**SOLUTIONS TO HOMEWORK PROBLEMS**

Contents



## 1. Homework Assignment # 1

1. Let $p\colon \widetilde{X} \to X$ be covering map which sends $\widetilde{x}_0 \in \widetilde{X}$ to $x_0 \in X$. Show that the induced map

$$p_*\colon \pi_n(\widetilde{X}, \widetilde{x}_0) \longrightarrow \pi_n(X, x_0)$$

is injective for $n \geq 1$. Hint: use the homotopy lifting property for the covering map $p$.

*Proof.* Since the induced map $p_*$ is a homomorphism, it suffices to show that the kernel of $p_*$ is trivial. So let $\widetilde{f}\colon (I^n, \partial I^n) \to (\widetilde{X}, \widetilde{x}_0)$ be a map of pairs representing an element in the kernel of $p_*$. In other words, there is a homotopy $H$ between the map $p \circ \widetilde{f}\colon (I^n, \partial I^n) \to (X, x_0)$ and the constant map $x_0\colon (I^n, \partial I^n) \to (X, x_0)$. More explicitly this means that we have that $H$ is a map $H\colon I^n \times I \to X$ with

$$\begin{aligned} H(s,0) &= p \circ \widetilde{f}(s) && \text{for } s \in I^n \\ H(s,t) &= x_0 && \text{for } s \in \partial I^n \text{ or } t = 1. \end{aligned}$$

In particular, the outer square of the diagram

$$\begin{tikzcd} I^n \times 0 \arrow[r, "\widetilde{f}"] \arrow[d, hook] & \widetilde{X} \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & \widetilde{X} \end{tikzcd}$$

commutes. By the homotopy lifting property for the covering space $\widetilde{X} \to X$, there is a map $\widetilde{H}$ making the whole diagram commutative. We claim that $\widetilde{H}$ is the desired homotopy between $\widetilde{f}\colon (I^n, \partial I^n) \to (\widetilde{X}, \widetilde{x}_0)$ and the constant map $\widetilde{x}_0\colon (I^n, \partial I^n) \to (\widetilde{X}, \widetilde{x}_0)$. So we need to show that

- $\widetilde{H}(s,t) = \widetilde{x}_0$ for all $s \in \partial I^n$ and $t \in I$.
- $\widetilde{H}(s,1) = \widetilde{x}_0$ for all $s \in I^n$ and

To prove the first property suppose that $F \subset \partial I^n$ is one of the $(n-1)$-dimensional faces that form the boundary of $I^n$. The map $H$ maps all of $F \times I$ to the base point $x_0 \in X$, and hence the lift $\widetilde{H}$ maps $F \times I$ to the fiber $p^{-1}(x_0) \subset \widetilde{X}$. Since $F \times I$ is connected, $\widetilde{H}$ maps all of $F \times I$ to a connected component of $p^{-1}(x_0)$, that is, to *one point* of the discrete space $p^{-1}(x_0)$. Since $\widetilde{H}$ maps $(s,0)$ to $\widetilde{x}_0$, it maps every point of $p^{-1}(x_0)$ to $\widetilde{x}_0$. This proves the first property above.

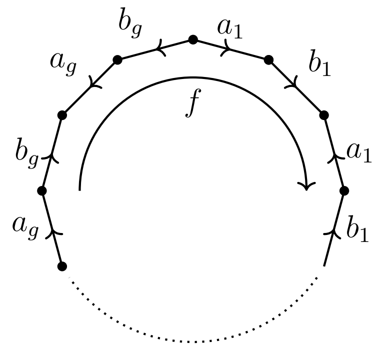
The second property follows by the same argument applied to $\widetilde{H}_{I^n \times \{1\}}$: since $\widetilde{H}$ is a lift of $H$ and $H$ maps $I^n \times \{1\}$ to the basepoint $x_0 \in X$, $\widetilde{H}$ maps the connected space $I^n \times \{1\}$ to one point of the fiber $p^{-1}(x_0)$. This point must be $\widetilde{x}_0$, since $\widetilde{H}$ maps the boundary of $I^n \times \{1\}$ to $\widetilde{x}_0$ by the first property. □

2. a) Calculate the homology groups of the surface $\Sigma_g$ of genus $g \geq 1$. Hint: Use the standard description of $\Sigma_g$, the connected sum $T \# \dots \# T$ of $g$ copies of the torus, as a polygon with boundary identifications.

b) Calculate the homology groups of the connected sum $X_k = \mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$ of $k$ copies of the real projective plane. Hint: Use the standard description of $X_k$ as a polygon with boundary identifications. For calculating the homology of the associated chain complex it will be useful to work with a convenient basis for the free $\mathbb{Z}$-module $C_1$, which is *not* the obvious basis given by the edges of polygonal pattern we use for the calculation.

c) According to the Classification Theorem for closed connected surfaces, every closed connected surface is homeomorphic to either $\Sigma_g$ for some $g \geq 0$ or homeomorphic to $X_k$ for some $k > 0$. Based on your calculations in part (a) and (b), make a conjecture on how the orientability of a closed connected surface $\Sigma$ is determined by its homology groups (we recall that $\Sigma_g$ is orientable, but $X_k$ is not).

*Proof.* **Part a).** We recall that the surface $\Sigma_g$ of genus $g$ can be described as the quotient of the $4g$-gon with edge identifications given by the picture




The resulting pattern of polygons $\Gamma$ on $\Sigma_g$ (enhanced by the orientations shown in the picture) has one vertex $v$, $2g$ edges $a_i, b_i$ for $1 \leq i \leq g$ and one face $f$. Hence the associated chain complex $C_*(\Sigma, \Gamma)$ looks as follows:

$$\mathbb{Z}v \xleftarrow{\partial_1} \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \cdots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_g \xleftarrow{\partial_2} \mathbb{Z}f$$

Since there is only one vertex $v$ involved, we have $\partial_1(e) = v - v = 0$ for any edge $e = a_i, b_i$ and hence $\partial_1 \equiv 0$. We note that

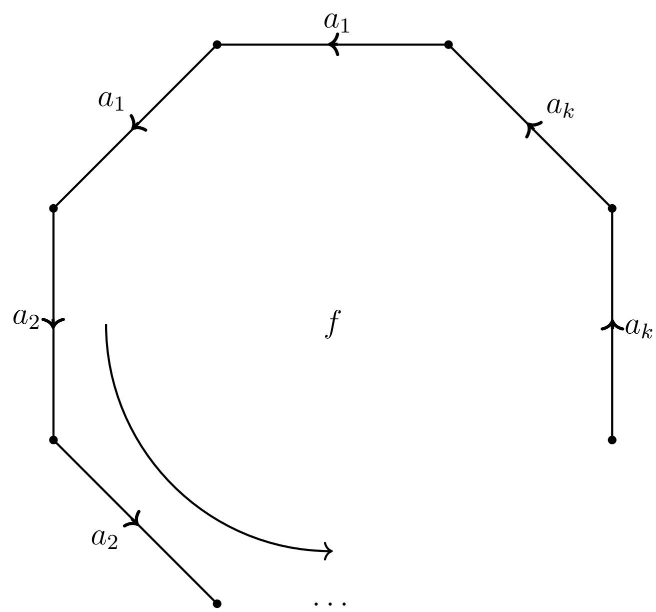
$$\partial_2(f) = a_1 + b_1 - a_1 - b_1 + \cdots + a_g + b_g - a_g - b_g = 0,$$

and hence $\partial_2 \equiv 0$ as well. In other words, $C_*(\Sigma_g, \Gamma)$ is a chain complex with trivial differential, and hence

$$\begin{aligned} H_k(\Sigma_g) =& H_k(C_*(\Sigma, \Gamma)) = C_k(\Sigma_g, \Gamma) \\ =& \begin{cases} \mathbb{Z}v & \text{for } k = 0 \\ \mathbb{Z}a_1 \oplus \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_g & \text{for } k = 1 \\ \mathbb{Z}f & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases} \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2 \\ \mathbb{Z}^{2g} & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

**Part b).** We recall that the connected sum $\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2$ of $k$ copies of the projective plane $\mathbb{RP}^2$ can be described as the quotient of the $2k$-gon with edge identifications given by the picture

$$\underbrace{\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k} \approx \tag{1}$$




The resulting pattern given by this polygon on the surface $P\#\ldots\#P$ has one vertex $v$, $k$ edges $a_1, \ldots, a_k$ and one face $f$. Hence the chain complex associated to this pattern looks like

$$\mathbb{Z}v \xleftarrow{\partial_1} \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k \xleftarrow{\partial_2} \mathbb{Z}f$$

Since there is only one vertex involved, we have $\partial_1(a_i) = v - v = 0$ for any edge $a_i$ and hence $\partial_1 \equiv 0$. To determine $\partial_2(f)$ we notice that every edge label occurs exactly twice, with arrows pointing in the same direction as the arrow for $f$. Hence $\partial_2(f) = 2a_1 + 2a_2 + \cdots + 2a_k$. It follows that $H_0 = \mathbb{Z}$, $H_2 = 0$, and

$$H_1 = \frac{\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k}{\mathbb{Z}2(a_1 + \cdots + a_k)}.$$

To identify this quotient group, we choose a different basis of the free abelian group $C_1$, namely $a_1, \ldots, a_{k-1}, c$, with $c = a_1 + \cdots + a_k$. Then we see

$$H_1 = \frac{\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_{k-1} \oplus \mathbb{Z}c}{\mathbb{Z}2c} \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

**Part c).** We observe that for the orientable surfaces $\Sigma_g$ the homology group $H_2(\Sigma_g)$ is isomorphic to $\mathbb{Z}$, while $H_2(X_k) = 0$ for the non-orientable surfaces $X_k$. $\square$

3. Let $\Sigma$ be a closed surface, which is not necessarily connected. Then $\Sigma$ has finitely many connected components $\Sigma_1, \ldots, \Sigma_k$ each of which is a closed connected surface.

   a) How can the homology group $H_q(\Sigma)$ be expressed in terms of the homology groups $H_q(\Sigma_i)$? Hint: If $\Gamma_i$ is a pattern of polygons on $\Sigma_i$ (enhanced by orientations), and $\Gamma$ is the resulting pattern of polygons on $\Sigma$, how is the chain complex $C_*(\Sigma, \Gamma)$ related to the chain complexes $C_*(\Sigma_i, \Gamma_i)$? What does that imply for the homology groups?

b) What is the group $H_0(\Sigma)$?
c) Taking the results of problem 2 and 3(a), you have now calculated the homology groups of *all* closed surfaces. We also know the Euler characteristic of all closed surfaces (note that the Euler characteristic of the disjoint union $\Sigma \amalg \Sigma'$ of two closed surfaces $\Sigma$, $\Sigma'$ is given by $\chi(\Sigma \amalg \Sigma') = \chi(\Sigma) + \chi(\Sigma')$). Based on these calculations, come up with a conjecture that expresses the Euler characteristic of a closed surface $\Sigma$ in terms of its homology groups $H_*(\Sigma)$, and prove that conjecture.

Remark: Later we will *define* the Euler characteristic of more general topological spaces in terms of their homology groups. We will generalize the description of the Euler characteristic as an alternating sum of the number of vertices, edges and faces to an important class of spaces known as "CW complexes".

*Proof.* **Part a).** We recall that the homology group $H_k(\Sigma)$ is defined as the $k$-th homology group of the chain complex $(C_*(\Sigma,\Gamma),\partial)$. Here $\Gamma$ is a pattern of polygons on the surface $\Sigma_i$, equipped with an orientation for all edges and faces. The group $C_k(\Sigma_i,\Gamma_i)$ is the free abelian group generated by the vertices (for $k=0$) resp. edges (for $k=1$) resp. faces (for $k=2$). For an oriented edge $e \in C_1(\Sigma)$ the boundary $\partial_1(e) \in C_0(\Sigma,\Gamma)$ is $\mathrm{tip}(e) - \mathrm{tail}(e)$. For an oriented face $f \in C_2(\Sigma)$ the boundary $\partial_2(f)$ is $\sum \pm e$, where the sum runs over all edges $e$ which are edges of the polygon $f$; the sign is positive if the orientations of $e$ and $f$ agree, and negative otherwise.

Given a pattern of polygons with orientations $\Gamma^i$ on each surface $\Sigma_i$, the union of all these vertices, oriented edges and faces can be viewed as providing us with a pattern of polygons $\Gamma$ on $\Sigma$ (with orientations on edges and faces). Denoting by $V^i$ the set of vertices of $\Gamma^i$, and by $V$ the corresponding sets for the pattern of polygons $\Gamma$ on $\Sigma$ whose restriction to $\Sigma_i \subset \Sigma$ is $\Gamma^i$. Then the set $V$ is the disjoint union $V^1 \amalg \cdots \amalg V_k$ of the sets $V^i$ and it follows that

$$C_0(\Sigma,\Gamma) = \mathbb{Z}[V] = \mathbb{Z}[V^1 \amalg \cdots \amalg V^k] \cong \bigoplus_{i=1}^k \mathbb{Z}[V^i] = \bigoplus_{i=1}^k C_0(\Sigma_i,\Gamma^i),$$

where the middle isomorphism comes from the map

$$\mathbb{Z}[V^1 \amalg \cdots \amalg V^k] \xrightarrow{\cong} \bigoplus_{i=1}^k \mathbb{Z}[V_i]$$

which sends the generator $v \in V^j \subset V^1 \amalg \cdots \amalg V^k \subset \mathbb{Z}[V^1 \amalg \cdots \amalg V^k]$ to the element $(c^1,\dots,c^k)$ where $c^i \in \mathbb{Z}[V^i]$ is the trivial element for $i \neq j$ and $c^i = v \in V^i \subset \mathbb{Z}[V^i]$ for $i = j$. Similarly, we have

$$C_1(\Sigma,\Gamma) \cong \bigoplus_{i=1}^k C_1(\Sigma_i,\Gamma^i) \qquad \text{and} \qquad C_2(\Sigma,\Gamma) \cong \bigoplus_{i=1}^k C_2(\Sigma_i,\Gamma^i)$$

for the 1-chains and 2-chains consisting of linear combinations of edges and faces.

Concerning the boundary homomorphisms $\partial_n \colon C_n(\Sigma,\Gamma) \to C_{n-1}(\Sigma,\Gamma)$ for $n=1,2$, if $e$ is an edge or $f$ is a face for the pattern $\Gamma$, then $e$ is an edge or $f$ is a face belonging to the pattern $\Gamma^i$ on $\Sigma^i$ for some $i$, and hence

$$\partial_1(e) = \partial_1^i(e) \qquad \text{and} \qquad \partial_2(f) = \partial_2^i(f)$$

where $\partial_n^i \colon C_n(\Sigma^i, \Gamma^i) \to C_{n-1}(\Sigma^i, \Gamma^i)$ is the boundary homomorphism for the pattern $\Gamma^i$ on $\Sigma_i$. It follows that the diagram

$$\begin{tikzcd} C_n(\Sigma,\Gamma) \arrow[r, "\partial_k"] \arrow[d, "\cong"'] & C_{n-1}(\Sigma,\Gamma) \arrow[d, "\cong"] \\ \bigoplus_{i=1}^k C_n(\Sigma_i,\Gamma^i) \arrow[r, "\bigoplus \partial_n^i"] & \bigoplus_{i=1}^k C_{n-1}(\Sigma_i,\Gamma^i) \end{tikzcd}$$

is commutative. We summarize this discussion by saying that the chain complex $(C_*(\Sigma, \Gamma), \partial_*)$ is isomorphic to the direct sum chain complex

$$\longleftarrow \bigoplus_{i\in I} C_{n-1}(\Sigma_i,\Gamma^i) \xleftarrow{\bigoplus \partial_n^i} \bigoplus_{i\in I} C_n(\Sigma_i,\Gamma^i) \xleftarrow{\bigoplus \partial_{n+1}^i} \bigoplus_{i\in I} C_{n+1}(\Sigma_i,\Gamma^i) \longleftarrow$$

**Lemma 1.** *Let $(C_*^i, \partial_*^i)$, $i \in I$ be a collection of chain complexes parametrized by some index set $I$ (not necessarily finite). Then the $n$-th homology group of the direct sum chain complex*

$$\xleftarrow{\bigoplus \partial_{n-1}^i} \bigoplus_{i\in I} C_{n-1}^i \xleftarrow{\bigoplus \partial_n^i} \bigoplus_{i\in I} C_n^i \xleftarrow{\bigoplus \partial_{n+1}^i} \bigoplus_{i\in I} C_{n+1}^i \xleftarrow{\bigoplus \partial_{n+1}^i}$$

*is isomorphic to $\bigoplus_{i\in I} H_n(C_*^i, \partial_*^i)$.*

*Proof.* Let us write $Z_n^i$ for the $n$-cycles of the chain complex $C_*^i$, and $Z_n$ for the $n$-cycles of the direct sum chain complex. Similarly, let $B_n^i$, $B_n$ be the $n$-boundaries of the chain complex $C_*^i$ resp. the direct sum chain complex. Then

$$Z_n = \ker\left(\bigoplus_{i\in I} \partial_n^i \colon \bigoplus_{i\in I} C_n^i \longrightarrow C_{n-1}^i\right) \cong \bigoplus_{i\in I} \ker \partial_n^i = \bigoplus_{i\in I} Z_n^i$$

Similarly, $B_n \cong \bigoplus_{i\in I} B_n^i$ and hence

$$H_n(\text{sum chain complex}) = \frac{Z_n}{B_n} \cong \frac{\bigoplus Z_n^i}{\bigoplus B_n^i} \cong \bigoplus \frac{Z_n^i}{B_n^i} = \bigoplus H_n(C_*^i)$$

□

This finishes the proof of part (a).

