Notes on point set topology, Fall 2010

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1 Pointset Topology

1.1 Metric spaces and topological spaces

We recall that a map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ between Euclidean spaces is *continuous* if and only if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \in X \quad d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon, \tag{1.1}$$

where $d(x,y) = |x-y| \in \mathbb{R}_{\geq 0}$ is the distance of two points x, y in some Euclidean space.

Example 1.2. (Examples of continuous maps.)

1. The addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 + x_2$;
2. The multiplication map $m\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 x_2$;

The proofs that these maps are continuous are simple estimates that you probably remember from calculus. Since the continuity of *all* the maps we'll look at in these notes is proved by expressing them in terms of the maps a and m, we include the proofs of continuity of a and m for completeness.

Proof. To prove that the addition map a is continuous, suppose $x = (x_1, x_2) \in \mathbb{R}^2$ and $\epsilon > 0$ are given. We claim that for $\delta := \epsilon/2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x,y) < \delta$ we have $d(a(x), a(y)) < \epsilon$ and hence a is a continuous function. To prove the claim, we note that

$$d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

and hence $|x_1 - y_1| \leq d(x,y)$, $|x_1 - y_1| \leq d(x,y)$. It follows that

$$d(a(x), a(y)) = |a(x) - a(y)| = |x_1 + x_2 - y_1 - y_2| \leq |x_1 - y_1| + |x_2 - y_2| \leq 2d(x,y) < 2\delta = \epsilon.$$

To prove that the multiplication map m is continuous, we claim that for

$$\delta := \min\{1, \epsilon/(|x_1| + |x_2| + 1)\}$$

and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(m(x), m(y)) < \epsilon$ and hence m is a continuous function. The claim follows from the following estimates:

$$\begin{aligned}
d(m(y), m(x)) &= |y_1y_2 - x_1x_2| = |y_1y_2 - x_1y_2 + x_1y_2 - x_1x_2| \\
&\leq |y_1y_2 - x_1y_2| + |x_1y_2 - x_1x_2| = |y_1 - x_1||y_2| + |x_1||y_2 - x_2| \\
&\leq d(x, y)(|y_2| + |x_1|) \leq d(x, y)(|x_2| + |y_2 - x_2| + |x_1|) \\
&\leq d(x, y)(|x_1| + |x_2| + 1) < \delta(|x_1| + |x_2| + 1) \leq \epsilon
\end{aligned}$$

□

Lemma 1.3. *The function* $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *has the following properties:*

1. $d(x, y) = 0$ *if and only if* $x = y$*;*
2. $d(x, y) = d(y, x)$ *(symmetry);*
3. $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Definition 1.4. A *metric space* is a set X equipped with a map

$$d\colon X \times X \to \mathbb{R}_{\geq 0}$$

with properties (1)-(3) above. A map $f\colon X \to Y$ between metric spaces X, Y is

an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$;

continuous if condition (1.1) is satisfied.

Two metric spaces X, Y are *isometric* (resp. *homeomorphic*) if there are isometries (resp. continuous maps) $f\colon X \to Y$ and $g\colon Y \to X$ which are inverses of each other.

Example 1.5. An important class of examples of metric spaces are subsets of $\mathbb{R}^n$. Here are particular examples we will be talking about during the semester:

1. The *n-disk* $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\} \subset \mathbb{R}^n$, and more generally, the *n-disk of radius r* $D^n_r := \{x \in \mathbb{R}^n \mid |x| \leq r\}$. We note that D^2_r is homeomorphic to D^2 for all r, but D^2_r is isometric to D^2 if and only if $r = 1$. (To see that D^n_r is not isometric to D^n_s we note if a metric space X is isometric to a metric space Y, then $\operatorname{diam}(X) = \operatorname{diam}(Y)$, where $\operatorname{diam}(X)$, the *diameter of X* is defined by $\operatorname{diam}(X) := \sup\{d(x, y) \mid x, y \in X\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. It is easy to see that $\operatorname{diam}(D^n_r) = 2r$.)
2. The *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \subset \mathbb{R}^{n+1}$.
3. The *torus* $T = \{v \in \mathbb{R}^3 \mid d(v, C) = r\}$ for $0 < r < 1$. Here $C = \{(x, y, 0) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$ is the standard circle, and $d(x, C) = \inf_{y \in C} d(x, y)$ is the distance between x and C.

