

Name:

Take Home Exam , Feb. 26, 2008

You can use any resources for this test (e.g., books, notes, internet) with the exception of fellow mathematicians: talking to anybody except me about this exam before you turn your exam in constitutes a violation of the honor code. Please return the exam to me in class on Wednesday, Feb. 27. Don't forget part (b) of problem 4 on the back page. I'll be around most of the day today, except 2-3:30 PM. Good Luck!

1. (10 points) Let A be a subspace of a topological space X and denote by $i: A \rightarrow X$ the inclusion map. Show that if A is a retract of X , then the induced map $i_*: H_q(A) \rightarrow H_q(X)$ is injective.

Proof. The assumption that A is a retract of X means that there is a map $r: X \rightarrow A$ making the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ & \searrow \mathbb{1} & \downarrow r \\ & & A \end{array}$$

commutative, where $\mathbb{1}$ denotes the identity map. Applying the homology functor, it follows that the diagram

$$\begin{array}{ccc} H_q(A) & \xrightarrow{i_*} & H_q(X) \\ & \searrow \mathbb{1} & \downarrow r_* \\ & & H_q(A) \end{array}$$

is commutative. Hence i_* is injective, since if $i_*(a) = i_*(a')$, then $a = r(i_*(a)) = r(i_*(a')) = a'$. □

2. (15 points) Suppose the following diagram of abelian groups and group homomorphisms is commutative with exact rows:

$$\begin{array}{ccccccccc} \longrightarrow & C_{q+1} & \xrightarrow{\partial_{q+1}} & A_q & \xrightarrow{f_q} & B_q & \xrightarrow{g_q} & C_q & \xrightarrow{\partial_q} & A_{q-1} & \longrightarrow \\ & c_{q+1} \downarrow & & a_q \downarrow & & b_q \downarrow & & c_q \downarrow & & a_{q-1} \downarrow & \\ \longrightarrow & C'_{q+1} & \xrightarrow{\partial'_{q+1}} & A'_q & \xrightarrow{f'_q} & B'_q & \xrightarrow{g'_q} & C'_q & \xrightarrow{\partial'_q} & A'_{q-1} & \longrightarrow \end{array}$$

Assuming in addition that the maps c_q are isomorphisms show that there is a long exact sequence of the form

$$\longrightarrow A_q \longrightarrow A'_q \oplus B_q \longrightarrow B'_q \longrightarrow A_{q-1} \longrightarrow A'_{q-1} \oplus B_{q-1} \longrightarrow B'_{q-1} \longrightarrow$$

First define carefully the homomorphisms in the above sequence. Then prove exactness at each location.

Proof. We define the maps in the above sequence as follows:

$$\begin{aligned} \alpha_q: A_q &\longrightarrow A'_q \oplus B_q & a &\mapsto (a_q(a), f_q(a)) \\ \beta_q: A'_q \oplus B_q &\longrightarrow B'_q & (a', b) &\mapsto f'_q(a') - b_q(b) \\ \gamma_q: B'_q &\longrightarrow A_{q-1} & b' &\mapsto \partial_q c_q^{-1} g'_q(b') \end{aligned}$$

Exactness at B'_q . First we show $\gamma_q \beta_q = 0$. For $(a', b) \in A'_q \oplus B_q$ we have

$$\gamma_q \beta_q(a', b) = \partial_q c_q^{-1} g'_q(f'_q a' - b_q b) = -\partial_q c_q^{-1} g'_q b_q b = \partial_q g_q b = 0$$

Here the second equality holds due to $g'_q f'_q = 0$, the third follows from the commutativity of the third square, and the last is due to $\partial_q g_q = 0$.

To show $\ker \gamma_q \subset \text{im } \beta_q$ let $b' \in B'_q$ with $\gamma_q b' = \partial_q c_q^{-1} g'_q b' = 0$. By exactness at C_q there is an element $b \in B_q$ such that $g_q b = c_q^{-1} g'_q b'$ or equivalently

$$g'_q b' = c_q g_q b = g'_q b_q b,$$

where the second equality follows from commutativity of the third square. It follows that $b' - b_q b$ is in the kernel of g'_q and hence by exactness at B'_q , there is an element $a' \in A'_q$ with $f'_q a' = b' - b_q b$. This implies

$$\beta_q(a', -b) = f'_q a' + b_q b = b'$$

which shows that b' is in the image of β_q .

