is a chain homotopy equivalence.

Now we consider the cochain complex $\widehat{C}^*(K; \mathbb{Z}/2) = \mathrm{Hom}_Z(\widehat{C}_*(K), \mathbb{Z}/2)$. Let $\alpha, \beta, \gamma \in \widehat{C}^1(K; \mathbb{Z}/2)$ be the basis dual to $a, b, c \in C_1(K)$, and let $\varphi_i, \varphi_2 \in \widehat{C}^2(K; \mathbb{Z}/2)$ be the basis dual to $\sigma_1, \sigma_2 \in \widehat{C}_2(K)$. Next we calculate the coboundary map

$$\delta \colon \widehat{C}^1(K; \mathbb{Z}/2) \longrightarrow \widehat{C}^2(K; \mathbb{Z}/2).$$

We note that

$$(\delta\alpha)(\sigma_i) = \alpha(\partial\sigma_i) = 1 \quad \text{for } i = 1, 2,$$

and hence $\delta\alpha = \varphi_1 + \varphi_2$. The *same* statement holds β and γ (we calculate with coefficients in $\mathbb{Z}/2$!) and so $\delta\beta = \delta\gamma = \varphi_1 + \varphi_2$. It follows that $\psi_1 := \alpha + \gamma$ and $\psi_2 := \beta + \gamma$ are 1-cocycles which represent a basis for $H^1(K;\mathbb{Z}/2)$. We compute:

$$\begin{aligned}
(\psi_1 \cup \psi_1)(\sigma_1) &= \psi_1(b)\cdot\psi_1(c) = 0\cdot 1 = 0\\
(\psi_1 \cup \psi_1)(\sigma_2) &= \psi_1(a)\cdot\psi_1(b) = 1\cdot 0 = 0\\
(\psi_1 \cup \psi_2)(\sigma_1) &= \psi_1(b)\cdot\psi_2(c) = 0\cdot 1 = 0\\
(\psi_1 \cup \psi_2)(\sigma_2) &= \psi_1(a)\cdot\psi_2(b) = 1\cdot 1 = 1\\
(\psi_2 \cup \psi_2)(\sigma_1) &= \psi_2(b)\cdot\psi_2(c) = 1\cdot 1 = 1\\
(\psi_2 \cup \psi_2)(\sigma_2) &= \psi_2(a)\cdot\psi_2(b) = 0\cdot 1 = 0
\end{aligned}$$

This implies

$$\psi_1 \cup \psi_1 = 0 \qquad \psi_1 \cup \psi_2 = \phi_2 \qquad \psi_2 \cup \psi_2 = \phi_1$$

Passing to cohomology, we note that the 2-cocycles ϕ_1 and ϕ_2 both represent the non-trivial element of $H^2(K;\mathbb{Z}/2) \cong \mathbb{Z}/2$. Hence the above calculation implies

$$[\psi_1] \cup [\psi_1] = 0 \quad \text{and} \quad [\psi_1] \cup [\psi_2] = [\psi_2] \cup [\psi_2] \neq 0$$

□

4. Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in the cohomology groups $H^q(M_g;\mathbb{Z})$ for M_g the closed orientable surface of genus g, by using the quotient map from M_g to a wedge-sum of g tori (this is problem # 1 on page 226 in Hatcher's book, where you can find a picture of this quotient map).

Proof. To describe the cup product on $H^*(M_g;\mathbb{Z})$ we need to use suitable generators of $H^1(M_g;\mathbb{Z})$. Our strategy is to use the cellular chain complex associated to the standard CW structure of M_g: we view M_g as a polygon with $4g$ edges, which are pairwise identified; the (equivalence class) of the vertices give one 0-cell v, the pairs of edges give $2g$ 1-cells $a_1, b_1, \dots, a_g, b_g$, and the polygon itself gives a 2-cell F. Hence the cellular chain complex $C_*^{CW}(M_g)$ has the following form:

$$C_0^{CW}(M_g) = \mathbb{Z}v \qquad C_1^{CW}(M_g) = \bigoplus_{i=1}^{g} \mathbb{Z}a_i \oplus \mathbb{Z}b_i \qquad C_2^{CW}(M_g) = \mathbb{Z}F$$

We recall that the boundary map in the cellular chain complex of M_g is zero and hence we can interpret the cells v, a_i, b_i, F as basis elements for the appropriate homology groups. Let $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g \in H^1(M_g;\mathbb{Z})$ the basis dual to $a_1, b_1, \dots, a_g, b_g$, and let $\gamma \in H^2(M_g;\mathbb{Z})$ be the element dual to $F \in H_2(M_g)$.

Claim. $\alpha_i \cup \beta_i = \gamma$; all cup products involving different elements are zero.

To prove this claim, we will use the map $f\colon M_g \to W_g := T \vee \cdots \vee T$ described above from M_g to a wedge of g copies of the torus $T = S^1 \times S^1$ to compare the cohomology ring of M_g with the cohomology ring of W_g via the ring homomorphism

$$f^*\colon H^*(W_g; \mathbb{Z}) \longrightarrow H^*(M_g; \mathbb{Z})$$

induced by f.