4. The *general linear group*

$$\begin{aligned} GL_n(\mathbb{R}) &= \{\text{vector space isomorphisms } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &\leftrightarrow \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) \neq 0\} \\ &= \{\text{invertible } n \times n\text{-matrices}\} \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} = \mathbb{R}^{n^2} \end{aligned}$$

Here the bijection sends $f\colon \mathbb{R}^n \to \mathbb{R}^n$ to $(f(e_1), \dots, f(e_n))$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.

5. The *special linear group*

$$SL_n(\mathbb{R}) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

6. The *orthogonal group*

$$\begin{aligned} O(n) &= \{\text{linear isometries } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{n^2} \end{aligned}$$

We recall that a collection of vectors $v_i \in \mathbb{R}^n$ is *orthonormal* if $|v_i| = 1$ for all i, and v_i is perpendicular to v_j for $i \neq j$.

7. The *special orthogonal group*

$$SO(n) = \{(v_1, \dots, v_n) \in O(n) \mid \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

8. The *Stiefel manifold*

$$\begin{aligned} V_k(\mathbb{R}^n) &= \{\text{linear isometries } f\colon \mathbb{R}^k \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{kn} \end{aligned}$$

Example 1.6. The following maps between metric spaces are continuous. While it is possible to prove their continuity using the definition of continuity, it will be much simpler to prove their continuity by 'building' these maps using compositions and products from the continuous maps a and m of Example 1.2. We will do this below in Lemma 1.17.

1. Every polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous. We recall that a polynomial function is of the form $f(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdot \cdots \cdot x_n^{i_n}$ for $a_{i_1, \dots, i_n} \in \mathbb{R}$.

2. Let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices. Then the map

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

given by matrix multiplication is continuous. Here we use the fact that a map to the product $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R} \times \cdots \times \mathbb{R}$ is continuous if and only if each component map is continuous (see Lemma 1.16), and each matrix entry of AB is a polynomial and hence a continuous function of the matrix entries of A and B. Restricting to the invertible matrices $GL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$, we see that the multiplication map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

is continuous. The same holds for the subgroups $SO(n) \subset O(n) \subset GL_n(\mathbb{R})$.

3. The map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous (this is a homework problem). The same statement follows for the subgroups of $GL_n(\mathbb{R})$.

Definition 1.7. Let X be a metric space. A subset $U \subset X$ is *open* if for every point $x \in U$ there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Here $B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\}$ is the *ball of radius ϵ around x*.

Lemma 1.8. *A map $f\colon X \to Y$ between metric space is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset $V \subset Y$.*

Proof: homework

Lemma 1.9. *Let X be a metric space, and $\mathcal{U}$ be the collection of open subsets of X. Then $\mathcal{U}$ has the following properties:*

1. *X and $\emptyset$ belong to $\mathcal{U}$.*
2. *The union of a collection in $\mathcal{U}$ belongs to $\mathcal{U}$.*
3. *The intersection of a finite collection of subsets in $\mathcal{U}$ belongs to $\mathcal{U}$.*

Definition 1.10. A *topology* on a set X is a collection $\mathcal{U}$ of subsets of X satisfying the properties of the previous lemma. A *topological space* is a pair $(X, \mathcal{U})$ consisting of a set X and a topology $\mathcal{U}$ on X. If $(X, \mathcal{U})$ is a topological space, a subset $U \subset X$ is *open* if U belongs to $\mathcal{U}$; it is *closed* if its complement $X \setminus U$ belongs to $\mathcal{U}$.

Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be topological spaces. A map $f\colon X \to Y$ is *continuous* if and only if $f^{-1}(U) \in \mathcal{V}$ for every $U \in \mathcal{U}$. It is easy to see that any composition of continuous maps is continuous.

Examples of topological spaces.