Exactness at $A'_q \oplus B_q$. First we show $\beta_q \alpha_q = 0$. For $a \in A_q$ we have

$$\beta_q \alpha_q a = \beta_q(a_q a, f_q a) = f'_q a_q a - b_q f_q a = 0$$

due to the commutativity of the second square.

To show $\ker \beta_q \subset \text{im } \alpha_q$, let $(a', b) \in A'_q \oplus B_q$ with

$$\beta_q(a', b) = f'_q a' - b_q b = 0.$$

Then we have

$$c_q g_q b = g'_q b_q b = g'_q f'_q a' = 0,$$

where the first equality is due to the commutativity of the third square, and the last is due to exactness at B'_q . Since c_q is an isomorphism, this implies $g_q b = 0$ and hence by exactness at B_q , there is an element $a \in A_q$ with $f_q a = b$. If we could show $a_q a = a'$, we would be done. However we can only say the following:

$$f'_q(a_q a - a') = f'_q a_q a - f'_q a' = b_q f_q a - b_q b = 0$$

where the second equality follows from the commutativity of the second square and our assumption $f'_q a' = b_q b$. Since f'_q is not necessarily injective, we can't conclude that $a_q a = a'$, but thanks to exactness at A'_q , it implies that there is an element $c' \in C'_{q+1}$ with $\partial_{q+1} c' = a_q a - a'$. Moreover, since c_{q+1} is an isomorphism, there is a $c \in C_{q+1}$ with $c_{q+1} c = c'$. Now we modify the element $a \in A_q$ by defining $\bar{a} \stackrel{\text{def}}{=} a - \partial_{q+1} c$. We calculate

$$\begin{aligned} f_q \bar{a} &= f_q(a - \partial_{q+1} c) = f_q a = b \\ a_q \bar{a} &= a_q(a - \partial_{q+1} c) = a_q a - \partial'_{q+1} c_{q+1} c = a_q a - (a_q a - a') = a' \end{aligned}$$

This shows that $\alpha_q(\bar{a}) = (a', b)$ as desired.

Exactness at A_q . First let us show $\alpha_q \circ \gamma_{q+1} = 0$. For $b' \in B'_{q+1}$ we have

$$\begin{aligned} \alpha_q \gamma_{q+1} b' &= \alpha_q(\partial_{q+1} c_{q+1}^{-1} g'_{q+1} b') \\ &= (a_q \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b', f_q \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b') \\ &= (\partial'_{q+1} g'_{q+1} b', 0) = (0, 0) \end{aligned}$$

since the compositions $f_q \partial_{q+1}$ and $\partial'_{q+1} g'_{q+1}$ are zero due to the exactness at A_q resp. A'_q .

To show $\ker \alpha_q \subset \text{im } \gamma_{q+1}$, let $a \in A_q$ with $\alpha_q a = (a_q a, f_q a) = (0, 0)$. By exactness at A_q there is an element $c \in C_{q+1}$ with $\partial_{q+1} c = a$. Then

$$\partial'_{q+1} c_{q+1} c = a_q \partial_{q+1} c = a_q a = 0$$

and hence by exactness at C'_{q+1} , there is an element $b' \in B'_{q+1}$ with $g'_{q+1}b' = c_{q+1}c$. This implies

$$\gamma_{q+1}b' = \partial_{q+1}c_{q+1}^{-1}g'_{q+1}b' = \partial_{q+1}c = a,$$

which shows that a is in the image of γ_{q+1} . □

3. (10 points) Let x_1, \dots, x_l be distinct points of \mathbb{R}^n . Calculate the reduced homology groups of the space $\mathbb{R}^n \setminus \{x_1, \dots, x_l\}$. Hint: Compare the homology groups of $\mathbb{R}^n \setminus \{x_1, \dots, x_l\}$ with those of \mathbb{R}^n by analyzing the long exact homology sequence of this pair of spaces.

Proof. Consider the following portion of the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus X)$, $X \stackrel{\text{def}}{=} \{x_1, \dots, x_l\}$:

$$\tilde{H}_{q+1}(\mathbb{R}^n) \longrightarrow \tilde{H}_{q+1}(\mathbb{R}^n, \mathbb{R}^n \setminus X) \xrightarrow{\partial} \tilde{H}_q(\mathbb{R}^n \setminus X) \longrightarrow \tilde{H}_q(\mathbb{R}^n)$$

The reduced homology groups of \mathbb{R}^n vanish since \mathbb{R}^n is contractible, and hence the map ∂ in the sequence above is an isomorphism.