**Part b).** By the Classification Theorem for surfaces, any *connected* closed surface is homeomorphic to either the sphere, or a connected sum of tori or real projective planes. Our calculations show that for all these surfaces the homology group $H_0(\Sigma)$ is isomorphic to $\mathbb{Z}$. Part (a) implies that $H_0(\Sigma)$ for a not necessarily connected surface is the direct sum of $\mathbb{Z}$'s with each copy of $\mathbb{Z}$ corresponding to a connected component of $\Sigma$. In other words, $H_0(\Sigma) \cong \mathbb{Z}^k$, where $k$ is the number of connected components of $\Sigma$.

**Part c).** Can the Euler characteristic of a compact connected surface be expressed in terms of its homology groups?

First let us look at the Euler characteristic and the homology groups of all *connected* closed surfaces $\Sigma$. By the Classification Theorem, it suffices to consider $\Sigma = \Sigma_g$ (the surface of genus $g$) and $\Sigma = X_k$ (the connected sum of $k$ copies of the real projective plane $\mathbb{RP}^2$).

| $\Sigma$ | $\chi(\Sigma)$ | $H_0(\Sigma)$ | $H_1(\Sigma)$ | $H_2(\Sigma)$ |
|---|---|---|---|---|
| $\Sigma_g$ | $2-2g$ | $\mathbb{Z}$ | $\mathbb{Z}^{2g}$ | $\mathbb{Z}$ |
| $X_k$ | $2-k$ | $\mathbb{Z}$ | $\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2$ | $0$ |

In the case of $\Sigma = \Sigma_g$, $\#V = \operatorname{rk} H_0(\Sigma)$, $\#E = \operatorname{rk} H_1(\Sigma)$, and $\#F = \operatorname{rk} H_2(\Sigma)$, and hence

$$\chi(\Sigma) = \#V - \#E + \#F = \operatorname{rk} H_0(\Sigma) - \operatorname{rk} H_1(\Sigma) + \operatorname{rk} H_2(\Sigma).$$

In the case of $\Sigma = X_k$, $\#V = 1 = \operatorname{rk} H_0(\Sigma)$, but $\#E = k \neq k-1 = \operatorname{rk} H_1(\Sigma)$, and $\#F = 1 \neq 0 = \operatorname{rk} H_2(\Sigma)$. However, the above equation is still true! So, summarizing, we have proved that for all connected closed surfaces $\Sigma$,

$$\chi(\Sigma) = \sum_{q=0}^{2} (-1)^q \operatorname{rk} H_q(\Sigma); \tag{2}$$

in other words, the Euler characteristic of $\Sigma$ is the alternating sum of the ranks of the homology groups of $\Sigma$.

If $\Sigma$ is the disjoint union of connected closed surfaces $\Sigma_1, \ldots, \Sigma_k$,

$$\chi(\Sigma) = \sum_{i=1}^{k} \chi(\Sigma_i) \qquad \text{and} \qquad H_q(\Sigma) \cong \bigoplus_{i=1}^{k} H_q(\Sigma_i).$$

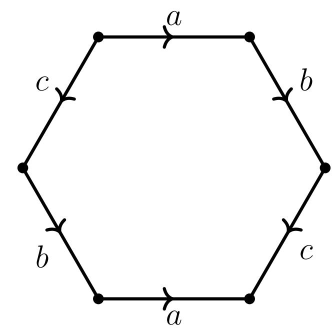
It follows that $\operatorname{rk} H_q(\Sigma) = \sum_{i=1}^{k} \operatorname{rk} H_q(\Sigma_i)$, and hence

$$\begin{aligned}
\sum_{q=0}^{2} (-1)^q \operatorname{rk} H_q(\Sigma) &= \sum_{q=0}^{2} (-1)^q \left( \sum_{i=1}^{k} \operatorname{rk} H_q(\Sigma_i) \right) \\
&= \sum_{i=1}^{k} \left( \sum_{q=0}^{2} (-1)^q \operatorname{rk} H_q(\Sigma_i) \right) \\
&= \sum_{i=1}^{k} \chi(\Sigma_i) = \chi(\Sigma)
\end{aligned}$$

which proves that equation (2) holds for any closed surface $\Sigma$.

$\square$

4. Let $\Sigma$ be the closed connected surface described as polygon with edge identification by the following picture:




a) Calculate the homology groups of $\Sigma$ using the enhanced pattern $\Gamma$ on $\Sigma$ given by the picture above (pick the orientation of the face to be clockwise). Describe an explicit basis $\{e_1, \ldots, e_k\}$ of the free part of $H_1(\Sigma)$ in terms of the edges $a, b, c$.

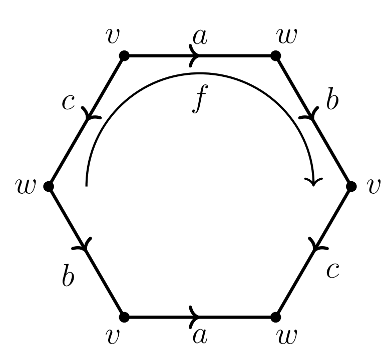
b) Consider the map $g \colon \Sigma \to \Sigma$ that is given by clockwise rotation by 60 degrees of the hexagon above. It maps the pattern $\Gamma$ to itself (reversing the orientation of some of the edges) and hence produces a commutative diagram

$$\begin{tikzcd} C_0(\Sigma,\Gamma) \arrow[d, "g_\#"] & C_1(\Sigma,\Gamma) \arrow[l, "\partial_1"'] \arrow[d, "g_\#"] & C_2(\Sigma,\Gamma) \arrow[l, "\partial_2"'] \arrow[d, "g_\#"] \\ C_0(\Sigma,\Gamma) & C_1(\Sigma,\Gamma) \arrow[l, "\partial_1"'] & C_2(\Sigma,\Gamma) \arrow[l, "\partial_2"'] \end{tikzcd}$$

where the homomorphisms $g_\#$ are determined by the effect of $g$ on vertices, edges and faces, respectively. The maps $g_\#$ then in turn determine homomorphisms $g_* \colon H_q(\Sigma) \to H_q(\Sigma)$ for $q = 0, 1, 2$. Write $g_*(e_i) \in H_1(\Sigma)$ as a linear combination of the basis $\{e_i\}$. Write down the $k \times k$-matrix corresponding to $g_*$.

c) What is the order of the automorphisms $g_* \colon H_1(\Sigma) \to H_1(\Sigma)$, i.e., the smallest natural number $n$ such that $g_*^n$ is the identity on $H_1(\Sigma)$?

*Proof.* **Part (a)** We note that up to equivalence, there are *two* vertices, which we denote $v$ resp. $w$. We also pick an orientation of the face $f$. All of these data are indicated in the following picture.



From this picture we can directly read off the chain complex $C_*(\Sigma, \Gamma)$ associated to this enhanced polygonal pattern on $\Sigma$:

$$\mathbb{Z}v \oplus \mathbb{Z}w \xleftarrow{\partial_1} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xleftarrow{\partial_2} \mathbb{Z}f;$$

the differentials are given by

$$\partial_1(a) = w - v \quad \partial_1(b) = v - w \quad \partial_1(c) = w - v \qquad \partial_2(f) = a + b + c - a - b - c = 0$$

It follows that

$$\begin{aligned}
H_0(\Sigma) =& \frac{Z_0}{B_0} = \frac{\mathbb{Z}v \oplus \mathbb{Z}w}{\mathbb{Z}(v-w)} = \mathbb{Z}[v] \\
H_1(\Sigma) =& \frac{Z_1}{B_1} = \frac{\mathbb{Z}(a-c) \oplus \mathbb{Z}(b+c)}{0} = \mathbb{Z}[a-c] \oplus \mathbb{Z}[b+c] \\
H_2(\Sigma) =& \frac{Z_2}{B_2} = \frac{\mathbb{Z}f}{0} = \mathbb{Z}[f]
\end{aligned}$$

Here we write e.g. $[a-c] \in H_1 = Z_1/B_1$ for the equivalence class of $a - c \in Z_1$. In particular, a basis for the free $\mathbb{Z}$-module $H_1(\Sigma)$ is given by the elements $\alpha := [a-c]$ and $\beta := [b+c]$. Here we write $[x] \in H_1 = Z_1/B_1$ for the homology class of a cocycle $x \in Z_1$ (a kind of picky notation in this particular case, since $B_1$ happens to be trivial...).

**Part (b).** The homomorphism $g_\#\colon C_q(\Sigma, \Gamma) \to C_q(\Sigma, \Gamma)$ is determined by the action of clockwise rotation of the hexagon by 60 degrees on vertices, edges and faces:

$$g_\#(v) = w \quad g_\#(w) = v \qquad g_\#(a) = b \quad g_\#(b) = c \quad g_\#(c) = -a \qquad g_\#(f) = f;$$

in particular

$$\begin{aligned}
g_*(\alpha) =& g_*([a-c]) = [g_\#(a-c)] = [g_\#(a) - g_\#(c)] = [b - (-a)] = [(a-c) + (b+c)] = \alpha + \beta \\
g_*(\beta) =& g_*([b+c]) = [g_\#(b+c)] = [g_\#(b) + g_\#(c)] = [c - a] = -\alpha
\end{aligned}$$

It follows that with respect to the basis $\{\alpha, \beta\}$ of $H_1(\Sigma)$ the matrix corresponding to $g_*$ is

$$G = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and hence} \qquad G^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad G^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

This shows that $G^2$ and $G^3$ are not the identity matrix, but $G^6 = (G^3)^2$ is, and hence $G$ and $g_*$ have order 6. □

## 2. Homework Assignment # 2

1. Show that the singular chain complex of a topological space X is in fact a chain complex; i.e., that $\partial_q \circ \partial_{q+1} = 0$, where $\partial_q : C_q(X) \to C_{q-1}(X)$ is the boundary map.

*Proof.* It will suffice to show that $\partial_q \circ \partial_{q+1} = 0$ on the generators of $C_{q+1}$ since $\partial_q$ and $\partial_{q+1}$ are both homomorphisms. Consider $\sigma : \Delta^{q+1} \to X$ in $C_{q+1}(X)$.

$$\begin{aligned}
&\partial_q \circ \partial_{q+1}(\sigma) \\
=&\partial_q \left( \sum_{j=0}^{q} (-1)^j \partial_{q+1}(\sigma) \circ [e_0, ..., \hat{e_j}, ..., e_{q+1}] \right) \\
=&\sum_{i=0}^{q} (-1)^i \left( \sum_{j=0}^{q+1} (-1)^j \sigma \circ [e_0, ..., \hat{e_j}, ..., e_{q+1}] \circ [e_0, ..., \hat{e_i}, ..., e_q] \right) \\
=&\sum_{0 \le i < j \le q+1} (-1)^{i+j} \sigma \circ [e_0, ..., \hat{e_i}, ..., \hat{e_j}, ..., e_{q+1}] \\
&+ \sum_{0 \le j < i \le q+1} (-1)^{i+j+1} \sigma \circ [e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_{q+1}] \\
=&\sum_{0 \le i < j \le q+1} (-1)^{i+j} \sigma \circ [e_0, ..., \hat{e_i}, ..., \hat{e_j}, ..., e_{q+1}] \\
&- \sum_{0 \le j < i \le q+1} (-1)^{i+j} \sigma \circ [e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_{q+1}] \\
=&0.
\end{aligned}$$

Hence, since $\partial_q \circ \partial_{q+1} = 0$ on the generators of $C_{q+1}(X)$, $\partial_q \circ \partial_{q+1} = 0$, and the singular chain complex of a topological space X is in fact a chain complex. ☐

2. Show that the Hurewicz map

$$h\colon \pi_1(X, x_0) \to H_1(X) \qquad \text{given by} \qquad [\gamma] \mapsto [[\gamma]]$$

is a homomorphism. Explanation: for a based loop $\gamma\colon (I, \partial I) \to (X, x_0)$, we denote by $[\gamma]$ is the homotopy class of $\gamma$. In other words $[\gamma]$ is the element of the fundamental group $\pi_1(X, x_0)$ represented by the loop $\gamma$. Considering $\gamma$ as a singular 1-simplex in $X$, we note that it is a cycle, since $\partial\gamma = \gamma(1) - \gamma(0) = x_0 - x_0 = 0$, and we denote by $[[\gamma]] \in Z_1(X)/B_1(X) = H_1(X)$ the homology class it represents.

*Proof.* Let $[\gamma], [\gamma'] \in \pi_1(X, x_0)$, and let $\gamma\gamma'$ denote the concatenation of paths $\gamma$ and $\gamma'$. To show that h is a homomorphism, we need to verify that $h([\gamma][\gamma']) = h([\gamma]) + h([\gamma'])$; i.e., $[[\gamma\gamma']] = [[\gamma]] + [[\gamma']]$. We will do so by showing that $\gamma + \gamma' - \gamma\gamma' \in B_1(X)$ which will then imply that $[[\gamma\gamma']] = [[\gamma + \gamma']] = [[\gamma]] + [[\gamma']]$, as desired. Consider the singular 2-simplex, $\sigma : \Delta^2 \to X$, defined as the composition

$$\Delta^2 \xrightarrow{[e_0, \frac{1}{2}(e_0+e_1), e_1]} \Delta^1 \xrightarrow{\gamma\gamma'} X$$

Notice that $\partial_2(\sigma) = \gamma' - \gamma\gamma' + \gamma$, so $\gamma' - \gamma\gamma' + \gamma \in B_1(X)$ as desired. Thus, h is indeed a homomorphism. ☐

3. The goal of this problem is to show that the homomorphism

$$\bar{h}\colon \pi_1^{ab}(X, x_0) \to H_1(X)$$

induced by the Hurewicz homomorphism $h$ is in fact an isomorphism for a path connected space $X$; here $\pi_1^{ab}(X, x_0)$ is the abelianized fundamental group of $X$. The idea is to construct an inverse to $\bar{h}$ as follows. Choose for every point $x \in X$ a path $\lambda_x$ from $x_0$ to $x$. Define the map

$$\Psi\colon C_1(X)/B_1(X) \longrightarrow \pi_1^{ab}(X, x_0) \qquad \text{by} \qquad [[\gamma]] \mapsto [\lambda_{\gamma(0)} * \gamma * \bar{\lambda}_{\gamma(1)}]$$

for any singular 1-simplex $\gamma$, also known as path $\gamma\colon I \to X$. Here $\lambda_{\gamma(0)} * \gamma * \bar{\lambda}_{\gamma(1)}$ is the concatenation of the path $\lambda_{\gamma(0)}$ (from $x_0$ to $\gamma(1)$), the path $\gamma$ (from $\gamma(0)$ to $\gamma(1)$) and the path $\bar{\lambda}_{\gamma(1)}$ (from $\gamma(1)$ to $x_0$, obtained by running the path $\lambda_{\gamma(1)}$ from $x_0$ to $\gamma(1)$ backwards; as in class we have chosen for every point $x \in X$ a path $\lambda_x$ from the basepoint $x_0$ to $x$). In class we proved that the map $\Psi$ above is well-defined. Show that the restriction of $\Psi$ to $H_1(X) \subset C_1(X)/B_1(X)$ provides an inverse to the map $\bar{h}$.

