Homework Assignment # 11, due April 16

1. Show $\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$ for cochains $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$.

2. Show that the cup product is compatible with pull-back of cohomology classes in the sense that for a map $f\colon X \to Y$ and cohomology classes $\alpha \in H^k(Y; R)$, $\beta \in H^l(Y; R)$ we have

$$f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta).$$

Hint: Show first the analogous statement for cochains.

3. We recall that the homology of the Klein bottle K is given by

$$H_q(K) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & q = 1 \end{cases}$$

(a) Use the UCT to determine the cohomology groups $H^q(K; \mathbb{Z}/2)$.
(b) Determine the cup products on cohomology with $\mathbb{Z}/2$ coefficients. Hint: proceed similarly to what we did in class for determining the cup products for the cohomology of the torus.

4. Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in the cohomology groups $H^q(M_g; \mathbb{Z})$ for M_g the closed orientable surface of genus g, by using the quotient map from M_g to a wedge-sum of g tori (this is problem # 1 on page 226 in Hatcher's book, where you can find a picture of this quotient map).