Let $B_i \subset \mathbb{R}^n$ be a collection of mutually disjoint open balls with center $x_i \in B_i$, and let $B \stackrel{\text{def}}{=} \bigcup_{i=1}^l B_i$ be the union of these balls. We note that the pair $(B, B \setminus X)$ is obtained from the larger pair $(\mathbb{R}^n, \mathbb{R}^n \setminus X)$ by excising $U = \mathbb{R}^n \setminus B$. We note that the closure of U is contained in the open set $\mathbb{R}^n \setminus X$ and hence we obtain the excision isomorphism

$$H_q(\mathbb{R}^n, \mathbb{R}^n \setminus X) \cong H_q(B, B \setminus X).$$

To calculate this homology group, we note that the pair $(B, B \setminus X)$ is the disjoint union of the pairs $(B_i, B_i \setminus \{x_i\})$, and hence

$$H_q(B, B \setminus X) \cong \bigoplus_{i=1}^l H_q(B_i, B_i \setminus \{x_i\}).$$

Finally we note that B_i is a manifold of dimension n and hence the local homology group $H_q(B_i, B_i \setminus \{x_i\})$ is isomorphic to \mathbb{Z} for $q = n$ and trivial for $q \neq n$. Putting the various isomorphisms together, we obtain:

$$\tilde{H}_q(\mathbb{R}^n \setminus X) = \begin{cases} \mathbb{Z}^l & q = n - 1 \\ 0 & q \neq n - 1 \end{cases}$$

□

4. a) (10 points) Show that if $f: S^n \rightarrow S^n$ is a continuous map of degree 0, then there are points $x, y \in S^n$ such that $f(x) = x$ and $f(y) = -y$. Hint: Show that if $f(y) \neq -y$ for all $y \in S^n$, you could construct a homotopy between f and the identity map. Use a similar argument to show that $f(x) \neq x$ for all $x \in S^n$ leads to a contradiction.

Proof. Let us assume that $f(y) \neq -y$ for all $y \in S^n$. This implies that for all $y \in S^n$ and $t \in [0, 1]$ the vector $(1-t)f(y) + ty$ is non-zero and hence

$$H: S^n \times [0, 1] \longrightarrow S^n \quad H(y, t) = \frac{(1-t)f(y) + ty}{\|(1-t)f(y) + ty\|}$$

is a homotopy between f and the identity map $\mathbb{1}$. This leads to the contradiction $0 = \deg(f) = \deg(\mathbb{1}) = 1$.

Similarly, if we assume that $f(x) \neq x$ for all $x \in S^n$, then for all $x \in S^n$ and $t \in [0, 1]$ the vector $(1-t)f(x) - tx$ is non-zero and hence

$$H: S^n \times [0, 1] \longrightarrow S^n \quad H(x, t) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

is a homotopy between f and $-\mathbb{1}$. This leads to the contradiction $0 = \deg(f) = \deg(-\mathbb{1}) = \pm 1$. \square

b) (5 points) Let F be a vector field on the disk $D^n \subset \mathbb{R}^n$; i.e., F is a continuous map $F: D^n \rightarrow \mathbb{R}^n$ which we picture by drawing the vector $F(x)$ with its tail at the point x . Show that if $F(x) \neq 0$ for all $x \in D^n$, then there must be some point x on ∂D^n where F points radially outward, and another point y on ∂D^n where F points radially inward (i.e., $F(x) = ax$ and $F(y) = -by$ for some positive real numbers a, b).

Proof. Due to the assumption $F(x) \neq 0$ for all $x \in D^n$, we can construct a homotopy

$$H: S^n \times [0, 1] \longrightarrow S^n \quad H(x, t) = \frac{F(tx)}{\|F(tx)\|}$$

between the map $f: S^n \rightarrow S^n$ given by $f(x) = \frac{F(x)}{\|F(x)\|}$ and the constant map $F_0: S^n \rightarrow S^n, x \mapsto \frac{F(x)}{\|F(x)\|}$. Since F_0 is not surjective, its degree is zero and hence $\deg(f) = \deg(F_0) = 0$. This allows us to apply part (a) to the map f and we conclude that there exists points $x, y \in S^n$ such that $f(x) = x, f(y) = -y$. In terms of the vector field F this means that $F(x) = \|F(x)\|f(x) = \|F(x)\|x$ and $F(y) = \|F(y)\|f(y) = -\|F(y)\|y$ as desired. \square