*Proof.* We need to show:

(a) $\Psi|_{H_1(X)} \circ \bar{h} = id_{\pi_1^{ab}(X,x_0)}$;

(b) $\bar{h} \circ \Psi|_{H_1(X)} = id_{H_1(X)}$.

To prove (a), let $[\gamma : (I, \partial I) \to (X, x_0)] \in \pi_1^{ab}(X, x_0)$. Then

$$\begin{aligned}
\Psi|_{H_1(X)} \circ \bar{h}([\gamma]) &= \Psi|_{H_1(X)}([[\gamma]]) \\
&= [\lambda_{\gamma(0)}\gamma\overline{\lambda_{\gamma(1)}}] = [\lambda_{x_0}\gamma\overline{\lambda_{x_0}}] \\
&= [\lambda_{x_0}][\gamma][\overline{\lambda_{x_0}}] = [\lambda_{x_0}][\overline{\lambda_{x_0}}][\gamma] = [\gamma],
\end{aligned}$$

which shows $\Psi|_{H_1(X)} \circ \bar{h} = id_{\pi_1^{ab}(X,x_0)}$.

To prove (b), we will use the following statements:

(i) If $\gamma, \gamma'\colon I \to X$ are paths with $\gamma(1) = \gamma'(0)$, and $\gamma\gamma'\colon I \to X$ their concatenation, then $\gamma' - \gamma\gamma' + \gamma \in B_1(X)$.

(ii) $\bar{\gamma} + \gamma \in B_1(X)$.

Statement (i) we proved in problem # 2 (note that the argument used there does not require $\gamma$ and $\gamma'$ to be based loops; it is only necessary that $\gamma(1) = \gamma'(0)$ in order to be able to have the concatenated path $\gamma\gamma'$). To deduce the second claim, we apply part (i) for $\gamma' = \bar{\gamma}$ which implies $\bar{\gamma} - \gamma\bar{\gamma} + \gamma \in B_1(X)$. We note that $\gamma\bar{\gamma}$ is a loop based at $\gamma(0)$ which is homotopic to the constant loop. Hence its image under the Hurewicz map $[[\gamma\bar{\gamma}]] \in H_1(X)$ is trivial. In other words, $\gamma\bar{\gamma} \in B_1(X)$, which proves (ii).

With these preliminaries, we can calculate for any path $\gamma$:

$$\begin{aligned}
\bar{h} \circ \Psi([[\gamma]]) &= \bar{h}([\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}] = [[\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}]] \\
&= [[\lambda_{\gamma(0)}]] + [[\gamma]] + [[\bar{\lambda}_{\gamma(1)}]] = [[\lambda_{\gamma(0)}]] + [[\gamma]] - [[\lambda_{\gamma(1)}]] \\
&= [[\gamma]] + [[\lambda(\partial(\gamma))]],
\end{aligned} \tag{3}$$

where $\lambda\colon C_0(X) \to C_1(X)$ is given on generators by $x \mapsto \lambda_x$ (and hence $\lambda$ maps $\partial\gamma = \gamma(1) - \gamma(0)$ to $\lambda_{\gamma(0)} - \lambda_{\gamma(1)}$). Since formula (3) holds for the generators $\gamma \in C_1(X)$, it hold for every element $z \in C_1(X)$; in particular, if $z$ is a *cycle*, we have $\bar{h} \circ \Psi([[z]]) = [[z]]$, which is what we wanted to prove. ∎

4. Calculate the reduced homology groups of the subspace $X \subset \mathbb{R}^3$ which is the union of the sphere $S^2$ and the $x$-axis.

*Proof.* The natural idea is to apply the Mayer-Vietoris sequence to $X = U \cup V$, where $U = S^2$, and $V$ is the $x$-axis. Unfortunately, this doesn't work since these aren't *open* subsets of $X$ as required by the Mayer-Vietoris Theorem. The way around this problem is to work with larger open subsets $U', V' \subset X$ that contain $U$ resp. $V$ as *deformation retracts*. This means that there is a map

$$r^U : U' \to U \qquad \text{such that} \qquad r^U \circ i^U = \mathrm{id}_U \quad \text{and} \quad i^U \circ r^U \sim \mathrm{id}_{U'},$$

where $i^U : U \to U'$ is the inclusion map, and similarly for $V$. There are many possible choices for $U'$, $V'$, for example we can take

$$U' := S^2 \cup x\text{-axis without the origin} \qquad V' := x\text{-axis} \cup \{(x,y,z) \in S^2 \mid x \neq 0\}.$$

The maps $r^U$, $r^V$, the homotopy between $i^U \circ r^U$ and $\mathrm{id}_{U'}$, and the homotopy between $i^V \circ r^V$ and $\mathrm{id}_{V'}$ can be written down explicitly in formulas, but it might be more helpful to describe them in a geometric way as follows: the subset $\{\pm 1\} \subset \mathbb{R} \setminus \{0\}$ is a deformation retract (take the linear homotopy between the identity on $\mathbb{R} \setminus \{0\}$ and the map $x \mapsto x/|x|$). It follows that $U = S^2$ is a deformation retract of $U' := S^2 \cup x$-axis without the origin.

Similarly, $V'$ is the union of the $x$-axis and the two open disks given by the left resp. right hemisphere of $S^2$. The center of each of these disks is a deformation retract of the disk. These deformation retractions and homotopies fit together to show that $V = x$-axis is a deformation retract of $V'$. Finally, using both retractions (for the $x$-axis without 0 and the two disks), we obtain the set $U \cap V = \{(-1,0,0),(1,0,0)\}$ as a deformation retract of $U' \cap V'$.

These homotopy equivalences allow us in particular to calculate the reduced homology groups of $U'$, $V'$ and $U' \cap V'$ as

$$\widetilde{H}_k(U') \cong \widetilde{H}_k(U) \cong \begin{cases} \mathbb{Z} & k = 2 \\ 0 & k \neq 2 \end{cases}$$

$$\widetilde{H}_k(V') \cong \widetilde{H}_k(V) \cong \widetilde{H}_k(\mathrm{pt}) = 0$$

$$\widetilde{H}_k(U' \cap V') \cong \widetilde{H}_k(U \cap V) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

These calculations show that in the Mayer-Vietoris sequence for $X = U' \cup V'$, there are only two non-trivial reduced homology groups of $U'$, $V'$ or $U' \cap V'$ that contribute. Here are the relevant portions of the Mayer-Vietoris sequence:

$$0 = \widetilde{H}_2(U' \cap V') \longrightarrow \widetilde{H}_2(U') \oplus H_2(V') \longrightarrow \widetilde{H}_2(X) \xrightarrow{\partial} \widetilde{H}_1(U' \cap V') = 0$$

The exactness of the sequence implies that the middle map is an isomorphism, and hence $\widetilde{H}_2(X) \cong \widetilde{H}_2(U') \oplus H_2(V') \cong \mathbb{Z}$.

$$0 = \widetilde{H}_1(U \cap V) \longrightarrow \widetilde{H}_1(X) \xrightarrow{\partial} \widetilde{H}_0(U' \cap V') \longrightarrow \widetilde{H}_0(U') \oplus H_0(V') = 0$$

The exactness of the sequence implies that $\partial$ is an isomorphism, and hence $\widetilde{H}_1(X) \cong \widetilde{H}_0(U' \cap V') \cong \mathbb{Z}$. For $k \neq 1, 2$ the exactness of the Mayer-Vietoris sequence implies that $\widetilde{H}_k(X) = 0$. Summarizing our result, we find

$$\widetilde{H}_k(X) \cong \begin{cases} \mathbb{Z} & k = 1, 2 \\ 0 & k \neq 1, 2 \end{cases}$$

□

## 3. Homework Assignment # 3

1. Let $x_1, \ldots, x_l$ be distinct points of $\mathbb{R}^n$. Calculate the reduced homology groups of the space $\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$. Hint: Compare the homology groups of $\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$ with those of $\mathbb{R}^n$ via the Mayer-Vietoris sequence.

*Proof.* Let $D_1, \ldots, D_l \subset \mathbb{R}^n$ be disjoint open disks with center $x_1, \ldots, x_l$. Let

$$U := D_1 \cup \cdots \cup D_l \qquad \text{and} \qquad V := \mathbb{R}^n \setminus \{x_1, \ldots, x_l\}.$$

$$H_k(U \cap V) = H_k(\amalg_{i=1}^l (D_i \setminus \{x_i\}) \cong \bigoplus_{i=1}^l H_k(D_i \setminus \{x_i\}) \cong \bigoplus_{i=1}^l H_k(S^{n-1}) \cong \begin{cases} \mathbb{Z}^l & k = 0, n-1 \\ 0 & k \neq 0, n-1 \end{cases}$$

$$H_k(U) = H_k(\amalg_{i=1}^l (D_i) \cong \bigoplus_{i=1}^l H_k(D_i) \cong \begin{cases} \mathbb{Z}^l & k = 0 \\ 0 & k \neq 0 \end{cases}$$

We note that that the inclusion map $i^U \colon U \cap V \to U$ is a bijection on connected components, and hence it induces an isomorphism on $H_0$. Next we consider the Mayer-Vietoris sequence for the decomposition of $\mathbb{R}^n$ as the union of the open subsets $U$ and $V$:

$$\longrightarrow \widetilde{H}_{k+1}(\mathbb{R}^n) \xrightarrow{\partial} \widetilde{H}_k(U \cap V) \xrightarrow{i^U_* \oplus i^V_*} \widetilde{H}_k(U) \oplus \widetilde{H}_k(V) \longrightarrow \widetilde{H}_k(\mathbb{R}^n) \longrightarrow$$

Since $\mathbb{R}^n$ is contractible, its the reduced homology groups $\widetilde{H}_k(\mathbb{R}^n)$ vanish, and hence the map $i^U_* \oplus i^V_*$ is an isomorphism by exactness of the Mayer-Vietoris sequence. For $k = 0$ the map $i^U_*$ is an isomorphism on the homology group $H_k$ as noted above, and hence also on $\widetilde{H}_k$. It follows that $\widetilde{H}_0(V) = 0$. For $k > 0$, the vanishing of $\widetilde{H}_k(U)$ implies

$$\widetilde{H}_k(V) \cong H_k(U \cap V) \cong \begin{cases} \mathbb{Z}^l & k = 0, n-1 \\ 0 & k \neq 0, n-1 \end{cases}$$

So, summarizing, we have

$$\widetilde{H}_k(\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}) \cong \begin{cases} \mathbb{Z}^l & k = n-1 \\ 0 & k \neq n-1 \end{cases}$$

□

2. (a) Show that $S^{n-1}$ is no retract of $D^n$, that is, there is no continuous map $r\colon D^n \to S^{n-1}$ whose restriction to $S^{n-1}$ is the identity.
(b) Prove the Brouwer Fixed Point Theorem: Any continuous map $f\colon D^n \to D^n$ from the closed $n$-disk to itself has a fixed point, that is, there is some point $x_0 \in D^n$ such that $f(x_0) = x_0$.

Hint: Prove by contradiction. More precisely, assuming that $f$ has no fixed point, construct a map $r\colon D^n \to S^{n-1}$ whose restriction to $S^{n-1}$ is the identity.

*Proof.* **Part (a)** Assume that $r\colon D^n \to S^{n-1}$ is a retraction. Then we have commutative diagrams

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend left, "\mathrm{id}"] & D^n \arrow[r, "r"] & S^{n-1} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \widetilde{H}_q(S^{n-1}) \arrow[r, "i_*"] \arrow[rr, bend left, "\mathrm{id}_*"] & \widetilde{H}_q(D^n) \arrow[r, "r_*"] & \widetilde{H}_q(S^{n-1}) \end{tikzcd} .$$

This is the desired contradiction, since $\widetilde{H}_{n-1}(S^{n-1})$ is isomorphic to $\mathbb{Z}$ and hence the identity map on $\widetilde{H}_{n-1}(S^{n-1})$ cannot factor through $\widetilde{H}_q(D^n) = 0$.

**Part (b)** Assuming that $f$ has no fixed point, the line through $x$ and $f(x)$ intersects the sphere $S^{n-1}$ in exactly two points. Let $r(x) \in S^{n-1}$ be the intersection point closer to $x$ than to $f(x)$. If $x$ is a point on the sphere, then $x$ is clearly that intersection point; in other words, the map $r\colon D^n \to S^{n-1}$ restricted to $S^{n-1}$ is the identity on $S^{n-1}$. It remains to show that $r$ is continuous.

The idea is to write down a formula for $r(x)$, and to argue that $r$ is a composition of basic functions that we know are continuous from calculus. To derive the formula for $r(x)$, we note that every point of the line through $f(x)$ and $x$ is of the form

$$x + t(x - f(x)) \qquad \text{for } t \in \mathbb{R}.$$

Moreover, the point $r(x) = x + t(x - f(x))$ is characterized by the two conditions $||r(x)||^2 = 1$ and $t \geq 0$. The first condition is a quadratic equation for $t$. Explicitly, setting $v = v(x) = x - f(x)$ we have

$$||r(x)||^2 = \langle r(x), r(x)\rangle = \langle tv + x, tv + x\rangle = ||v||^2t^2 + 2\langle v, x\rangle t + ||x||^2$$

and hence the quadratic formula gives us the following non-negative solution for $t$:

$$t = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{where } a = ||v(x)||^2,\ b = 2\langle v(x), x\rangle,\ c = ||x||^2 - 1. \tag{4}$$

We note that the functions $a(x)$, $b(x)$, $c(x)$ are *continuous* maps $D^n \to \mathbb{R}$, since they are expressed as compositions of functions that are well-known to be continuous. This implies that the function $t = t(x)$ defined by equation (4) is a continuous function: the only thing to make sure is that the denominator function $a(x) = ||v(x)||^2$ is nowhere 0 (this is guaranteed by the assumption that $v(x) = x - f(x) \neq 0$ for all $x \in D^n$), and that the expression under the square root is positive (this follows geometrically since we have always *two* intersection points of the line through $x$ and $f(x)$ and the sphere; this is equivalent to having two solutions of the quadratic equation; this in turn is equivalent to a positive expression under the square root in the quadratic formula). Putting everything together, we conclude that the map

$$r\colon D^n \longrightarrow S^{n-1} \qquad \text{given by} \qquad r(x) = x + t(x)(x - f(x))$$

is continuous since $t(x)$ and $f(x)$ are continuous maps. □

3. Let $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ be a short exact sequence of chain complexes. Show that the sequence

$$\ldots \longrightarrow H_q(A) \xrightarrow{f_*} H_q(B) \xrightarrow{g_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \ldots$$

is exact at $H_q(C)$ and $H_q(B)$.

