Homework Assignment # 6, due March 12

1. Let $x_1, \dots, x_l$ be points in S^n, and let $U_1, \dots, U_l$ be mutually disjoint neighborhoods of these points. Fix a generator $\alpha \in H_n(S^n) \cong \mathbb{Z}$ and let $\alpha_k \in H_n(U_k, U_k \setminus \{x_k\})$ be the image of α via the isomorphism

$$\widetilde{H}_n(S^n) \xrightarrow[\cong]{(j_k)_*} H_n(S^n, S^n \setminus \{x_k\}) \xleftarrow[\cong]{(i_k)_*} H_n(U_k, U_k \setminus \{x_k\})$$

Show that the composition

$$\begin{array}{ccccc}
\widetilde{H}_n(S^n) & \xrightarrow[\cong]{j_*} & H_n(S^n, S^n \setminus \{x_1, \dots, x_l\}) & \xleftarrow[\cong]{(i_k)_*} & H_n(\amalg U_k, \amalg\,(U_k \setminus \{x_k\})) \\
 & & & & \Big\downarrow \cong \\
 & & & & \bigoplus_{k=1}^{l} H_n(U_k, U_k \setminus \{x_k\})
\end{array}$$

maps α to $\alpha_1 + \dots + \alpha_l$.

2. Let

$$\begin{array}{ccccc}
A_* & \xrightarrow{f_*} & B_* & \xrightarrow{g_*} & C_* \\
\Big\downarrow{\scriptstyle a_*} & & \Big\downarrow{\scriptstyle b_*} & & \Big\downarrow{\scriptstyle c_*} \\
A'_* & \xrightarrow{f_*} & B'_* & \xrightarrow{g_*} & C'_*
\end{array}$$

be a commutative diagram of chain complexes and chain maps whose rows are short exact. Show that the following diagram is commutative:

$$\begin{array}{ccc}
H_q(C_*) & \xrightarrow{\partial} & H_q(A_*) \\
\Big\downarrow{\scriptstyle c_*} & & \Big\downarrow{\scriptstyle a_*} \\
H_q(C'_*) & \xrightarrow{\partial} & H_{q-1}(A'_*)
\end{array}$$

This statement is referred to as the *naturality of the connecting homomorphism*. In particular, if $f\colon (X, A) \to (X', A')$ is a map between pairs of topological spaces, then the singular chain complexes of A, X, (X, A), A', X', (X', A') fit together in a diagram as above and we conclude that the diagram

$$\begin{array}{ccc}
H_q(X, A) & \xrightarrow{\partial} & H_{q-1}(A) \\
\Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\
H_q(X', A') & \xrightarrow{\partial} & H_{q-1}(A')
\end{array}$$

is commutative.

3. Let $p(z)$ be a non-constant polynomial with roots $z_1, \dots, z_l$. Show that the local degree $\deg(p, z_k)$ is equal to the multiplicity of the root z_k.

4. Show that the complex projective space $\mathbb{CP}^n$ is a CW complex with one cell of dimension $2i$ for $0 \leq i \leq n$.