1. Let X be a metric space. Then $\mathcal{U} = \{\text{open subsets of } X\}$ is a topology on X, the *metric topology.*
2. Let X be a set. Then $\mathcal{U} = \{\text{all subsets of } X\}$ is a topology, the *discrete topology.* We note that *any* map $f\colon X \to Y$ to a topological space Y is continuous. We will see later that the only continuous maps $\mathbb{R}^n \to X$ are the constant maps.
3. Let X be a set. Then $\mathcal{U} = \{\emptyset, X\}$ is a topology, the *indiscrete topology.*

1.2 Constructions with topological spaces

The subspace topology. Let X be a topological space, and $A \subset X$ a subset. Then

$$\mathcal{U} = \{A \cup U \mid U \underset{open}{\subset} X\}$$

is a topology on A called the *subspace topology.*

Lemma 1.11. *Let X be a metric space and $A \subset X$. Then the metric topology on A agrees with the subspace topology on A (as a subset of X equipped with the metric topology).*

Lemma 1.12. *Let X, Y be topological spaces and let A be a subset of X equipped with the subspace topology. Then the inclusion map $i\colon A \to X$ is continuous and a map $f\colon Y \to A$ is continuous if and only if the composition $i \circ f\colon Y \to X$ is continuous.*

Basis for a topology. Sometimes it is convenient to define a topology $\mathcal{U}$ on a set X by first describing a smaller collection $\mathcal{B}$ of subsets of X, and then defining $\mathcal{U}$ to be those subsets of X that can be written as *unions* of subsets belonging to $\mathcal{B}$. We've done this already when defining the metric topology: Let X be a metric space and let $\mathcal{B}$ be the collection of subsets of X of the form $B_\epsilon(x) := \{y \in X \mid d(y,x) < \epsilon\}$ (we call these subsets *balls* in X). A subset of X is open (in the sense of Definition 1.7) if and only if it is a union of balls in X.

Lemma 1.13. *Let $\mathcal{B}$ be a collection of subsets of a set X satisfying the following conditions*

1. *Every point $x \in X$ belongs to some subset $B \in \mathcal{B}$.*
2. *If $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$ there is some $B \in \mathcal{B}$ with $x \in B$ and $B \subset B_1 \cap B_2$.*

Then $\mathcal{U} := \{$unions of subsets belonging to $\mathcal{B}\}$ is a topology on X.

If the above conditions are satisfied, the collection $\mathcal{B}$ is called a *basis for the topology* $\mathcal{U}$ or $\mathcal{B}$ *generates the topology* $\mathcal{U}$. It is easy to check that the collection of balls in a metric space satisfies the above conditions and hence the collection of open subsets is a topology as claimed by Lemma 1.9.

The Product topology

Definition 1.14. The *product topology* on the Cartesian product $X \times Y = \{(x,y) \mid x \in X,\, y \in Y\}$ of topological spaces X, Y is the topology with basis

$$\mathcal{B} = \{U \times V \mid U \underset{open}{\subset} X,\, V \underset{open}{\subset} Y\}$$

The collection $\mathcal{B}$ obviously satisfies property (1) of a basis; property (2) holds since $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$. We note that the collection $\mathcal{B}$ is *not* a topology since the union of $U \times V$ and $U' \times V'$ is typically not a Cartesian product (e.g., draw a picture for the case where $X = Y = \mathbb{R}$ and U, U', V, V' are open intervals).

Lemma 1.15. *The product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (with each factor equipped with the metric topology) agrees with the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.*

Proof: homework.

Lemma 1.16. *Let X, Y_1, Y_2 be topological spaces. Then the projection maps $p_i\colon Y_1 \times Y_2 \to Y_i$ is continuous and a map $f\colon X \to Y_1 \times Y_2$ is continuous if and only if the* component maps

$$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{p_i} Y_i$$

are continuous for $i = 1, 2$.

Proof: homework

Lemma 1.17. *1. Let $A \subset \mathbb{R}^n$ and let $f, g\colon A \to \mathbb{R}$ be continuous maps. Then $f + g$ and $f \cdot g$ continuous maps from A to $\mathbb{R}$. If $g(x) \neq 0$ for all $x \in A$, then also f/g is continuous.*

2. Any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.

3. The multiplication map $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is continuous.

Proof. To prove part (1) we note that the map $f + g\colon A \to \mathbb{R}$ can be factored in the form

$$A \xrightarrow{f\times g} \mathbb{R} \times \mathbb{R} \xrightarrow{a} \mathbb{R}$$