*Proof.* **Exactness at $H_q(C)$.** We need to show im $g_* =$ ker $\partial$. Recall that for $q \in \mathbb{Z}$, the chain map $g_*$ induces a map $g_* : H_q(B) \longrightarrow H_q(C)$ where $g_*([b]) = [g(b)]$. Let $g_*([b]) \in$ im $g_* \subset H_q(C)$, where $[b] \in H_q(B)$ (so $b \in Z_q(B)$), then $\partial g_*([b]) = \partial([g(b)])$. Recall the construction of $\partial$: $\partial$ map the homology class represented by a cycle $c$ (which is $g(b)$ in our case) to the homology class $[a]$ of an element $a \in A_{q-1}$ for which $f(a) = \partial(b')$, with $b'$ chosen so that $c = g(b) = g(b')$ for $b \in B_q$. Since we showed that the choice of such a $b'$ is irrelevant, we can then choose $b'$ to be $b$. Hence, $f(a) = \partial(b) = 0$ since $b \in Z_q(B)$. The map $f_q$ is injective (by the exactness of the given sequence), so we have that $a = 0$, whence $\partial g_*([b]) = \partial([g(b)]) = [0]$. Therefore, im $g_* \subset$ ker $\partial$ at $H_q(C)$.

To show im $g_* \supset$ ker $\partial$, let $[c] \in$ ker $\partial \subset H_q(C)$. Using the notation from the construction of $\partial$, there exists a $b \in B_q$ such that $g(b) = c$ and an $a \in A_{q-1}$ for which $f(a) = \partial(b)$, and since $[c] \in$ ker $\partial$, $\partial a' = a$ for some $a' \in A_q$. We notice that the element $b - f(a') \in B_q$ is a q-cycle since

$$\partial(b - f(a')) = \partial b - \partial f(a') = f(a) - f\partial a' = 0,$$

whence $[b - f(a')] \in H_q(B)$. Furthermore, by the exactness of the original sequence, $gf = 0$, so

$$g(b - f(a')) = g(b) - gf(a') = g(b) - 0 = g(b) = c.$$

Thus, $[c] = [g(b - f(a'))] = g_*([b - f(a')]) \in$ im $g_*$, and im $g_* \supset$ ker $\partial$.

**Exactness at $H_q(B)$.** To show im $f_* \subset$ ker $g_*$, let $f_*([a]) = [f(a)] \in$ im $f_* \subset H_q(B)$, where $a \in Z_q(A)$. Then, by the exactness of the original sequence, $gf = 0$, so $g_*([f(a)]) = [gf(a)] = [0]$. Hence, im $f_* \subset$ ker $g_*$.

To show im $f_* \supset$ ker $g_*$, let $[b] \in H_q(B)$ such that $g_*([b]) = [0]$; i.e., $g(b) \in \mathrm{im}(\partial : C_{q+1} \to C_q)$. Therefore, $\exists c \in C_{q+1}$ such that $\partial(c) = g(b)$. Since $g$ is onto (by the exactness of the given sequence), $\exists b' \in B_{q+1}$ such that $g(b') = c$. The commutativity of the diagram (of the chain complexes, chain maps, and boundary maps) yields:

$$g\partial(b') = \partial g(b') = \partial(c) = g(b).$$

Since each $g$ is a homomorphism, we then have $g(b-\partial(b'))=0$, so $b-\partial(b')\in \ker\ g = \mathrm{im}\ f$. Thus, $\exists a\in A_q$ such that $f(a)=b-\partial(b')$. Again, by the commutativity of the diagram, $f(\partial a)=\partial f(a)=\partial b-\partial\partial b'=\partial b=0$, so $a=0$ since f is injective (ker $f=0$). Thus, $a\in Z_q(A)$ and $[a]\in H_q(A)$. Moreover, since $b-f(a)=\partial(b')$, $f_*([a])=[f(a)]=[b]\in H_q(B)$. Thus, im $f_*\supset$ ker $g_*$.

To recap, we now have that:

$$\ldots \longrightarrow H_q(A)\xrightarrow{f_*}H_q(B)\xrightarrow{g_*}H_q(C)\xrightarrow{\partial}H_{q-1}(A)\longrightarrow\ldots$$

is exact at $H_q(C)$ and $H_q(B)$. □

4. Suppose the following diagram of abelian groups and group homomorphisms is commutative with exact rows:

$$\begin{tikzcd}
{} \arrow[r] & C_{q+1} \arrow[r, "\partial_{q+1}"] \arrow[d, "c_{q+1}"'] & A_q \arrow[r, "f_q"] \arrow[d, "a_q"'] & B_q \arrow[r, "g_q"] \arrow[d, "b_q"'] & C_q \arrow[r, "\partial_q"] \arrow[d, "c_q"'] & A_{q-1} \arrow[r] \arrow[d, "a_{q-1}"'] & {} \\
{} \arrow[r] & C'_{q+1} \arrow[r, "\partial'_{q+1}"] & A'_q \arrow[r, "f'_q"] & B'_q \arrow[r, "g'_q"] & C'_q \arrow[r, "\partial'_q"] & A'_{q-1} \arrow[r] & {}
\end{tikzcd}$$

Assuming in addition that the maps $c_q$ are isomorphisms show that there is a long exact sequence of the form

$$\longrightarrow A_q\xrightarrow{\alpha_q}A'_q\oplus B_q\xrightarrow{\beta_q}B'_q\xrightarrow{\gamma_q}A_{q-1}\xrightarrow{\alpha_{q-1}}A'_{q-1}\oplus B_{q-1}\xrightarrow{\beta_{q-1}}B'_{q-1}\longrightarrow$$

First define carefully the homomorphisms in the above sequence. Then prove exactness at each location.

*Proof.* We define the maps in the above sequence as follows:

$$\begin{aligned}
&\alpha_q\colon A_q\longrightarrow A'_q\oplus B_q && a\mapsto (a_q(a),f_q(a))\\
&\beta_q\colon A'_q\oplus B_q\longrightarrow B'_q && (a',b)\mapsto f'_q(a')-b_q(b)\\
&\gamma_q\colon B'_q\longrightarrow A_{q-1} && b'\mapsto \partial_q c_q^{-1}g'_q(b')
\end{aligned}$$

**Exactness at $B'_q$.** First we show $\gamma_q\beta_q=0$. For $(a',b)\in A'_q\oplus B_q$ we have

$$\gamma_q\beta_q(a',b)=\partial_q c_q^{-1}g'_q(f'_qa'-b_qb)=-\partial_q c_q^{-1}g'_qb_qb=\partial_q g_q b=0$$

Here the second equality holds due to $g'_qf'_q=0$, the third follows from the commutativity of the third square, and the last is due to $\partial_q g_q=0$.

To show $\ker\gamma_q\subset \mathrm{im}\beta_q$ let $b'\in B'_q$ with $\gamma_q b'=\partial_q c_q^{-1}g'_qb'=0$. By exactness at $C_q$ there is an element $b\in B_q$ such that $g_qb=c_q^{-1}g'_qb'$ or equivalently

$$g'_qb'=c_qg_qb=g'_qb_qb,$$

where the second equality follows from commutativity of the third square. It follows that $b'-b_qb$ is in the kernel of $g'_q$ and hence by exactness at $B'_q$, there is an element $a'\in A'_q$ with $f'_qa'=b'-b_qb$. This implies

$$\beta_q(a',-b)=f'_qa'+b_qb=b'$$

which shows that $b'$ is in the image of $\beta_q$.

**Exactness at $A'_q \oplus B_q$.** First we show $\beta_q\alpha_q = 0$. For $a \in A_q$ we have

$$\beta_q\alpha_q a = \beta_q(a_q a, f_q a) = f'_q a_q a - b_q f_q a = 0$$

due to the commutativity of the second square.

To show $\ker\beta_q \subset \mathrm{im}\alpha_q$, let $(a', b) \in A'_q \oplus B_q$ with

$$\beta_q(a', b) = f'_q a' - b_q b = 0.$$

Then we have

$$c_q g_q b = g'_q b_q b = g'_q f'_q a' = 0,$$

where the first equality is due to the commutativity of the third square, and the last is due to exactness at $B'_q$. Since $c_q$ is an isomorphism, this implies $g_q b = 0$ and hence by exactness at $B_q$, there is an element $a \in A_q$ with $f_q a = b$. If we could show $a_q a = a'$, we would be done. However we can only say the following:

$$f'_q(a_q a - a') = f'_q a_q a - f'_q a' = b_q f_q a - b_q b = 0$$

where the second equality follows from the commutativity of the second square and our assumption $f'_q a' = b_q b$. Since $f'_q$ is not necessarily injective, we can't conclude that $a_q a = a'$, but thanks to exactness at $A'_q$, it implies that there is an element $c' \in C'_{q+1}$ with $\partial_{q+1} c' = a_q a - a'$. Moreover, since $c_{q+1}$ is an isomorphism, there is a $c \in C_{q+1}$ with $c_{q+1} c = c'$. Now we modify the element $a \in A_q$ by defining $\bar{a} := a - \partial_{q+1} c$. We calculate

$$\begin{aligned} f_q\bar{a} &= f_q(a - \partial_{q+1}c) = f_q a = b \\ a_q\bar{a} &= a_q(a - \partial_{q+1}c) = a_q a - \partial'_{q+1} c_{q+1} c = a_q a - (a_q a - a') = a' \end{aligned}$$

This shows that $\alpha_q(\bar{a}) = (a', b)$ as desired.

**Exactness at $A_q$.** First let us show $\alpha_q \circ \gamma_{q+1} = 0$. For $b' \in B'_{q+1}$ we have

$$\begin{aligned} \alpha_q\gamma_{q+1} b' &= \alpha_q(\partial_{q+1} c^{-1}_{q+1} g'_{q+1} b') \\ &= (a_q \partial_{q+1} c^{-1}_{q+1} g'_{q+1} b', f_q \partial_{q+1} c^{-1}_{q+1} g'_{q+1} b') \\ &= (\partial'_{q+1} g'_{q+1} b', 0) = (0, 0) \end{aligned}$$

since the compositions $f_q\partial_{q+1}$ and $\partial'_{q+1} g'_{q+1}$ are zero due to the exactness at $A_q$ resp. $A'_q$.

To show $\ker\alpha_q \subset \mathrm{im}\gamma_{q+1}$, let $a \in A_q$ with $\alpha_q a = (a_q a, f_q a) = (0, 0)$. By exactness at $A_q$ there is an element $c \in C_{q+1}$ with $\partial_{q+1} c = a$. Then

$$\partial'_{q+1} c_{q+1} c = a_q \partial_{q+1} c = a_q a = 0$$

and hence by exactness at $C'_{q+1}$, there is an element $b' \in B'_{q+1}$ with $g'_{q+1} b' = c_{q+1} c$. This implies

$$\gamma_{q+1} b' = \partial_{q+1} c^{-1}_{q+1} g'_{q+1} b' = \partial_{q+1} c = a,$$

which shows that $a$ is in the image of $\gamma_{q+1}$. $\square$

## 4. Homework Assignment # 4

1. Let $X$ be a topological space and let $\Sigma X$ be the *suspension of* $X$ which is defined as the quotient space $X \times [0,1]/\sim$, where the equivalence relation is generated by $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$.

(a) Show that $\Sigma S^n$ is homeomorphic to $S^{n+1}$.

(b) Construct an isomorphism $H_{q+1}(\Sigma X) \xrightarrow{\cong} H_q(X)$ (this is called the *suspension isomorphism*).

Hint for (b): Think of the suspension isomorphism as a generalization of the isomorphism $\widetilde{H}_{q+1}(S^{n+1}) \cong \widetilde{H}_q(S^n)$ proved in class. That proof used the decomposition of $S^{n+1}$ as a union of $U = S^{n+1} \setminus$ north pole and $V = S^{n+1} \setminus$ south pole. In this more general situation, the subspaces $U = \{[x,t] \in \Sigma \mid 0 \le t < 1\}$ and $V = \{[x,t] \in \Sigma \mid 0 < t \le 1\}$ of $\Sigma X$ play an analogous role.

*Proof.* **Part (a)** The map $\mathbb{R}^{n+1} \times [0,1] \to \mathbb{R}^{n+2}$ given by $(x,t) \mapsto (x \sin \pi t, \cos \pi t) \in \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$ is continuous, since all its components are continuous. It restricts to a continuous map $f\colon \mathbb{R}^{n+1} \times [0,1] \supset S^n \times [0,1] \to S^{n+1} \subset \mathbb{R}^{n+2}$. Moreover, $f$ maps the subspace $S^n \times \{0\}$ to the "north pole" $(0,1) \in S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ and $S^n \times \{1\}$ to the "south pole" $(0,-1)$. Hence $f$ induces a well-defined continuous map $\bar{f}\colon \Sigma S^n \to S^{n+1}$.

It is easy to check that this map is a bijection. Moreover, $\bar{f}$ is an *open map* (i.e., it sends open sets of the domain to open sets of the codomain), since its domain $\Sigma S^n$ is compact (as quotient of the product of compact spaces $S^n \times I$), and its codomain $S^{n+1}$ is Hausdorff. Hence the inverse of $\bar{f}$ is continuous and consequently $\bar{f}$ is a homeomorphism.

**Part (b)** We apply the Mayer-Vietoris sequence for the decomposition of the suspension $\Sigma X$ as the union of the two open subspaces $U$ and $V$. We note that the spaces $U$ and $V$ (can be pictured as "cones") are both contractible, i.e., homotopy equivalent to the 1-point space pt. To see this, let $i\colon \mathrm{pt} \to \Sigma X$ be the inclusion map that sends pt to the "cone point" $[x,0] \in U$, and let $r\colon \Sigma X \to \mathrm{pt}$ be the unique projection map. Then $r \circ i = \mathrm{id}_{\mathrm{pt}}$, and $i \circ r\colon U \to U$ sends every point to the cone point, and it remains to show that this map is homotopic to the constant map. A homotopy

$$H\colon \Sigma X \times I \longrightarrow \Sigma X \qquad \text{is given by} \qquad ([x,t],s) \mapsto [x,st]$$

The argument for $V$ is analogous. We claim that the inclusion map $i\colon X \to U \cap V$, $x \mapsto [x,1/2]$, is a homotopy equivalence. To see this, let $r\colon U \cap V \to X$ be the projection map given by $[x,t] \mapsto x$. Then $r \circ i = \mathrm{id}_X$, and $i \circ r$ is homotopic to $\mathrm{id}_{U\cap V}$ via the homotopy

$$H\colon (U \cap V) \times I \longrightarrow U \cap V \qquad H([x,t],s) = [x,(1-s)\frac{1}{2} + st]$$

We remark that $(1-s)\frac{1}{2} + st$ is the linear path from $1/2$ (for $s=0$) to $t$ (for $s=1$). Writing down the Mayer-Vietoris sequence for reduced homology groups we have

$$\begin{tikzcd}
{} \arrow[r] & \widetilde{H}_k(U) \oplus \widetilde{H}_k(V) \arrow[r] \arrow[d, equal] & \widetilde{H}_k(\Sigma X) \arrow[r, "\partial"] & \widetilde{H}_{k-1}(U \cap V) \arrow[r] \arrow[d, "r_*", "\cong"'] & \widetilde{H}_{k-1}(U) \oplus \widetilde{H}_{k-1}(V) \arrow[r] \arrow[d, equal] & {} \\
& 0 & & \widetilde{H}_{k-1}(X) & 0 &
\end{tikzcd}$$

The exactness of the sequence implies that $\partial$ is an isomorphism, and hence the composition $r_* \circ \partial\colon \widetilde{H}_k(\Sigma X) \to \widetilde{H}_{k-1}(X)$ is the desired isomorphism. $\square$

2. A map $f\colon X \to Y$ determines a map $\Sigma f\colon \Sigma X \to \Sigma Y$ between the suspensions of $X$ resp. $Y$, defined by $\Sigma X \ni [x,t] \mapsto [f(x),t] \in \Sigma Y$.
   (a) We recall that the suspension $\Sigma S^n$ is homeomorphic to $S^{n+1}$. Show that for a map $f\colon S^n \to S^n$, the degree of the map $S^{n+1} = \Sigma S^n \xrightarrow{\Sigma f} \Sigma S^n = S^{n+1}$ is equal to the degree of $f$.
   (b) Let $r_n\colon S^n \to S^n$ be the reflection map $(x_0,\dots,x_n) \mapsto (-x_0,\dots,x_n)$. Show that $\deg(r_n) = -1$. Hint: Using part (a), proceed by induction over $n$, starting at $n = 0$.

