Homework Assignment # 4, due Feb. 13

1. Let $A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$ be a short exact sequence of chain complexes. Show that the sequence

$$\longrightarrow H_q(A) \xrightarrow{f_*} H_q(B) \xrightarrow{g_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{f_*} H_{q-1}(B) \longrightarrow$$

is exact at $H_q(C)$ and $H_{q-1}(B)$.

2. Recall that the *reduced homology groups* $\widetilde{H}_q(X)$ of a space X are the homology groups of the *augmented chain complex*

$$\ldots \qquad 0 \longleftarrow \mathbb{Z} \xleftarrow{\epsilon} C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_2} \qquad \ldots$$

For a pair (X, A) with $A \neq \emptyset$, we define $\widetilde{H}_q(X, A) = H_q(X, A)$.
a) Show that $\widetilde{H}_q(X) \cong H_q(X)$ for $q \neq 0$, and that there is a short exact sequence

$$0 \to \widetilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0. \tag{1}$$

Hint: one way of doing this is to construct a chain map from the augmented chain complex to the chain complex of X, try to fit it in a short exact sequence of chain complexes, and use the exact homology sequence of problem 1.

We note that equation (1) implies in particular that there is an isomorphism $\widetilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$, but this is not *canonical*.
b) Show that there is an exact sequence of reduced homology groups

$$\longrightarrow \widetilde{H}_q(A) \xrightarrow{i_*} \widetilde{H}_q(X) \xrightarrow{j_*} \widetilde{H}_q(X, A) \xrightarrow{\partial} \widetilde{H}_{q-1}(A) \xrightarrow{i_*} \ldots$$

3. Prove the following statement which is known as the 5-*lemma*. Suppose we have a commutative diagram of abelian groups and group homomorphisms

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

such that the rows are exact sequences. Show that if the outer two vertical maps h_1, h_2, h_4, and h_5 are isomorphisms, then also the middle map h_3 is an isomorphism.

Remark: the assumptions that the maps h_1, h_2, h_4, h_5 are all isomorphisms are slightly stronger than needed for the proof. What weaker assumptions will do?