The map $f \times g$ is continuous by Lemma 1.16 since its component maps f, g are continuous; the map a is continuous by Example 1.2, and hence the composition $f + g$ is continuous. The argument for $f \cdot g$ is the same, with a replaced by m. To prove that f/g is continuous, we factor it in the form

$$A \xrightarrow{\quad f\times g \quad} \mathbb{R} \times \mathbb{R}^\times \xrightarrow{p_1\times(I\circ p_2)} \mathbb{R} \times \mathbb{R}^\times \xrightarrow{\quad m \quad} \mathbb{R},$$

where p_1 (resp. p_2) is the projection to the first (resp. second) factor of $\mathbb{R} \times \mathbb{R}^\times$ and $I\colon \mathbb{R}^\times \to \mathbb{R}^\times$ is the inversion map $x \mapsto x^{-1}$. By Lemma 1.16 the p_i's are continuous, in calculus we learned that I is continuous, and hence again by Lemma 1.16 the map $p_1 \times (I \circ p_2)$ is continuous.

To prove part (2), we note that the constant map $\mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto a$ is obviously continuous, and that the projection map $p_i\colon \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_i$ is continuous by Lemma 1.16. Hence by part (1) of this lemma, the monomial function $x \mapsto a x_1^{i_1} \cdots x_n^{i_n}$ is continuous. Any polynomial function is a sum of monomial functions and hence continuous.

For the proof of (3), let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices and let

$$\mu\colon M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

be the map given by matrix multiplication. By Lemma 1.16 the map μ is continuous if and only if the composition

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \xrightarrow{\mu} M_{n\times n}(\mathbb{R}) \xrightarrow{p_{ij}} \mathbb{R}$$

is continuous for all $1 \leq i, j \leq n$, where p_{ij} is the projection map that sends a matrix A to its entry $A_{ij} \in \mathbb{R}$. Since the $p_{ij}(\mu(A, B)) = (A \cdot B)_{ij}$ is a *polynomial* in the entries of the matrices A and B, this is a continuous map by part (2) and hence μ is continuous.

Restricting μ to invertible matrices, we obtain the multiplication map

$$\mu_|\colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

that we want to show is continuous. We will argue that in general if $f\colon X \to Y$ is a continuous map with $f(A) \subset B$ for subsets $A \subset X$, $B \subset Y$, then the *restriction* $f_{|A}\colon A \to B$ is continuous. To prove this, consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_{|A}} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

where i, j are the obvious inclusion maps. These inclusion maps are continuous w.r.t. the subspace topology on A, B by Lemma 1.12. The continuity of f and i implies the continuity of $f \circ i = j \circ f_{|A}$ which again by Lemma 1.12 implies the continuity of $f_{|A}$. □

Quotient topology. Let X be a topological space, let $\sim$ be an equivalence relation on X, let $X/\sim$ be the set of equivalence classes and let

$$p\colon X \to X/\sim \qquad x \mapsto [x]$$

be the projection map that sends a point $x \in X$ to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is the collection of subsets $\mathcal{U} = \{U \subset X/\sim \mid p^{-1}(U) \text{ is an open subset of } X\}$. The set $X/\sim$ equipped with the quotient topology is called the *quotient space*.

Lemma 1.18. *The projection map $p\colon X \to X/\sim$ is continuous and a map $f\colon X/\sim \to Y$ to a topological space Y is continuous if and only if the composition $p \circ f\colon X \to Y$ is continuous.*

Example 1.19. 1. Let A be a subset of a topological space X. Define a equivalence relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. We use the notation X/A for the quotient space $X/\sim$.

   (a) We claim that the quotient space $[-1,+1]/\{\pm 1\}$ is homeomorphic to S^1 via the map $f\colon [-1,+1]/\{\pm 1\} \to S^1$ given by $[t] \mapsto e^{\pi i t}$. Here we use that a continuous bijection $f\colon X \to Y$ from a compact space to a Hausdorff space is a homeomorphism.

   (b) More generally, D^n/S^{n-1} is homeomorphic to S^n. (proof: homework)

2. quotients of the square by various equivalence relations gives: torus, Klein bottle, real projective plane $D^2/\sim = S^2/\sim$. We can obtain a surface of genus 2 from an 8-gon with suitable boundary identifications (first redraw 8-gon as a union of squares with a corner chipped off; identifying boundaries on each square leads to punctured torus).