*Proof.* **Part (a)** The key arguments of problem 1(b) show that the maps

$$\widetilde{H}_{n+1}(\Sigma X) \xrightarrow[\cong]{\partial} \widetilde{H}_n(U\cap V) \xrightarrow[\cong]{r_*} \widetilde{H}_n(X)$$

are both isomorphisms (we use the same notation as in the solution of problem # 1). Let $f\colon X \to X$ be a map, and $\Sigma f\colon \Sigma X \to \Sigma X$ its suspension. Then inspection of the definition of $\Sigma f$ shows that $\Sigma f$ maps the subspace $U \cap V \subset \Sigma X$ to itself, and is compatible with the retraction map $r\colon U\cap V \to X$ in the sense that the diagram

$$\begin{tikzcd} U\cap V \arrow[r, "r"] \arrow[d, "\Sigma f_{|U\cap V}"'] & X \arrow[d, "f"] \\ U\cap V \arrow[r, "r"] & X \end{tikzcd}$$

is commutative. By functoriality of homology and the naturality of the boundary homomorphism of the Mayer-Vietoris sequence we then obtain the following commutative diagram

$$\begin{tikzcd} \widetilde{H}_{n+1}(\Sigma X) \arrow[r, "\partial", "\cong"'] \arrow[d, "(\Sigma f)_*"] & \widetilde{H}_n(U\cap V) \arrow[r, "r_*", "\cong"'] \arrow[d, "(\Sigma f_{|U\cap V})_*"] & \widetilde{H}_n(X) \arrow[d, "f"] \\ \widetilde{H}_{n+1}(\Sigma X) \arrow[r, "\partial", "\cong"'] & \widetilde{H}_n(U\cap V) \arrow[r, "r_*", "\cong"'] & \widetilde{H}_n(X) \end{tikzcd}$$

Specializing to $X = S^n$ and hence $\Sigma X \approx S^{n+1}$ (by part (a)), we conclude from the above diagram that $\deg(\Sigma f) = \deg(f)$.

$\square$

3. Prove the following statements for the local degree.
   (a) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry (i.e., $f$ belongs to the orthogonal group $O_n$), then $\deg(f,0) = \deg(f_{|S^{n-1}})$. Hint: apply the statement $\deg(\Sigma g) = \deg(g)$ from the previous problem to $g = f_{|S^{n-1}}$, and calculate the degree of $\Sigma g$ using the theorem expressing the degree of a map as a sum of local degrees.
   (b) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, then $\deg(f,0) = \operatorname{sign}\det(f)$. Hint: By homotopy invariance of $\deg(f,0)$, it only depends on $[f] \in \pi_0\mathrm{GL}_n$ (here $\pi_0\mathrm{GL}_n$ denotes the set of connected components of the space $\mathrm{GL}_n$, and $[f] \in \pi_0\mathrm{GL}_n$ is the connected component that contains the point $f$). Recall that $\mathrm{GL}_n$ has two

components; moreover, there is a bijection $\pi_0\mathrm{GL}_n \to \{\pm 1\}$ given by $[f] \mapsto \operatorname{sign}\det(f)$. Use part (a) and problem 2(b) to construct a linear isometry $f$ with $\deg(f, 0) = -1$.

(c) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map which is differentiable at the point $x_0 \in \mathbb{R}^n$. Let $Df_{x_0}$ be the derivative at $x_0$ (which is a linear map $Df_{x_0}\colon \mathbb{R}^n \to \mathbb{R}^n$; the corresponding matrix is the Jacobian of $f$ at the point $x_0$). Show that if $Df_{x_0}$ is invertible, then

$$\deg(f, x_0) = \operatorname{sign}\det(Df_{x_0}).$$

Hint: Use the assumption that $f$ is differentiable at $x_0$ to write $f(x)$ in the form $f(x) = f(x_0) + Df_{x_0}(x - x_0) + e(x - x_0)$ where the error term $e(h)$ is $o(h)$ for $h \to 0$. Use the homotopy $f_t(x) := f(x_0) + Df_{x_0}(x - x_0) + te(x - x_0)$ to argue (carefully!) that the local degrees of $f_0$ and $f_1 = f$ at $x_0$ agree.

*Proof.* **Part (a).** Following the hint,

$$\deg(f_{|S^{n-1}}) = \deg(g) = \deg(\Sigma g) = \deg(\Sigma g, c_-),$$

where $\Sigma g\colon \Sigma S^{n-1} \to \Sigma S^{n-1}$ is the suspension of $g$, and $c_- = [x, 0] \in \Sigma S^{n-1}$ is the bottom cone point (I'm thinking of the suspension $\Sigma S^{n-1}$ as the union of the two cones $C_- = \{[x,t] \in \Sigma S^{n-1} \mid t \in [0, 1/2]\}$ with cone point $c_- = [x, 0]$ and $C_- = \{[x,t] \in \Sigma S^{n-1} \mid t \in [1/2, 1]\}$ with cone point $c_+ = [x, 1]$). The last equality follows from the theorem expressing the degree of the map $\Sigma g$ as the sum of the local degree of $\Sigma g$ at the points of $(\Sigma g)^{-1}(c_-) = \{c_-\}$. To show that $\deg(\Sigma g, c_-) = \deg(f, 0)$ we will construct a homeomorphism $h\colon C_- \xrightarrow{\approx} D^n$ which maps $c_-$ to 0 such that the diagram

$$\begin{tikzcd} C_- \arrow[r, "h", "\approx"'] \arrow[d, "\Sigma g_{|C_-}"'] & D^n \arrow[d, "f_{|D^n}"] \\ C_- \arrow[r, "h", "\approx"'] & D^n \end{tikzcd}$$

commutes. The induced diagram of local homology groups

$$\begin{tikzcd} H_n(C_-, C_- \setminus c_-) \arrow[r, "h_*", "\cong"'] \arrow[d, "(\Sigma g_{|C_-})_*"'] & H_n(D^n, D^n \setminus 0) \arrow[d, "(f_{|D^n})_*"] \\ H_n(C_-, C_- \setminus c_-) \arrow[r, "h_*", "\cong"'] & H_n(D^n, D^n \setminus 0) \end{tikzcd}$$

then implies $\deg(\Sigma g, c_-) = \deg(f, 0)$. We define the map

$$h\colon C_- \longrightarrow D^n \qquad \text{by} \qquad h([x,t]) = 2tx$$

This is evidently a continuous bijection. Since $C_-$ is compact (as quotient of the product of the compact spaces $S^{n-1}$ and $[0, 1/2]$) and $D^n$ is Hausdorff, it follows that $h$ is a homeomorphism.

**Part(b).** Let $r\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear isometry given by $(x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$. Its restriction to $S^{n-1} \subset \mathbb{R}^n$ according to problem 2(b) has degree $-1$ ($r_{|S^{n-1}}$ is the reflection map denoted $r_{n-1}$ in problem 2(b)). Hence by part (a) it follows that $\deg(r, 0) =$

$\deg(r_{|S^{n-1}}) = -1$. This is sufficient to conclude the equation

$$\deg(f, 0) = \operatorname{sign}\det(f) \qquad \text{for every } f \in \mathrm{GL}_n,$$

since both, $\deg(f, 0)$ and $\operatorname{sign}\det(f)$ depend only on the connected component $[f] \in \mathrm{GL}_n$, and hence it suffices to check the equality for *one element* $f$ in each of the two components of $\mathrm{GL}_n$. It is clear that the above equation holds for $f = \mathrm{id}_{\mathbb{R}^n}$, and we just proved that it also holds for $f$ the reflection map, which lives in the other component.

**Part(c).** To prove part (c), we note that the assumption that $f$ is differentiable at $x_0$ means that $f(x)$ can be written in the form

$$f(x) = f(x_0) + Df_{x_0}(x - x_0) + e(x - x_0),$$

where the 'error term' $e(h)$ is $o(h)$ for $h \to 0$, which means

$$\lim_{h\to 0} \frac{e(h)}{|h|} = 0. \tag{5}$$

We define

$$f(x)_t := f(x_0) + Df_{x_0}(x - x_0) + te(x - x_0),$$

and want to argue that $f_t$ is a map of pairs

$$f_t \colon (B_\epsilon(x_0), B_\epsilon(x_0) \setminus \{x_0\}) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus f(x_0)),$$

for sufficiently small $\epsilon > 0$, where $B_\epsilon(x_0)$ is the ball of radius $\epsilon$ around $x_0$. In other words, we want to argue that $f_t^{-1}(f(x_0)) \cap B_\epsilon(x_0) = \{x_0\}$, or equivalently, that $Df_{x_0}(h) + te(h) \neq 0$ for all $h$ with $0 < h < \epsilon$. The idea is to show that for $0 < h < \epsilon$ the norm of $Df_{x_0}(h)$ is large compared to the norm of $te(h)$.

To make this precise, let $m := \min_{h\in S^{n-1}} ||Df_{x_0}(h)||$. We note that $m > 0$, since $m = ||Df_{x_0}(h_0)||$ for some $h_0 \in S^{n-1}$, and $Df_{x_0}(h_0) \neq 0$ due to our assumption that $Df_{x_0}$ is invertible. Now the statement (5) allows us to choose $\epsilon > 0$ such that $\frac{||e(h)||}{|h|} < m$ for $||h|| < \epsilon$. This implies that for $0 < ||h|| < \epsilon$ we have

$$||te(h)|| \leq ||e(t)|| < m||h|| \leq ||Df_{x_0}(\frac{h}{||h||})||\,||h|| = ||Df_{x_0}(h)||$$

and hence $Df_{x_0}(h) + te(h) \neq 0$ as desired. We conclude that $f = f_1$ is homotopic to $g = f_0$ as maps from $(B_\epsilon(x_0), B_\epsilon(x_0) \setminus \{x_0\})$ to $(\mathbb{R}^n, \mathbb{R}^n \setminus f(x_0))$ and hence $\deg(f, x_0) = \deg(g, x_0)$.

Finally, we want to compare $\deg(g, x_0)$ and $\deg(D, 0)$, where $D = Df_{x_0}$. We note that by construction of $g$ the following diagram is commutative

$$\begin{tikzcd} B_\epsilon(0) \arrow[r, "Df_{x_0}"] \arrow[d, "T_{x_0}"', "\approx"] & \mathbb{R}^n \arrow[d, "T_{f(x_0)}"', "\approx"] \\ B_\epsilon(x_0) \arrow[r, "g"] & \mathbb{R}^n \end{tikzcd}$$

where $T_{x_0}$, $T_{f(x_0)}$ are translation maps defined for $v \in \mathbb{R}^n$ by $T_v \colon \mathbb{R}^n \to \mathbb{R}^n$, $w \mapsto w + v$. The corresponding diagram of local homology groups then implies $\deg(Df_{x_0}, 0) = \deg(g, x_0)$.

$\square$

4. Let $(X, V, A)$ be a triple of topological spaces (i.e., $A \subset V \subset X$). Show that there is a long exact sequence of homology groups

$$\dots \longrightarrow H_q(V, A) \longrightarrow H_q(X, A) \longrightarrow H_q(X, V) \xrightarrow{\partial} H_{q-1}(V, A) \longrightarrow \dots$$

Hint: use the algebraic fact that a short exact sequence of chain complexes leads to a long exact sequence of homology groups.

*Proof.* Let $i\colon (V, A) \to (X, A)$ and $j\colon (X, A) \to (X, V)$ be the maps of pairs induced by the inclusion map $V \to X$ resp. the identity on $X$. The induced maps on singular $q$-chains

$$\begin{tikzcd} C_q(V,A) \arrow[r, "i_q"] \arrow[d, equal] & C_q(X,A) \arrow[r, "j_q"] \arrow[d, equal] & C_q(X,V) \arrow[d, equal] \\ C_q(V)/C_q(A) & C_q(X)/C_q(A) & C_q(X)/C_q(V) \end{tikzcd}$$

form a short exact sequence since $j_q$ is an epimorphism whose kernel is equal to $C_q(V)/C_q(A) \subset C_q(X)/C_q(A)$.

This implies that

$$C_*(V, A) \xrightarrow{i_*} C_*(X, A) \xrightarrow{j_*} C_*(X, V)$$

is a short exact sequence of chain complexes which implies the desired long exact sequence of homology groups. □

## 5. Homework Assignment # 5

2. Show that the complex projective space $\mathbb{CP}^n$ is a CW complex with one cell of dimension $2i$ for $0 \le i \le n$.

*Proof.* It suffices to show that $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell of dimension $2n$. Define

$$\Phi\colon D^{2n} \longrightarrow \mathbb{C}P^n$$
$$z = (z_0, \dots, z_{n-1}) \mapsto [z_0, z_1, \dots, z_{n-1}, \sqrt{1 - ||z||^2}],$$

and let $\varphi\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ be the natural projection map given by $(z_0, \dots, z_{n-1}) \mapsto [z_0, z_1, \dots, z_{n-1}]$. We note that these maps are compatible in the sense that the following diagram is commutative

$$\begin{tikzcd} S^{2n-1} \arrow[r, "\varphi"] \arrow[d, "j"] & \mathbb{C}P^{n-1} \arrow[d, "i"] \\ D^{2n} \arrow[r, "\Phi"] & \mathbb{C}P^n, \end{tikzcd}$$

where $i$, $j$ are the obvious inclusion maps. It follows that the map

$$\mathbb{C}P^{n-1} \cup_\varphi D^{2n} \xrightarrow{i \cup \Phi} \mathbb{C}P^n$$

is well-defined and continuous. We note that this map is surjective, since if $[z_0, \dots, z_n] \in \mathbb{C}P^n$ with $z_n \neq 0$, then multiplying all components by the unit complex number $z_n^{-1}||z_n||$, we can assume w.l.o.g. that $z_n$ is a positive real number. Injectivity is obvious and it follows that $f = i \cup \Phi$ is a continuous bijection. It follows that $f$ is a homeomorphism since the domain of $f$ is compact and its range is Hausdorff. □

2. (a) Show that $S^m \times S^n$ is a CW-complex. There are many choices; try to be economical in the sense of trying to minimize the numbers of cells of the decomposition. Hint: think about the CW-structure on $S^1 \times S^1$ (here we can visualize things...).

(b) Compute the homology groups of $S^m \times S^n$ assuming that $m \geq 2$ and $n \geq m+2$.

(c) If $X$ and $Y$ are CW complexes (with finitely many cells to avoid point set topology issues), is there an associated CW structure for the product $X \times Y$? Hint: show that $X \times Y$ has a CW structure whose cells correspond to pairs of cells (with one cell from $X$ and the other cell from $Y$). More precisely, if $e^k_\alpha$ is a $k$-cell of $X$ and $e^\ell_\beta$ is an $\ell$-cell of $Y$, show that there is corresponding cell denoted $e^k_\alpha \times e^\ell_\beta$ of dimension $k+\ell$ for the product $X \times Y$. It is useful here to replace disks $D^n$ by cubes $I^n$, thanks to the pleasant property $I^m \times I^n = I^{m+n}$ (which is only true up to homeomorphism for disks).

*Proof.* **Part (a).** It will be convenient to identify $S^m$ with $D^m/\partial D^m$, and $S^n$ with $D^n/\partial D^n$. Moreover, let $x_0 \in S^m = D^m/\partial D^m$ be the basepoint obtained by collapsing all the points in $\partial D^m$ to one point, and similarly let $y_0$ be the basepoint of $S^n = D^n/\partial D^n$.

We define the skeleta $X^k$ of $X := S^m \times S^n$, by setting

$$X^0 = \{(x_0, y_0)\} \quad X^m = S^m \times \{y_0\} \quad X^n = S^m \times \{y_0\} \cup \{x_0\} \times S^n \quad X^{m+n} = S^m \times S^n.$$

For $k \notin S := \{0, m, n, m+n\}$, we define $X^k := X^\ell$, where $\ell = \min\{s \in S \mid s \leq k\}$. It remains to show that

1. $X^m$ is obtained by attaching an $m$-cell to $X^0$,
2. $X^n$ is obtained by attaching an $n$-cell to $X^m$, and
3. $X^{m+n}$ is obtained by attaching an $(m+n)$-cell to $X^n$.

