Homework Assignment # 11, due April 16

1. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of left modules over a ring R. Let $\epsilon^A\colon A_* \to A$ and $\epsilon^C\colon C_* \to C$ be free resolutions. Show that there exists a free resolution $\epsilon^B\colon B_* \to B$ and chain maps $f*\colon A_* \to B_*$, $g_*\colon B_* \to C_*$ such that the diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_2^A} & & \downarrow{\scriptstyle d_2^B} & & \downarrow{\scriptstyle d_2^C} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_1^A} & & \downarrow{\scriptstyle d_1^B} & & \downarrow{\scriptstyle d_1^C} & & \\
0 & \longrightarrow & A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon^A} & & \downarrow{\scriptstyle \epsilon^B} & & \downarrow{\scriptstyle \epsilon^C} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

is commutative and that the first row is an exact sequence of chain complexes. Hint: make the following Ansatz for B_* (this German word doesn't have an english translation and is used in english; if you don't know it, look it up in wikipedia): let $B_q = A_q \oplus C_q$, let $f_q\colon A_q \to A_q \oplus C_q$ be the inclusion, and let $g_q\colon A_q \oplus C_q \to C_q$ be the projection map. Then construct the homomorphism ϵ^B and then inductively the boundary maps $d_q^B\colon B_q \to B_{q-1}$ in such a way that $d_{q-1}^B \circ d_q^B = 0$ (here we interpret d_0^B as ϵ^B) and the above diagram is commutative. Show that the exactness of the middle column of the above diagram is a consequence of the exactness of the left and right column.

2. Prove the Universal Coefficient Theorem for cohomology groups; i.e., show that if C_* is a chain complex of left modules over a PID R, and P is a left R-module, then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(H_{q-1}(C_*), P) \longrightarrow H^q(C_*, P) \longrightarrow \mathrm{Hom}_R(H_q(C_*), P) \longrightarrow 0$$

Hint: Adapt the proof of the homology Universal Coefficient Theorem we did in class to the cohomology case.

3. Use our knowledge of the homology groups of the real projective space $\mathbb{RP}^n$ to calculate

(a) the homology groups $H_q(\mathbb{RP}^n; \mathbb{Z}/2)$;

(b) the cohomology groups $H^q(\mathbb{RP}^n; \mathbb{Z})$;
(c) the cohomology groups $H^q(\mathbb{RP}^n; \mathbb{Z}/2)$.
(d) Calculate the cohomology groups $H^q(\mathbb{RP}^n; \mathbb{Z}/2)$ using only the result you obtained in part (a), but no knowledge of the homology groups $H_q(\mathbb{RP}^n)$.

4. Let X be a topological space whose homology groups $H_q(X)$ are all finitely generated. Then the Universal Coefficient Theorem implies that for any field $\mathbb{F}$ the vector spaces $H_q(X; \mathbb{F})$ are finite dimensional. A convenient way to encode *all* the $\mathbb{F}$-homology groups is the *Poincaré series*

$$P(X; \mathbb{F})(z) := \sum_{q=0}^{\infty} \dim_{\mathbb{F}} H_q(X; \mathbb{F})\, z^q$$

Show that

$$P(X \amalg Y; \mathbb{F}) = P(X; \mathbb{F}) + P(Y; \mathbb{F})$$
$$P(X \times Y; \mathbb{F}) = P(X; \mathbb{F}) \cdot P(Y; \mathbb{F})$$