3. The real projective space

$$\mathbb{RP}^n := \{\text{1-dimensional subspaces of } \mathbb{R}^{n+1}\} = S^n/v \sim \pm v$$

   Homework: $\mathbb{RP}^1 \approx S^1$; $\mathbb{RP}^3 \approx SO(3)$

4. The complex projective space

$$\mathbb{CP}^n := \{\text{1-dimensional subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/v \sim zv, \qquad z \in S^1$$

   homework: $\mathbb{CP}^1 \approx S^2$

5. The Grassmann manifold $G_k(\mathbb{R}^{n+k}) := \{k\text{-dimensional subspaces of } \mathbb{R}^{n+k}\}$. There is a surjective map

$$V_k(\mathbb{R}^{n+k}) = \{\text{isometries } f\colon \mathbb{R}^k \to \mathbb{R}^{n+k}\} \twoheadrightarrow G_k(\mathbb{R}^{n+k}) \qquad f \mapsto \text{im}(f)$$

   Two isometries f, f' have the same image if and only if there is some isometry $g\colon \mathbb{R}^k \to \mathbb{R}^k$ such that $f' = f \circ g$. In other words, we get a bijection $V_k(\mathbb{R}^{n+k})/\sim \leftrightarrow G_k(\mathbb{R}^{n+k})$ if we define an equivalence relation $\sim$ on the Stiefel manifold by $f \sim f'$ if and only if there is some isometry $g\colon \mathbb{R}^k \to \mathbb{R}^k$ such that $f' = f \circ g$. This the quotient topology on $V_k(\mathbb{R}^{n+k})/\sim$ then gives $G_k(\mathbb{R}^{n+k})$ a topology (note that for $k = 1$, $V_k(\mathbb{R}^{n+k}) = S^n$, and this agrees with how we put a topology on the projective space $\mathbb{RP}^n = G_1(\mathbb{R}^{n+1})$.

6. If X is a topological space and a group H acts X (say from the right via $X \times H \to X$, $(x,h) \mapsto xh$; requirement: $(xh)h' = x(hh')$ for $x \in X$, $h, h' \in H$). The group action defines an equivalence relation $\sim$ on X via $x' \sim x$ if and only if there is some $h \in H$ such that $x' = xh$. Equivalence classes are called the *orbits* of the action; the quotient space $X/\sim$ is the *orbit space*, denoted X/H.

   (a) $G_k(\mathbb{R}^{n+k}) = V_k(\mathbb{R}^{n+k})/O(k)$

   (b) homogeneous spaces G/H for topological groups G. Explanation: a *topological group* is a group G equipped with a topology such that the multiplication map $G \times G \to G$ and the inversion map $G \to G$, $g \mapsto g^{-1}$ are continuous. A subgroup $H \leq G$ act on G via the multiplication map $G \times H \to G$, $(g,h) \mapsto gh$. The orbit space is denoted G/H (or $H \backslash G$ if we use the corresponding left H-action on G), and is called *homogeneous space*. Warning: there is difference between the homogeneous space G/H and the quotient space of G obtained by collapsing the subspace H to a point (Example 1.19 (1)), which we also would denote by G/H (unfortunately, both notations are standard; fortunately, it is usually clear from the context which version of G/H we are talking about, since the homogeneous space makes only sense if H is a subgroup of a topological group G).

We want to show that many topological spaces we've discussed so far are actually homogeneous spaces. To do that we use the following result.

Proposition 1.20. (Recognition principle for homogeneous spaces) *Let G be a compact topological group that acts continuously and transitively on a topological space X. Then X is homeomorphic to the homogeneous space G/H where $H = \{g \in G \mid gx_0 = x_0\}$ is the isotropy subgroup of some point $x_0 \in X$.*

Proof. Let

$$f \colon G/H \longrightarrow X \quad \text{be defined by} \quad [g] \mapsto gx_0$$

This map is *surjective* by the transitivity assumption; it is *injective* since if $gx_0 = g'x_0$, then $x_0 = g^{-1}g'x_0$ and hence $h := g^{-1}g'$ belongs to the isotropy subgroup H. This implies $g' = gh$, and hence $[g'] = [g] \in G/H$.