Since $X^m$ is an $m$-sphere, (1) is clear. Similarly, $X^n$ is a wedge of an $m$-sphere and an $n$-sphere which implies (2). To show (3), we need to write down the *characteristic map* $\Phi\colon D^{m+n} \to X$ for the alleged $(m+n)$-cell. To construct this map it is useful to identify the disk $D^{m+n}$ with the product $D^m \times D^n$ by any homeomorphism. We define

$$\Phi\colon D^{m+n} = D^m \times D^n \longrightarrow X = D^m/\partial D^m \times D^n/\partial D^n \qquad (x,y) \mapsto [x,y].$$

This map sends $\partial(D^m \times D^n) = (D^m \times \partial D^n) \cup (\partial D^m \times D^n)$ to $X^n$, and hence we can define the *attaching map*

$$\phi := \Phi_{\partial D^{m+n}}\colon \partial(D^m \times D^n) \longrightarrow X^n.$$

By construction, the diagram

$$\begin{tikzcd} \partial(D^m \times D^n) \arrow[r, "\phi"] \arrow[d, hook] & X^n \arrow[d, "i"] \\ D^m \times D^n \arrow[r, "\Phi"] & X \end{tikzcd}$$

commutes, and hence the continuous map $i \amalg \Phi \colon X^n \amalg D^m \times D^n \longrightarrow X$ induces a well-defined continuous map $h \colon X^n \cup_\phi (D^m \times D^n) \longrightarrow X$. It is clear that $h$ is a bijection, and hence a homeomorphism, since the domain of $h$ is compact and its codomain is Hausdorff.

**Part (b).** By part (a), $X = S^m \times S^n$ has a CW structure with one cell each in dimension $0, m, n, m+n$. Hence the cellular chain complex $C_*^{\mathrm{CW}}(X)$ is given by

$$C_k^{\mathrm{CW}}(X) = \begin{cases} \mathbb{Z}e^k & \text{for } k = 0, m, n, m+n \\ 0 & \text{otherwise} \end{cases}$$

The assumptions on $m$, $n$ guarantee that if $C_k^{\mathrm{CW}}(X)$ is non-trivial, then $C_{k-1}^{\mathrm{CW}}(X)$ is zero. Hence all differentials in this chain complex are zero and hence $H_k^{\mathrm{CW}}(X) = C_k^{\mathrm{CW}}(X)$. The singular homology groups $H_k(X)$ are isomorphic to the cellular homology groups $H_k^{\mathrm{CW}}(X)$ by the Theorem from class. It follows that

$$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, m, n, m+n \\ 0 & \text{otherwise} \end{cases}$$

**Part (c).** Let $e_\alpha^k$ be a $k$-cell of $X$, let $e_\beta^\ell$ be an $\ell$-cell of $Y$, and let

$$\Phi_\alpha^k \colon I^k \longrightarrow X^k \qquad \Phi_\alpha^\ell \colon I^\ell \longrightarrow X^\ell$$

be their characteristic maps. This suggests that the characteristic map for the product cell $e_\alpha^k \times e_\beta^\ell$ should be the product map

$$I^{k+\ell} = I^k \times I^\ell \xrightarrow{\Phi_\alpha^k \times \Phi_\beta^\ell} X^k \times Y^\ell$$

The requirement that the characteristic map of an $n$-cell has the $n$-skeleton as codomain then forces us to define the $n$-skeleton of $X \times Y$ as

$$(X \times Y)^n := \bigcup_{k+\ell=n} X^k \times Y^\ell \subset X \times Y.$$

Then $(X \times Y)^0 = X^0 \times Y^0$ is the product of two discrete topological spaces and hence discrete.

It remains to show that the characteristic maps $\Phi_\alpha^k \times \Phi_\beta^\ell$ for $k + \ell = n$ indeed provide a homeomorphism from the $(n-1)$-skeleton of $X \times Y$ with $n$-cells attached via attaching maps $\varphi_{\alpha,\beta}^{k,\ell} := (\Phi_\alpha^k \times \Phi_\beta^\ell)_{|\partial(I^k \times I^\ell)}$ to the $n$-skeleton of $X \times Y$. More precisely, let $A_k$ be the set that parametrizes $k$-cells of $X$, and let $B_\ell$ be the set that parametrizes $\ell$-cells of $Y$. Consider the following commutative diagram.

$$\begin{tikzcd}
(X\times Y)^{n-1} \coprod\limits_{\substack{k+\ell=n\\ \alpha\in A_k,\ \beta\in B_\ell}} I^k_\alpha\times I^\ell_\beta \arrow[rr, "i_{n-1}\coprod \Phi^k_\alpha\times\Phi^\ell_\beta"] \arrow[d, two heads] & & (X\times Y)^n \\
\left((X\times Y)^{n-1} \coprod\limits_{\substack{k+\ell=n\\ \alpha\in A_k,\ \beta\in B_\ell}} I^k_\alpha\times I^\ell_\beta\right)/\sim \arrow[urr, "h"'] & &
\end{tikzcd}$$

Here $\sim$ is the equivalence relations that identifies a point in $\partial(I^k_\alpha\times^\ell_\beta)$ with its image in $(X\times Y)^{n-1}$ under the attaching map $\varphi^{k,\ell}_{\alpha,\beta}$. The map $i_{n-1}$ is the inclusion map of the $(n-1)$-skeleton of $X\times Y$ to the $n$-skeleton. The horizontal map is by construction a continuous surjective map. The points identified by the equivalence relation $\sim$ map to the same points in $(X\times Y)^n$, and hence the horizontal map induces a well-defined continuous surjection $h$. Moreover, $h$ is injective, since by construction of the equivalence relation, two points in the disjoint union get identified if and only if they map to the same point in $(X\times Y)^n$. Hence $h$ is a continuous bijection. Induction over skeleta can be used to argue that the domain of $h$ is compact and the codomain is Hausdorff. This implies that $h$ is a homeomorphism as required. $\square$

3. (a) Show that if $0\longrightarrow C_k\xrightarrow{\partial} C_{k-1}\xrightarrow{\partial}\dots\xrightarrow{\partial} C_0\longrightarrow 0$ is a chain complex of finitely generated abelian groups with homology groups $H_q:=H_q(C_*)$, then

$$\sum_{q=0}^k(-1)^q\operatorname{rk}C_q=\sum_{q=0}^k(-1)^q\operatorname{rk}H_q.$$

Hint: Use the fact that if $0\to A\to B\to C\to 0$ is a short exact sequence of finitely generated abelian groups, then $\operatorname{rk}B=\operatorname{rk}A+\operatorname{rk}C$. Show that if $Z_q$ (resp. $B_q$) are the $q$-cycles (resp. $q$-boundaries) of $C_*$, then there are short exact sequences

$$0\to Z_q\to C_q\to B_{q-1}\to 0\qquad\text{and}\qquad 0\to B_q\to Z_q\to H_q\to 0.$$

(b) Show that if $X$ is a finite CW complex, then

$$\sum_{q=0}^k(-1)^q\operatorname{rk}H_q(X)=\sum_{q=0}^k(-1)^q\#\{q\text{-cells of }X\}.$$

The alternating sum on the left is known as the *Euler characteristic* of the topological space $X$, denoted $\chi(X)$. The statement above shows that if $X$ is a surface, this definition agrees with our previous definition as the alternating sum of the number of vertices, edges and faces of a pattern of polygons on $X$ (since we can now interpret such a pattern as providing a CW structure for the surface $X$).

*Proof.* **Part (a).** By definition of $B_{q-1}$ it is the image of the homomorphism $\partial\colon C_q \to C_{q-1}$; the kernel of the surjective map $\partial\colon C_q \to B_{q-1}$ is, again by definition, the group of $q$-cycles $Z_q$. This gives the first exact sequence. The second sequence is exact since the homology group $H_q$ by definition is the quotient group $Z_q/B_q$. Now we calculate ranks

$$\operatorname{rk} C_q = \operatorname{rk} Z_q + \operatorname{rk} B_{q-1} = \operatorname{rk} H_q + \operatorname{rk} B_q + \operatorname{rk} B_{q-1}.$$

Hence

$$\sum_q (-1)^q \operatorname{rk} C_q = \sum_q (-1)^q \operatorname{rk} H_q + \sum_q (-1)^q \operatorname{rk} B_q + \sum_q (-1)^q \operatorname{rk} B_{q-1} = \sum_q (-1)^q \operatorname{rk} H_q,$$

since the last two sums cancel.

**Part (b).** Applying part (a) to the cellular chain complex $C_*^{CW}(X)$, we have

$$\begin{aligned}\sum_{q=0}^{k} (-1)^q \operatorname{rk} H_q(X) &= \sum_{q=0}^{k} (-1)^q \operatorname{rk} H_q(C_*^{CW}(X)) \\ = \sum_{q=0}^{k} (-1)^q \operatorname{rk} C_q^{CW}(X) &= \sum_{q=0}^{k} (-1)^q \#\{q\text{-cells of } X\}.\end{aligned}$$

□

4. (a) Let $\Sigma$ be a surface of genus $g$, and let $\widetilde{\Sigma}$ be a surface which is a $d$-fold covering of $\Sigma$. Show that $\chi(\widetilde{\Sigma}) = d\chi(\Sigma)$. Hint: Use our definition of the Euler characteristic $\chi(\Sigma)$ of a surface $\Sigma$ as the alternating sum of the number of vertices, edges and faces of a pattern of polygons on $\Sigma$.

(b) For $\Sigma$ and $\widetilde{\Sigma}$ as in (a), show that the genus $\widetilde{g}$ of $\widetilde{\Sigma}$ is given by the formula $\widetilde{g} = dg - d + 1$.

(c) According to the formula above, the surface of genus 4 is a 3-fold covering of the surface of genus 2; in particular, there is a free action of the cyclic group $C_3$ of order three on the surface of genus 4. Draw a picture of a surface of genus 4 which has an obvious $C_3$-symmetry.

*Proof.* **Part (a).** Let $\Gamma$ be a pattern of polygons on $\Sigma$. Let $V$ (resp. $E$ resp. $F$) be the set of vertices (resp. edges resp. faces) of $\Gamma$. Then $\Gamma$ determines a pattern of polygons on the covering space $\widetilde{\Sigma}$ whose vertices (resp. edges resp. faces) are the preimages of the vertices/edges/faces of $\Gamma$ under the the $d$-fold covering map $\pi\colon \widetilde{\Sigma} \to \Sigma$. There are exactly $d$ vertices/edges/faces in $\widetilde{\Gamma}$ which are the preimages of a given vertex/edge/face of $\Gamma$, and hence

$$\#\widetilde{V} = d\#V \qquad \#\widetilde{E} = d\#E \qquad \#\widetilde{F} = d\#F,$$

where $\widetilde{V}$ (resp. $\widetilde{E}$ resp. $\widetilde{F}$) denotes the set of vertices (resp. edges resp. faces) of $\widetilde{\Gamma}$. It follows that

$$\chi(\widetilde{\Sigma}) = \#\widetilde{V} - \#\widetilde{E} + \#\widetilde{F} = d\#V - d\#E + d\#F = d\chi(\Sigma).$$

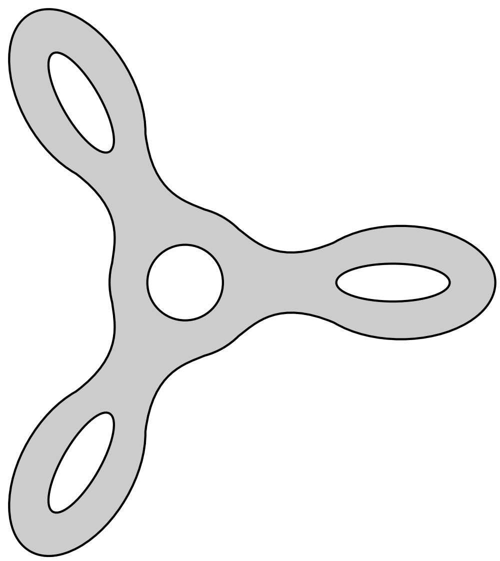
**Part (b).** From class we know that the Euler characteristic $\chi(\Sigma)$ of a closed connected surface of genus $g$ is given by

$$\chi(\Sigma) = 2 - 2g.$$

Hence using part (a), it follows that

$$\widetilde{g} = 1 - \frac{\chi(\widetilde{\Sigma}))}{2} = 1 - \frac{d\chi(\Sigma)}{2} = 1 - \frac{d(2-2g)}{2} = 1 - d + dg.$$

**Part (c).** Here is the picture of a surface $\widetilde{\Sigma}$ of genus 4 with a rotational $C_3$-symmetry. Note that the $C_3$-action on this surface is *free*, and that the quotient space $\Sigma = \widetilde{\Sigma}/C_3$ is a surface of genus 2. Thinking of the surface of genus 2 as the connected sum to two tori, the circle separating the two tori is triply covered by the three circles in $\widetilde{\Sigma}$ which separate the "central hole" from the three "outer holes".



□

## 6. Homework Assignment # 6

1. Prove the following statement which is known as the 5-*lemma*. Suppose we have a commutative diagram of abelian groups and group homomorphisms

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5 \end{tikzcd}$$

such that the rows are exact sequences. Show that if the vertical maps $h_1$, $h_2$, $h_4$, and $h_5$ are isomorphisms, then also the middle map $h_3$ is an isomorphism.

Remark: the assumptions that the maps $h_1$, $h_2$, $h_4$, $h_5$ are all isomorphisms are slightly stronger than needed for the proof. What weaker assumptions will do?

*Proof.* We first prove that $h_3$ is injective. Let $a_3 \in A_3$ such that $h_3a_3 = 0 \in B_3$. Then by the commutativity of the diagram, we have $h_4f_3a_3 = g_3h_3a_3 = g_3(0) = 0$, so $f_3(a_3) \in \ker\ h_4 = 0$ since $h_4$ is injective. Hence, $a_3 \in \ker\ f_3 = \operatorname{im}\ f_2$, so there is an $a_2 \in A_2$ such that $f_2(a_2) = a_3$.

Again, by the commutativity of the diagram, we have $g_2h_2(a_2) = h_3f_2(a_2) = h_3(a_3) = 0$, so $h_2(a_2) \in \ker\ g_2 = \operatorname{im}\ g_1$. Thus, there is an element $b_1 \in B_1$ such that $g_1(b_1) = h_2(a_2)$. Moreover, since $h_1$ is surjective, there is some $a_1 \in A_1$ with $h_1a_1 = b_1$. It follows that $h_2f_1a_1 = g_1h_1a_1 = g_1b_1 = h_2a_2$. Thus, since $h_2$ is injective, $a_2 = f_1a_1$, so $a_3 = f_2a_2 = f_2f_1a_1 = 0$. Therefore, $h_3$ is injectiv.

To show surjectivity of $h_3$, let $b_3 \in B_3$. Since $h_4$ is surjective, there is some $a_4 \in A_4$ with $h_4a_4 = g_3b_3 \in A_4$ and by commutativity, $h_5f_4a_4 = g_4h_4a_4 = g_4g_3b_3 = 0$. Since $h_5$ is injective, this implies $f_4a_4 = 0$. By exactness of the top row at $A_4$, there is an $a_3 \in A_3$ such that $f_3(a_3) = a_4$. Hence,

$$g_3(h_3(a_3) - b_3) = h_4f_3a_3 - g_3b_3 = h_4a_4 - g_3b_3 = 0.$$

By the exactness of the lower row at $B_3$, this implies that there exists $b_2 \in B_2$ such that $g_2(b_2) = h_3(a_3) - b_3$. Since $h_2$ is surjective, there is some $a_2 \in A_2$ with $h_2a_2 = b_2$ and hence

$$h_3(a_3 - f_2a_2) = h_3a_3 - h_3f_2a_2 = h_3a_3 - g_2h_2a_2 = b_3,$$

which shows that $b_3$ is in the image of $h_3$. Since $b_3$ was arbitrary, this shows that $h_3$ is surjective.

