Homework Assignment # 9, due April 2

1. Let Y be a convex subset of a euclidean space, and let $AC_*(Y) \subset C_*(Y)$ be the subcomplex generated by affine linear simplices. Let

$$(1) \qquad S\colon AC_q(Y) \longrightarrow AC_q(Y) \qquad T\colon AC_q(Y) \longrightarrow AC_{q+1}(Y)$$

be the linear maps constructed in class. We recall that the 'barycentric subdivision map' is a chain map $AC_*(Y) \to AC_*(Y)$, and that T is a chain homotopy from S to $\mathbb{1}$, i.e.,

$$T\partial + \partial T = \mathbb{1} - S.$$

We extend S and T to linear maps

$$S\colon C_q(X) \longrightarrow C_q(X) \qquad T\colon C_q(X) \longrightarrow C_{q+1}(X)$$

on chains on any space X by defining for a simplex $\sigma\colon \Delta^q \to X$

$$S(\sigma) = \sigma_\# S[e_0, \dots, e_q] \qquad T(\sigma) = \sigma_\# T[e_0, \dots, e_q].$$

Here $[e_0, \dots, e_q]\colon \Delta^n \to \Delta^n$ is the affine linear simplex given by the identity map; in particular, we can apply the maps (1) to it (for $Y = \Delta^q$) and obtain affine linear chains $S[e_0, \dots, e_q] \in AC_q(\Delta^q) \subset C_q(\Delta^q)$ (resp. $T[e_0, \dots, e_q] \in AC_{q+1}(\Delta^q) \subset C_{q+1}(\Delta^q)$). Then we can apply the chain map

$$\sigma_\#\colon C_*(\Delta^q) \longrightarrow C_*(X)$$

induced by $\sigma\colon \Delta^q \to X$ to obtain a chain in $C_*(X)$.
(a) Show that $S\colon C_*(X) \longrightarrow C_*(X)$ is a chain map.
(b) Show that T is a chain homotopy from S to the identity of $C_*(X)$.

2. Show that for $k, l \geq 1$

$$\widetilde{H}_q((S^k \times D^l)/(S^k \times S^{l-1})) = \begin{cases} \mathbb{Z} & q = l, k+l \\ 0 & q \neq l, k+l \end{cases}$$

3. Suppose M is a compact manifold of dimension n, and suppose $f\colon S^k \times D^{n-k} \to M$ is an embedding (i.e., a homeomorphism onto its image). Let $M' := M \setminus f(S^k \times \operatorname{int}(D^{n-k}))$, and let

$$\widehat{M} := M' \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1};$$

in other words, $\widehat{M}$ is obtained from the disjoint union

$$M' \amalg D^{k+1} \times S^{n-k-1}$$

by identifying a point $(x, y) \in S^k \times S^{n-k-1} \subset D^{k+1} \times S^{n-k-1}$ with $f(x, y) \in M'$. It is not hard to show that $\widehat{M}$ is again a closed n-manifold. This is an important way to modify a manifold M called *surgery*. More precisely we say that $\widehat{M}$ *is obtained by a k-surgery* from M. The effect on homology groups is easiest to determine for $k < m$ for $n = 2m$ or $n = 2m+1$ (m is called the 'middle dimension of M' and consequently this is called 'surgery below the middle dimension').

(a) Show that the inclusion $M' \to M$ induces an isomorphism on H_q for $q < m$. Hint: consider the long exact sequence of the pair (M, M'), show that this is a good pair, and identify M/M'.

(b) Show that the inclusion $M' \to \widehat{M}$ induces an isomorphism on H_q for $q < n-1$, $q \neq k, k+1$. Hint: consider the long exact sequence of the pair $(\widehat{M}, M')$, show that this is a good pair, and identify $\widehat{M}/M'$.

(c) Let g be the composition $\mathbb{Z} \cong H_k(S^k) \cong H_k(S^k \times D^{n-k}) \xrightarrow{f_*} H_k(M)$. Show that $H_k(\widehat{M}) \cong H_k(M)/g(\mathbb{Z})$ and

$$H_{k+1}(\widehat{M}) = \begin{cases} H_{k+1}(M) & g \text{ injective} \\ H_{k+1}(M) \oplus \mathbb{Z} & g \text{ not injective} \end{cases}$$

Hint: in order to identify the boundary homomorphisms in the exact sequence of the pair $(\widehat{M}, M')$, compare with the long exact sequence of the pair (D^{k+1}, S^k).