To show that f is continuous it suffices to show that the composition $f \circ p \colon G \to X$, $g \mapsto gx_0$ is continuous. To see this, we factor $f \circ p$ in the form

$$G = G \times \{x_0\} \hookrightarrow G \times X \xrightarrow{\mu} X$$

where μ is the action map □

Examples of homogeneous spaces.

1. spheres $S^n \approx O(n+1)/O(n)$
   (take the action $O(n+1) \times S^n \to S^n$, $(f,v) \mapsto f(v)$ and $x_0 = (0,\dots,0,1) \in S^n$)

2. Stiefel manifold $V_k(\mathbb{R}^{n+k}) \approx O(n+k)/O(n)$
   (take the action $O(n+k) \times V_k(\mathbb{R}^{n+k}) \to V_k(\mathbb{R}^{n+k})$, $(g,f) \mapsto g \circ f$ and $x_0 \colon \mathbb{R}^k \to \mathbb{R}^{n+k}$, $v \mapsto (0,v)$).

3. Grassmann manifold $G_k(\mathbb{R}^{n+k}) \approx O(n+k)/O(n) \times O(k)$ (homework problem).

1.3 Properties of topological spaces

Definition 1.21. Let X be a topological space, $x_i \in X$, $i = 1, 2, \dots$ a sequence in X and $x \in X$. Then x *is the limit of the* x_i*'s* if for all open subsets $U \subset X$ containing x there is some N such that $x_i \in U$ for all $i \geq N$.

Caveat: If X is a topological space with the indiscrete topology, *every point* is the limit of every sequence. The limit is *unique* if the topological space has the following property:

Definition 1.22. A topological space X is *Hausdorff* if for every $x, y \in X$, $x \neq y$, there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$.

Note: if X is a metric space, then the metric topology on X is Hausdorff (since for $x \neq y$ and $\epsilon = d(x, y)/2$, the balls $B_\epsilon(x)$, $B_\epsilon(y)$ are disjoint open subsets).

Warning: The notion of *Cauchy sequences* can be defined in metric spaces, but not in general for topological spaces (even when they are Hausdorff).

Lemma 1.23. *Let X be a topological space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.*

Proof. Assume $x \notin A$. Then x is a point in the open subset $X \setminus A$ and hence by the definition of limit, all but finitely many elements x_n must belong to $X \setminus A$, contradicting our assumptions. □

Definition 1.24. An *open cover* of a topological space X is a collection of open subsets of X whose union is X. If for every open cover of X there is a finite subcollection which also covers X, then X is called *compact.*

Some books (like Munkres' *Topology*) refer to open covers as *open coverings*, while newer books (and wikipedia) seem to prefer to above terminology, probably for the same reasons as me: to avoid confusions with *covering spaces*, a notion we'll introduce soon.

Now we'll prove some useful properties of compact spaces and maps between them, which will lead to the important Corollaries **??** and 1.27.

Lemma 1.25. *If $f\colon X \to Y$ is a continuous map and X is compact, then the image $f(X)$ is compact.*

In particular, if X is compact, then any quotient space $X/\sim$ is compact, since the projection map $X \to X/\sim$ is continuous with image $X/\sim$.

Proof. To show that $f(X)$ is compact assume that $\{U_a\}$, $a \in A$ is an open cover of the subspace $f(X)$. Then each U_a is of the form $U_a = V_a \cap f(X)$ for some open subset $V_a \in Y$. Then $\{f^{-1}(V_a)\}$, $a \in A$ is an open cover of X. Since X is compact, there is a finite subset A' of A such that $\{f^{-1}(V_a)\}$, $a \in A'$ is a cover of X. This implies that $\{U_a\}$, $a \in A'$ is a finite cover of $f(X)$, and hence $f(X)$ is compact. □

Lemma 1.26.