We see that we've used the assumptions that $h_4$, $h_2$ are injective, and that $h_1$ is surjective to show injectivity of $h_3$. Our proof that $h_3$ is surjective required the assumptions that $h_2$, $h_4$ are surjective, and that $h_5$ is injective. So it is sufficient to assume that $h_2$ and $h_4$ are isomorphisms, that $h_1$ is an epimorphism, and that $h_5$ is a monomorphism. $\square$

2. (a) Show that the relation "chain homotopic" is transitive. In other words, show that if $f, g, h\colon C_* \to D_*$ are chain maps, and $f$ is chain homotopic to $g$, and $g$ is chain homotopic to $h$, then $f$ is chain homotopic to $h$.

(b) Let $f, g\colon C_* \to D_*$ and $h, k\colon D_* \to E_*$ be chain maps. Show that if $f$ is chain homotopic to $g$ and $h$ is chain homotopic to $k$, then $h \circ f$ is chain homotopic to $k \circ g$.

*Proof.* Let $S\colon C_q \to D_{q+1}$ be a chain homotopy from $f$ to $g$ and let $T\colon C_q \to D_{q+1}$ be a chain homotopy from $g$ to $h$, that is,

$$\partial S + S\partial = g - f \qquad \text{and} \qquad \partial T + T\partial = h - g.$$

Then $\partial(S + T) + (S + T)\partial = \partial S + \partial T + S\partial + T\partial = g - f + h - g = h - f$, which shows that $S + T\colon C_q \to D_{q+1}$ is a chain homotopy from $f$ to $h$. This proves part (a).

To prove part (b), let $S\colon C_q \to D_{q+1}$ be a chain homotopy from $f$ to $g$, and let $T\colon D_q \to E_{q+1}$ be a chain homotopy from $h$ to $k$, that is

$$\partial S + S\partial = g - f \qquad \text{and} \qquad \partial T + T\partial = k - h.$$

Then

$$\begin{aligned} \partial hS + hS\partial &= h\partial S + hS\partial = h(\partial S + S\partial) = h(g - f) = hg - hf \\ \partial Tg + Tg\partial &= \partial Tg + T\partial g = (\partial T + T\partial)g = (k - h)g = kg - hg. \end{aligned}$$

Here both second equalities follow from our assumption that $h$ (resp. $g$) are chain maps. Adding both terms, we obtain

$$\partial(hS + Tg) + (hS + Tg)\partial = kg - hf,$$

which proves that $hS + Tg\colon C_q \to E_{q+1}$ is a chain homotopy from $hf$ to $kg$. □

3. Let $f, g\colon C_* \to D_*$ be two chain maps which are chain homotopic. Show that the induced maps in homology $f_*, g_*\colon H_q(C_*) \to H_q(D_*)$ are equal.

*Proof.* Let $T\colon C_q \to D_{q+1}$ be a chain homotopy from $g$ to $f$, that is, $\partial T + T\partial = f - g$. Let $z \in Z_q(C_*)$ be a $q$-cycle representing the homology class $[z] \in H_q(C_*)$. Then

$$f_*([z]) = [fz] = [\partial Tz + T\partial z + gz] = [\partial Tz + gz] = [gz] = g_*([z]).$$

Here the second equation holds since $\partial z = 0$, and the third equation holds since adding the boundary $\partial Tz$ to the cycle $gz$ does not change the homology class it represents. □

4. In class we outlined the proof of the statement that the inclusion map $\iota\colon C_*^{\mathcal{U}}(X) \to C_*(X)$ is a chain homotopy equivalence. Here $C_*^{\mathcal{U}}(X)$ is the subchain complex of $C_*(X)$ generated by all simplices which are contained in one of the open subsets of the open cover $\mathcal{U}$. The idea is to use barycentric subdivision to construct a chain map $\rho\colon C_*(X) \to C_*^{\mathcal{U}}(X)$ such that $\rho \circ \iota = \mathrm{id}$ and $\iota \circ \rho$ is chain homotopic to id. The purpose of this problem is to provide some of the steps in the proof of this statement (more precisely, to prove Lemmas A, B and C we mentioned in class). As in class let $\widetilde{LC}_*(Y) \subset \widetilde{C}_*(Y)$ be the augmented chain complex generated by affine linear simplices $\lambda = [w_0, \dots, w_q]\colon \Delta^q \to Y$, $w_i \in Y$ for a convex subset $Y$ of some vector space. Recall that for any point $b \in Y$, we define the linear map $b\colon \widetilde{LC}_q(Y) \to \widetilde{LC}_{q+1}(Y)$ which sends an affine linear simplex $[w_0, \dots, w_q]$ to the "cone" $[b, w_0, \dots, w_q]$. Recall further that $\partial b = \mathrm{id} - b\partial$; in other words, $b$ is a chain homotopy from 0 to id.

(a) Show that the linear map $S\colon \widetilde{LC}_q(Y) \to \widetilde{LC}_q(Y)$ defined inductively by $S\lambda := \lambda$ for $q = -1, 0$ and $S\lambda := b_\lambda(S\partial\lambda)$ is a chain map. Here $b_\lambda = \frac{1}{q+1}(w_0, \dots, w_q) \in Y$ is the barycenter of the affine linear simplex $\lambda = [w_0, \dots, w_q]$.

(b) Show that the linear map $T\colon \widetilde{LC}_q(Y) \to \widetilde{LC}_{q+1}(Y)$ defined inductively by $T\lambda := 0$ for $q = -1$ and $T\lambda := b_\lambda(\lambda - T\partial\lambda)$ is a chain homotopy from $S$ to id.

(c) Extend the subdivision operator $S$ and the chain homotopy $T$ to the singular chain complex of any topological space $X$ by defining for any singular simplex $\sigma\colon \Delta^q \longrightarrow X$

$$S\sigma = \sigma_\#(S\lambda_q) \qquad \text{and} \qquad T\sigma = \sigma_\#(T\lambda_q),$$

where $\lambda_q\colon \Delta^q \longrightarrow \Delta^q$ is the tautological affine $q$-simplex in $\Delta^q$ given by the identity map. Show that $S\colon C_q(X) \to C_q(X)$ is a chain map and $T\colon C_q(X) \to C_{q+1}(X)$ is a chain homotopy from $S$ to the identity.

*Proof.* **Part (a).** We will prove that $\partial S\lambda = S\partial\lambda$ for all $\lambda \in \widetilde{LC}_q(Y)$ by induction over $q$. So let us assume that this holds for all elements of $\widetilde{LC}_k(Y)$ for $k < q$ and let $\lambda \in \widetilde{LC}_q(Y)$. Then

$$\partial S\lambda = \partial(b_\lambda S\partial\lambda) = S\partial\lambda - b_\lambda\partial S\partial\lambda = S\partial\lambda - b_\lambda(S\partial^2\lambda) = S\partial\lambda,$$

where the second equality follows from the equation $\partial b = \mathrm{id} - b\partial$, and the third equality follows from our inductive assumption since $\partial\lambda \in \widetilde{LC}_{q-1}(Y)$.

**Part (b).** Again we use induction to prove this statement. So let us assume that we have the equality $\partial T + T\partial = \mathrm{id} - S$ of endomorphisms of $\widetilde{LC}_k(Y)$ for $k < q$. Then for an affine linear $q$-simplex $\lambda = [w_0, \dots, w_q]$ we have

$$\begin{aligned} \partial T\lambda &= \partial(b_\lambda(\lambda - T\partial\lambda)) = \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial T\partial\lambda) \\ &= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda - T\partial^2\lambda) = \lambda - T\partial\lambda - b_\lambda S\partial\lambda = \lambda - T\partial\lambda - S\lambda. \end{aligned}$$

Here the second equality follows from the equation $\partial b = \mathrm{id} - b\partial$, the third equality from our inductive assumption since $\partial\lambda \in \widetilde{LC}_{q-1}(Y)$, and the last equality from the inductive definition of $S\lambda$.

**Part (c).** We observe first that the linear maps $S$, $T$ constructed this way are *natural* in the sense that for any map $f\colon X \to Y$ we have

$$Sf_\# = f_\# S \qquad \text{and} \qquad Tf_\# = f_\# T,$$

where $f_\#\colon C_*(X) \to C_*(Y)$ is the chain map induced by $f$. To prove the first equation we calculate for a $q$-simplex $\sigma\colon \Delta^q \to X$

$$Sf_\#\sigma = S(f\circ\sigma) = (f\circ\sigma)_\# S\lambda_q = f_\#\sigma_\# S\lambda_q = f_\# S\sigma. \tag{6}$$

Here the first equality is the definition of $f_\#(\sigma)$, the second and last is the definition of $S$ on general simplices and the third holds by the functor property of the assignment $X \mapsto C_*(X)$. The equation $Tf_\# = f_\# T$ holds by the same argument, we just need to replace the letter $S$ by the letter $T$.

Next we prove that $S$ is a chain map. Let $\sigma\colon \Delta^q \to X$ be a $q$-simplex. Then

$$\begin{aligned} \partial S\sigma &= \partial\sigma_\# S\lambda_q && \text{by definition of } S \text{ on } C_*(X) \\ &= \sigma_\#\partial S\lambda_q && \text{since } \sigma_\# \text{ is a chain map} \\ &= \sigma_\# S\partial\lambda_q && \text{since } S \text{ is a chain map on } \widetilde{LC}_*(\Delta^q) \\ &= S\sigma_\#\partial\lambda_q && \text{by equation (6)} \\ &= S\partial\sigma_\#\lambda_q && \text{since } \sigma_\# \text{ is a chain map} \\ &= S\partial\sigma && \text{by definition of the induced map } \sigma_\# \end{aligned}$$

This shows that $S$ is a chain map. Similarly,

$$\begin{aligned} \partial T\sigma &= \partial\sigma_\# T\lambda_q && \text{by definition of } T \text{ on } C_*(X) \\ &= \sigma_\#\partial T\lambda_q && \text{since } \sigma_\# \text{ is a chain map} \\ &= \sigma_\#(T\partial\lambda_q - \lambda_q + S\lambda_q) && \text{since } \partial T + T\partial = \mathrm{id} - S \text{ on } \widetilde{LC}_*(\Delta^q) \\ &= T\sigma_\#\partial\lambda_q - \sigma_\#\lambda_q + S\sigma_\#\lambda_q && \text{by equation (6)} \\ &= T\partial\sigma_\#\lambda_q - \sigma_\#\lambda_q + S\sigma_\#\lambda_q && \text{since } \sigma_\# \text{ is a chain map} \\ &= T\partial\sigma - \sigma + S\sigma && \text{by definition of the induced map } \sigma_\# \end{aligned}$$

□

## 7. Homework Assignment # 7

1. (a) Show that $\mathbb{Z}/k \otimes \mathbb{Z}/\ell \cong \mathbb{Z}/\gcd(k,\ell)$.
   (b) Write down a free resolution $M \xleftarrow{\epsilon} M_*$ for the $\mathbb{Z}$-module $M = \mathbb{Z}/k$.
   (c) Calculate $\operatorname{Tor}_q^{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/\ell) := H_q(M_* \otimes \mathbb{Z}/\ell)$ (this turns out to be independent of the choice of the resolution $M_*$).

*Proof.* **Proof of part (a).** The homomorphism

$$\mathbb{Z}/k \otimes \mathbb{Z}/\ell \longrightarrow \mathbb{Z}/\gcd(k,\ell) \qquad \text{given by} \qquad m \otimes n \mapsto mn$$

is a well defined surjective map. To show that this is in fact an isomorphism, it remains to show that the element $1 \otimes 1 \in \mathbb{Z}/k \otimes \mathbb{Z}/\ell$ multiplied by $g := \gcd(k,\ell)$ is zero in $\mathbb{Z}/k \otimes \mathbb{Z}/\ell$. To show this, write $g$ in the form $g = ak + b\ell$ for $k, \ell \in \mathbb{Z}$. Then

$$g(1 \otimes 1) = ak(1 \otimes 1) + b\ell(1 \otimes 1) = a(k \otimes 1) + b(1 \otimes \ell) = 0.$$

**Proof of part (b).** A free resolution $M_*$ of $\mathbb{Z}/k$ is given by

$$\mathbb{Z}/k \xleftarrow{\ \epsilon\ } M_0 = \mathbb{Z} \xleftarrow{\ k\ } M_1 = \mathbb{Z} \longleftarrow M_2 = 0 \longleftarrow$$

**Proof of part (c).** Tensoring the free resolution $M_*$ from part (b) with $\mathbb{Z}/\ell$ results in the chain complex

$$\begin{array}{cccc} \text{degree} & 0 & 1 & 2 \\ & & & \\ & \mathbb{Z} \otimes \mathbb{Z}/\ell & \xleftarrow{k \otimes \mathrm{id}} \ \mathbb{Z} \otimes \mathbb{Z}/\ell & \longleftarrow 0 \longleftarrow \end{array}$$

Let $\Phi\colon \mathbb{Z} \otimes \mathbb{Z}/\ell \xrightarrow{\cong} \mathbb{Z}/\ell$ be the isomorphism given by $\Phi(m \otimes [n]) := [mn] \in \mathbb{Z}/\ell$, and write $k = gk'$, $\ell = g\ell'$ for integers $k', \ell'$ which are relatively prime. Then we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \otimes \mathbb{Z}/\ell \arrow[d, "\Phi"', "\cong"] & \mathbb{Z} \otimes \mathbb{Z}/\ell \arrow[l, "k \otimes \mathrm{id}"'] \arrow[d, "\Phi"', "\cong"] \\
\mathbb{Z}/\ell \arrow[d, equal] & \mathbb{Z}/\ell \arrow[l, "\cdot k"'] \arrow[d, "\cdot k'"', "\cong"] \\
\mathbb{Z}/\ell & \mathbb{Z}/\ell \arrow[l, "\cdot g"']
\end{tikzcd}$$

whose vertical maps are isomorphisms (multiplication by $k'$ is an isomorphism of $\mathbb{Z}/\ell$, since $\gcd(k', \ell) = 1$. The kernel and cokernel of the bottom horizontal map are both cyclic groups of order $g$ (generated by $[\ell'] \in \mathbb{Z}/\ell$ resp. $[1] \in \mathbb{Z}/\ell$). Hence

$$\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/\ell) = \operatorname{coker}(k \otimes \mathrm{id}) \cong \operatorname{coker}(\cdot g) \cong \mathbb{Z}/g$$
$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/\ell) = \ker(k \otimes \mathrm{id}) \cong \ker(\cdot g) \cong \mathbb{Z}/g$$

□

2. Let $X_k$ be the connected sum of $k$ copies of the real projective plane $\mathbb{RP}^2$, $k \geq 1$. Compute the homology groups $H_q(X_k; \mathbb{Z}/2)$ in the following two different ways:
(a) As the homology groups of $C_*^{CW}(X_k) \otimes \mathbb{Z}/2$, where $C_*^{CW}(X_k)$ is the cellular chain complex associated to the standard CW structure on $X_k$ obtained by regarding $X_k$ as a quotient of the $2k$-gon.
(b) Via the Universal Coefficient Theorem, using the fact that

$$H_q(X_k) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2 & q = 1 \\ 0 & \text{otherwise} \end{cases}.$$

*Proof.* **Part (a).** We recall that the connected sum of $k$ copies of $\mathbb{RP}^2$ is homeomorphic to the quotient obtained from a $2k$-gon with edges labeled $a_1a_1a_2a_2 \dots a_ka_k$. As discussed in class, we interpret this as a CW decomposition of $X_k$ with 0-skeleton $X_k^{(0)}$ given by the vertex $v$. The 1-skeleton $X_k^{(1)}$ consists of the quotient $\partial P/\sim$ of the boundary $\partial P$ of the polygon $P$ obtained by identifying those edges with the same label; this gives a bouquet $\bigvee_{i=1}^k S_i^1$ of $k$ circles labeled $a_i$ that have the point $v$ in common. The open 2-cell $e$ is the interior of the polygon; its attaching map is given by the projection map

$$S^1 = \partial P \to \partial P/\sim \;=\; X_k^{(1)}.$$

We note that the composition of this map with the projection onto $S_i^1$ has degree 2 (since this loop runs twice through each of the circles $S^1$). Hence according to our theorem describing the boundary map $\partial$ of the cellular chain complex, we have

$$\partial e = 2a_1 + 2a_2 + \cdots + 2a_k.$$

Summarizing, the cellular chain complex $C_*^{CW}(X_k)$ looks as follows.