To determine the ring structure on $H^*(W_g; \mathbb{Z})$, we first recall the cohomology ring of the torus T. We take the standard CW structure on T (given by thinking of the torus as a square with opposite edges identified), consisting of one 0-cell v, two 1-cells a, b and one 2-cell F'. The boundary maps in the cellular chain complex are zero so that we can interpret these cells as homology classes. Let α, β be the basis of $H^1(T; \mathbb{Z})$ which is dual to the basis a, b of $H_1(T; \mathbb{Z})$, and let $\gamma' \in H^2(T; \mathbb{Z})$ be the element dual to $F' \in H_2(T; \mathbb{Z})$. In class, we've shown that

$$\alpha \cup \beta = -\beta \cup \alpha = \gamma'$$

To prove that $\alpha_i \cup \beta_i = \gamma$, we will compare the cohomology ring of M_g with that of the torus T via the map induced by $f_i := p_i \circ f\colon M_g \to T$. The map f_i is cellular; i.e., for every n, it maps the n-skeleton M_g^n to the n-skeleton T^n of T. It follows that f_i induces homomorphisms

$$(f_i)_*\colon C_n^{CW}(M_g) = H_n(M_g^n, M_g^{n-1}) \longrightarrow C_n^{CW}(M_g) = H_n(T^n, T^{n-1})$$

which fit together to give a chain map

$$(f_i)_\#\colon C_*^{CW}(M_g) \longrightarrow C_*^{CW}(T)$$

We note that f_i maps the 1-cells a_i, b_i homeomorphically onto the 1-cells a, b and maps all other 1-cells of M_g to the base point. We claim that the induced map

$$(f_i)_\#\colon C_2^{CW}(M_g) = H_2(M_g^2, M_g^1) \longrightarrow C_2^{CW}(T) = H_2(T^2, T^1)$$

is an isomorphism. To prove this we find a point $x \in M_g$ such that f_i is a local homeomorphism near x. Hence f_i induces an an isomorphism on local homology groups and the claim follows from the commutative diagram

$$\begin{tikzcd}
H_2(M_g^2, M_g^1) \arrow[r, "(f_i)_*"] \arrow[d, "\cong"'] & H_2(T^2, T^1) \arrow[d, "\cong"] \\
H_2(M_g, M_g \setminus x) \arrow[r, "(f_i)_*", "\cong"'] & H_2(T, T \setminus f_i(x))
\end{tikzcd}$$

Since the boundary maps in both cellular chain complexes are zero, we can interpret the above statements as calculating the induced homomorphism $(f_i)_*\colon H_*(M_g) \to H_*(T)$. This implies the following statements concerning the induced homomorphism $f_i^*\colon H^*(T;\mathbb{Z}) \to H^*(M_g;\mathbb{Z})$ in cohomology:

$$f_i^*(\alpha) = \alpha_i \qquad f_i^*(\beta) = \beta_i \qquad f_i^*(\gamma') = \gamma$$

It follows that

$$\begin{aligned} \alpha_i \cup \beta_i = f_i^*(\alpha) \cup f_i^*(\beta) = f_i^*(\alpha \cup \beta) = f_i^*(\gamma') = \gamma \\ \alpha_i \cup \alpha_i = f_i^*(\alpha) \cup f_i^*(\alpha) = f_i^*(\alpha \cup \alpha) = 0 \end{aligned}$$

Similarly, $\beta_i \cup \beta_i = 0$. To show that the cup product of classes with different subscripts is always zero, we will use the following result.

Lemma 3. *Let $\alpha \in H^k(X;R)$ and $\beta \in H^l(Y;\mathbb{R})$ be cohomology classes with $k > 0$, $l > 0$, and let $p_1\colon X \vee Y \to X$, $p_2\colon X \vee Y \to Y$ be the projection maps. Then $p_1^*\alpha \cup p_2^*\beta = 0$.*

Before proving the lemma, let us use it to show that the cup product of classes in $H^1(M_g;\mathbb{Z})$ with different subscripts vanish; e.g.,

$$\alpha_i \cup \beta_j = f_i^*\alpha \cup f_j^*\beta = f^*p_i^*\alpha \cup f^*p_j^*\beta = f^*(p_i^*\alpha \cup p_j^*\beta) = 0.$$

To prove the lemma, we note that the inclusion maps $i_1\colon X \to X \vee Y$, $i_2\colon Y \to X \vee Y$ give us an isomorphism

$$i_1^* \oplus i_2^*\colon H^q(X \vee Y;\mathbb{R}) \xrightarrow{\cong} H^q(X;\mathbb{R}) \oplus H_q(Y;\mathbb{R})$$

for $q > 0$. We've proved this for homology groups; the same proof works for cohomology. Alternatively, we could deduce the cohomology statement from the homology statement via the UCT. It follows that for the proof of the lemma, it suffices to show that $i_j^*(p_1^*\alpha \cup p_2^*\beta) = 0$ for $j = 1, 2$.

$$i_1^*(p_1^*\alpha \cup p_2^*\beta) = (i_1^*p_1^*\alpha) \cup (i_1^*p_2^*\beta) = \alpha \cup (p_2 i_1)^*\beta$$

We observe that $p_2 i_1\colon X \to Y$ factors through the base point $\mathrm{pt} \in X \vee Y$. Since $H^q(\mathrm{pt};R) = 0$ for $q > 0$ and β is a class in positive degree, it follows that $i_1^*p_2^*\beta = 0$. The argument that $i_2^*(p_1^*\alpha \cup p_2^*\beta)$ vanishes is completely analogous. This proves the lemma. □