Homework Assignment # 7, due March 19

1. The *Moore space* $M(\mathbb{Z}/p,n)$ for $n \geq 1$ is defined to by $M(\mathbb{Z}/p,n) = S^n \cup e^{n+1}$, where the attaching map $\varphi\colon S^n \to S^n$ of the $n+1$-cell has degree p. Calculate the reduced homology groups of the Moore space $M(\mathbb{Z}/p,n)$.

2. If X is a topological space, we can define its *Euler characteristic* as

$$\chi(X) = \sum_i (-1)^i \operatorname{rk} H_i(X),$$

provided each homology group $H_i(X)$ is finitely generated and all but finitely many homology groups are zero (in this case we say that X is of *bounded finite type*). Let X be a topological space and $A, B \subset X$ subspaces with $X = \operatorname{int}(A) \cup \operatorname{int}(B)$.
a) Show that if A, B and $A \cap B$ are of bounded finite type, then so is X.
b) Show that in this case

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

We remark that the Euler characteristic of a topological space X of bounded finite type is a generalization of the cardinality of a finite set, since if X is a finite set, then its Euler characteristic equals its cardinality. We note that the formula to be proved in (b) obviously holds if X is a finite set.

3. If X, Y are finite CW complexes, then the product $X \times Y$ again has a CW structure whose cells correspond to products of cells of X and Y. More precisely, if e^m_α is an m-cell of X, and e^n_β is an n-cell of Y, then these determine a $m+n$-cell of $X \times Y$ denoted $e^m_\alpha \times e^n_\beta$. In particular, $C^{CW}_q(X \times Y)$ is the free abelian group generated by products cells $e^m_\alpha \times e^n_\beta$ with $m+n = q$. The cellular boundary map is determined by the formula

$$d(e^m_\alpha \times e^n_\beta) = (de^m_\alpha) \times e^n_\beta + (-1)^m e^m_\alpha \times d(e^n_\beta).$$

a) Calculate $H_*(S^m \times S^n)$.
b) Calculate the homology groups of $\underbrace{S^n \times \cdots \times S^n}_{k}$.

4. a) Calculate the homology groups of $\mathbb{R}P^2 \times \mathbb{R}P^2$.
b) Generalizing part a), calculate the homology groups of a product of two Moore spaces $M(\mathbb{Z}/p,m) \times M(\mathbb{Z}/q,n)$.