1. *If K is a closed subspace of a compact space X, then K is compact.*
2. *If K is compact subspace of a Hausdorff space X, then K is closed.*

Proof. To prove (1), assume that $\{U_a\}$, $a \in A$ is an open covering of K. Since the U_a's are open w.r.t. the subspace topology of K, there are open subsets V_a of X such that $U_a = V_a \cap K$. Then the V_a's together with the open subset $X \setminus K$ form an open covering of X. The compactness of X implies that there is a finite subset $A' \subset A$ such that the subsets V_a for $a \in A'$, together with $X \setminus K$ still cover X. It follows that U_a, $a \in A'$ is a finite cover of K, showing that K is compact.

The proof of part (2) is a homework problem. □

Corollary 1.27. *If $f\colon X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. We need to show that the map $g\colon Y \to X$ inverse to f is continuous, i.e., that $g^{-1}(U) = f(U)$ is an open subset of Y for any open subset U of X. Equivalently (by passing to complements), it suffices to show that $g^{-1}(C) = f(C)$ is a closed subset of Y for any closed subset C of C.

Now the assumption that X is compact implies that the closed subset $C \subset X$ is compact by part (1) of Lemma 1.26 and hence $f(C) \subset Y$ is compact by Lemma 1.25. The assumption that Y is Hausdorff then implies by part (2) of Lemma 1.26 that $f(C)$ is closed. □

Lemma 1.28. *Let K be a compact subset of $\mathbb{R}^n$. Then K is* bounded*, meaning that there is some $r > 0$ such that K is contained in the open ball $B_r(0) := \{x \in \mathbb{R}^n \mid d(x, 0) < r\}$.*

Proof. The collection $B_r(0) \cap K$, $r \in (0, \infty)$, is an open cover of K. By compactness, K is covered by a *finite* number of these balls; if R is the maximum of the radii of these finitely many balls, this implies $K \subset B_R(0)$ as desired. □

Corollary 1.29. *If $f\colon X \to \mathbb{R}$ is a continuous function on a compact space X, then f has a maximum and a minimum.*

Proof. $K = f(X)$ is a compact subset of $\mathbb{R}$. Hence K is bounded, and thus K has an infimum $a := \inf K \in \mathbb{R}$ and a supremum $b := \sup K \in \mathbb{R}$. The infimum (resp. supremum) of K is the limit of a sequence of elements in K; since K is closed (by Lemma 1.26 (2)), the limit points a and b belong to K by Lemma 1.23. In other words, there are elements $x_{min}, x_{max} \in X$ with $f(x_{min}) = a \leq f(x)$ for all $x \in X$ and $f(x_{max}) = b \geq f(x)$ for all $x \in X$. □

In order to use Corollaries 1.27 and 1.29, we need to be able to show that topological spaces we are interested in, are in fact compact. Note that this is *quite difficult* just working from the definition of compactness: you need to ensure that *every* open cover has a finite subcover. That sounds like a lot of work...

Fortunately, there is a very simple classical characterization of compact subspaces of Euclidean spaces:

Theorem 1.30. (Heine-Borel Theorem) *A subspace $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

We note that we've already proved that if $K \subset \mathbb{R}^n$ is compact, then K is a closed subset of $\mathbb{R}^n$ (Lemma 1.26(2)), and K is bounded (Lemma 1.28).

There two important ingredients to the proof of the converse, namely the following two results:

Lemma 1.31. *A closed interval $[a, b]$ is compact.*

This lemma has a short proof that can be found in any pointset topology book, e.g., [Mu].

Theorem 1.32. *If $X_1, \ldots, X_n$ are compact topological spaces, then their product $X_1 \times \cdots \times X_n$ is compact.*

For a proof see e.g. [Mu, Ch. 3, Thm. 5.7]. The statement is true more generally for a product of *infinitely many* compact space (as discussed in [Mu, p. 113], the correct definition of the product topology for infinite products requires some care), and this result is called *Tychonoff's Theorem*, see [Mu, Ch. 5, Thm. 1.1].

Proof of the Heine-Borel Theorem. Let $K \subset \mathbb{R}^n$ be closed and bounded, say $K \subset B_r(0)$. We note that $B_r(0)$ is contained in the n-fold product

$$P := [-r, r] \times \cdots \times [-r, r] \subset \mathbb{R}^n$$

which is compact by Theorem 1.32. So K is a closed subset of P and hence compact by Lemma 1.26(1). □

References

[Mu] Munkres, James R. *Topology: a first course*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975. xvi+413 pp.