$$\begin{array}{cccccc}
\text{degree} & 0 & & 1 & & 2 \\[2em]
C_*^{CW}(X_k) & \mathbb{Z}v & \xleftarrow{\;\partial\;} & \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k & \xleftarrow{\;\partial\;} & \mathbb{Z}e \\[2em]
 & 0 \longleftarrow\!\shortmid\, a_i & & 2(a_1 + \cdots + a_k) & \longleftarrow\!\shortmid & e
\end{array}$$

Tensoring with $\mathbb{Z}/2$ we obtain the chain complex

$$C_*^{CW}(X_k) \otimes \mathbb{Z}/2 \quad \cong \quad \mathbb{Z}/2 \longleftarrow \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \longleftarrow \mathbb{Z}/2$$

with trivial differential. It follows that

$$H_q(X_k) = \begin{cases} \mathbb{Z}/2 & q = 0, 2 \\ (\mathbb{Z}/2)^k & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

**Part (b).** According to the Universal Coefficient Theorem,

$$H_*(X_k;\mathbb{Z}/2) \;\cong\; H_*(X_k)\otimes\mathbb{Z}/2 \;\oplus\; \mathrm{Tor}_1(H_*(X_k),\mathbb{Z}/2).$$

The following table gives all relevant graded $\mathbb{Z}$-modules by showing all their non-trivial homogeneous pieces sorted by degree.

| $q$ | $H_q(X_k)$ | $H_q(X_k)\otimes\mathbb{Z}/2$ | $\mathrm{Tor}_1(H_{q-1}(X_k),\mathbb{Z}/2)$ | $H_q(X_k;\mathbb{Z}/2)$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ |
| 1 | $\mathbb{Z}^{k-1}\oplus\mathbb{Z}/2$ | $(\mathbb{Z}/2)^k$ | 0 | $(\mathbb{Z}/2)^k$ |
| 2 | 0 | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |

□

3. Let $0\to A\to B\to C\to 0$ be a short exact sequence of $\mathbb{Z}$-modules. Show that for any topological space $X$ there is a corresponding long exact sequence of homology groups

$$\longrightarrow H_q(X;A)\longrightarrow H_q(X;B)\longrightarrow H_q(X;C)\xrightarrow{\partial} H_{q-1}(X;A)\longrightarrow H_{q-1}(X;B)\longrightarrow .$$

Hint: Recall that a short exact sequence of chain complexes induces a long exact sequence of homology groups. Use the fact that the tensor product of a short exact sequence of $\mathbb{Z}$-modules with a *free* $\mathbb{Z}$-module is again exact.

*Proof.* Tensoring the short exact sequence $0\to A\to B\to C\to 0$ with the chain complex $C_*(X)$ leads to the sequence

$$0\longrightarrow C_*(X)\otimes A\longrightarrow C_*(X)\otimes B\longrightarrow C_*(X)\otimes C\longrightarrow 0 \tag{7}$$

of chain complexes. Looking at the submodule of homogenous elements of degree $q$ of these chain complexes we have the sequence

$$0\longrightarrow C_q(X)\otimes A\longrightarrow C_q(X)\otimes B\longrightarrow C_q(X)\otimes C\longrightarrow 0 \tag{8}$$

Since tensoring with any $\mathbb{Z}$-module is right-exact, this sequence is exact except possibly at the term $C_q(X)\otimes A$. However, since the $\mathbb{Z}$-module $C_q(X)$ is *free*, with basis $S\subset C_q(X)$ (given by the set $S=S_q(X)$ of singular $q$-simplices in $X$), we can write $C_q(X)=\mathbb{Z}[S]=\bigoplus_{s\in S}\mathbb{Z}_s$ (where the subscript $s$ just distinguishes the many copies of $\mathbb{Z}$). It follows that

$$C_q(X)\otimes A\cong\left(\bigoplus_{s\in S}\mathbb{Z}_s\right)\otimes A=\bigoplus_{s\in S}\mathbb{Z}_s\otimes A\cong\bigoplus_{s\in S}A_s.$$

Hence the map $\mathrm{id}_{C_q(X)}\otimes f\colon C_q(X)\otimes A\longrightarrow C_q(X)\otimes A$ can be identified with the map

$$\bigoplus_{s\in S}f_s\colon\;\bigoplus_{s\in S}A_s\longrightarrow\bigoplus_{s\in S}B_s.$$

The injectivity of $f\colon A\to B$ implies that the direct sum of many copies of $f$ (parametrized by $s\in S$) is also injective, and hence is the map $\mathrm{id}_{C_q(X)}\otimes f$. In other words, (7) is a short exact sequence of chain complexes which results in the desired long exact sequence of homology groups. □

4. (a) Let $M$, $N$ be right $R$-modules, and let $\epsilon^M\colon M_* \to M$, $\epsilon^N\colon N_* \to N$ be free resolutions. Show that if $f\colon M \to N$ is an $R$-linear map, then one can construct a morphism $f_*\colon M_* \to N_*$ of dg $R$-modules such that the diagram

(9)
$$\begin{tikzcd} M \arrow[d, "f"'] & M_* \arrow[l, "\epsilon^M"'] \arrow[d, "f_*"] \\ N & N_* \arrow[l, "\epsilon^N"'] \end{tikzcd}$$

is commutative. Hint: Unpacking the definitions, the morphism $f_*$ amounts to a collection of $R$-linear maps $f_q\colon M_q \to N_q$ such that the diagram

$$\begin{tikzcd} M \arrow[d, "f"] & M_0 \arrow[l, "\epsilon^M"'] \arrow[d, "f_0"] & M_1 \arrow[l] \arrow[d, "f_1"] & M_2 \arrow[l] \arrow[d, "f_2"] & \dots \arrow[l] \\ N & N_0 \arrow[l, "\epsilon^N"'] & N_1 \arrow[l] & N_2 \arrow[l] & \dots \arrow[l] \end{tikzcd}$$

is commutative. Note that the top and bottom row are exact sequences of $R$-modules by definition of "free resolution". Construct the maps $f_q$ inductively using the following property of a free module: if $g\colon A \to B$ is an $R$-module map whose domain $A$ is a *free* module, then $g$ factors through any $R$-linear surjection $h\colon C \to B$; i.e., there is an $R$-linear map $\widehat{g}\colon A \to C$ making the following diagram commutative:

$$\begin{tikzcd} & C \arrow[d, "h"] \\ A \arrow[ur, "\widehat{g}"] \arrow[r, "g"] & B \end{tikzcd}$$

(b) Show that the $R$-linear chain map $f_*\colon M_* \to N_*$ constructed in part (a) is unique up to $R$-linear chain homotopies, i.e., if $f'_*\colon M_* \to N_*$ is another solution to (a), show that there is a chain homotopy $T$ between them.

(c) Show that if $M$ is a right $R$-module, and $P$ is a left $R$-module, then the $\mathbb{Z}$-module $\operatorname{Tor}^q_R(M, P)$ is independent of the choice of a free resolution of $M$ in the sense that if $M_* \to M$ and $M'_* \to M$ are free resolutions of $M$, then there is an isomorphism between $H_q(M_* \otimes_R P)$ and $H_q(M'_* \otimes_R P)$ which is natural in $M$ and $P$.

*Proof.* **Part (a).** Since $M_0$ is a free $R$-module, the module map $f \circ \epsilon^M\colon M_0 \to N$ factors through the surjective map $\epsilon^N\colon N_0 \to N$; i.e., there is an $R$-linear map $f_0\colon M_0\colon N_0$ making the first square commutative. We will construct the $f_q$'s by induction. Let us assume that we already constructed $R$-linear maps $f_0, \dots, f_q$ making all diagrams to the left of $f_q$ commutative. We note that this implies in particular that $f_q$ maps $\ker(M_q \to M_{q-1})$ to

$\ker(N_q \to N_{q-1})$. Now we want to construct $f_{q+1}$ such that the following diagram commutes:

$$\begin{tikzcd}
\ker(M_q \to M_{q-1}) \arrow[d, "f_q"] & M_{q+1} \arrow[l, "d^M_{q+1}"'] \arrow[d, "f_{q+1}"] \\
\ker(N_q \to N_{q-1}) & N_{q+1} \arrow[l, "d^N_{q+1}"']
\end{tikzcd}$$

This map exists since $M_{q+1}$ is free and the map $d^N_{q+1}$ is surjective by exactness of the resolution $N_*$ at $N_q$.

**Part (b).** Assume that $f'_*\colon M_* \to N_*$ is another chain map lifting the map $f$. Our goal is to construct a chain homotopy $T$ between them; i.e., we want $R$-linear maps $T_q\colon M_q \to N_{q+1}$ with

$$d^N_{q+1}T_q + T_{q-1}d^M_q = f_q - f'_q \tag{10}$$

where the modules $M_q$, $N_q$ are interpreted as the trivial modules for $q < 0$. We will construct the $T_q$'s inductively. To construct $T_0$, we note that

$$\epsilon^N \circ f_0 = f \circ \epsilon^M = \epsilon^N \circ f'_0$$

implies that the range of $f_0 - f'_0$ is contained in $\ker \epsilon^N$, and hence there is a map $T_0$ making the diagram

$$\begin{tikzcd}
M_0 \arrow[d, "f_0 - f'_0"'] \arrow[dr, "T_0"] & \\
\ker \epsilon^N & N_1 \arrow[l, "d^N_1"]
\end{tikzcd}$$

commutative since $M_0$ is free and the horizontal map is surjective.

Now let us assume that we have constructed $T_0, \dots, T_{k-1}$ satisfying equation (10) for $q < k$. To construct $T_k$, we consider this equation for $q = k$ and put the term $T_{k-1}d^M_k$ on the right side of the above equation and try to solve for $T_k$. We note that the image of

$$g := f_k - f'_k - T_{k-1}d^M_k\colon M_k \to N_k$$

is contained in the kernel of $d^N_k$ since

$$\begin{aligned}
d^N_k(f_k - f'_k - T_{k-1}d^M_k) &= f_{k-1}d^M_k - f'_{k-1}d^M_k - d^N_k T_{k-1}d^M_k \\
= &f_{k-1}d^M_k - f'_{k-1}d^M_k - (T_{k-2}d^M_{k-1}d^M_k - f_{k-1}d^M_k - f'_{k-1}d^M_k) = 0
\end{aligned}$$

Here the first equation holds since $f_*$, $f'_*$ are chain maps, and the second equation follows from the inductive assumption.

Now we can construct $T_k$ making the diagram

$$\begin{tikzcd}
M_k \arrow[d, "g"'] \arrow[dr, "T_k"] & \\
\ker d^N_k & N_{k+1} \arrow[l, "d^N_{k+1}"]
\end{tikzcd}$$

commutative, since $M_k$ is free and $d^N_{k+1}$ is surjective onto the kernel of $d^N_k$ by the exactness of $N_*$. □

## 8. Homework Assignment # 8

1. Let $X_k$ be the the connected sum of $k$ copies of the real projective plane $\mathbb{RP}^2$. Calculate the homology groups of $X_k \times X_\ell$ in two ways:
(a) As the homology groups of the cellular chain complex

$$C^{CW}_*(X_k \times X_\ell) \cong C^{CW}_*(X_k) \otimes C^{CW}_*(X_\ell).$$

Hint: The chain complex $C^{CW}_*(X_k)$ was calculated in problem 2(a) of assignment # 7. Decompose $C^{CW}_*(X_k)$ as the direct sum of $k+1$ chain complexes of the type $\Sigma^m\mathbb{Z}$ and $\Sigma^m M\mathbb{Z}/n$ defined in class to make the tensor product $C^{CW}_*(X_k) \otimes C^{CW}_*(X_\ell)$ manageable.
(b) Via the Künneth Theorem.

*Proof.* **Part (a).** By inspection of the chain complex $C^{CW}_*(X_k)$ written down explicitly in problem 2(a) in homework assignment # 7, we see that it decomposes as a direct sum of sub chain complexes

$$C^{CW}_*(X_k) \cong \mathbb{Z} \oplus \underbrace{\Sigma\mathbb{Z} \oplus \cdots \oplus \Sigma\mathbb{Z}}_{k-1} \oplus \Sigma M\mathbb{Z}/2.$$

Here $\mathbb{Z}$-modules are interpreted as a chain complex concentrated in degree 0 (like the summand $\mathbb{Z}$ in the decomposition above, generated by the 0-cell $v$). For any chain complex $C_*$ we denote by $\Sigma C_*$ the *suspension of* $C_*$, the dg $\mathbb{Z}$-module obtained from $C_*$ by shifting all degrees by declaring $(\Sigma C_*)_q := C_{q-1}$; for example, $\Sigma\mathbb{Z}$ is a chain complex concentrated in degree $+1$. The $k-1$ summands $\Sigma\mathbb{Z}$ in the decomposition above are generated by $a_1, \ldots, a_{k-1}$. Finally, $M\mathbb{Z}/\ell$ denotes the chain complex consisting of one copy of $\mathbb{Z}$ in degree 0 and 1 with a differential that is multiplication by $\ell$; $M\mathbb{Z}/\ell$ is trivial in all other degrees. The summand $\Sigma M\mathbb{Z}/2$ is generated by $e$ in degree 2 and $a_1 + \cdots + a_k$ in degree 1.

Then we have the following isomorphisms of dg $\mathbb{Z}$-modules:

$$\begin{aligned}
&C^{CW}_*(X_k) \otimes C^{CW}_*(X_\ell)\\
\cong\ &\left(\mathbb{Z} \oplus \underbrace{\Sigma\mathbb{Z} \oplus \cdots \oplus \Sigma\mathbb{Z}}_{k-1} \oplus \Sigma M\mathbb{Z}/2\right) \otimes \left(\mathbb{Z} \oplus \underbrace{\Sigma\mathbb{Z} \oplus \cdots \oplus \Sigma\mathbb{Z}}_{\ell-1} \oplus \Sigma M\mathbb{Z}/2\right)\\
\cong\ &\mathbb{Z} \oplus \bigoplus_{k+\ell-2} \Sigma\mathbb{Z} \ \oplus\ \bigoplus_{2} \Sigma M\mathbb{Z}/2 \ \oplus\ \bigoplus_{(k-1)(\ell-1)} \Sigma^2\mathbb{Z} \ \oplus\ \bigoplus_{k+\ell-2} \Sigma^2 M\mathbb{Z}/2 \ \oplus\ \Sigma M\mathbb{Z}/2 \otimes \Sigma M\mathbb{Z}/2
\end{aligned}$$

The homology groups of the chain complexes $\mathbb{Z}$ and $M/\mathbb{Z}/2$ can be read off directly to obtain

$$H_q(\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \neq 0 \end{cases} \qquad H_q(M\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & q = 0 \\ 0 & q \neq 0 \end{cases}.$$

This implies

$$H_q(\Sigma^k\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = k \\ 0 & q \neq k \end{cases} \qquad H_q(\Sigma^M\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & q = k \\ 0 & q \neq k \end{cases}.$$

To it only remains to understand the homology of the chain complex $\Sigma M\mathbb{Z}/2 \otimes \Sigma M\mathbb{Z}/2 \cong \Sigma^2(M\mathbb{Z}/2 \otimes M\mathbb{Z}/2)$. One way to calculate this is via the Künneth Theorem.

| $q$ | $H_q(M\mathbb{Z}/2)$ | $(H_*(M\mathbb{Z}/2) \otimes H_*(M\mathbb{Z}/2))_q$ | $\mathrm{Tor}_1(H_*(M\mathbb{Z}/2), H_*(M\mathbb{Z}/2))_{q-1}$ | $H_q(M\mathbb{Z}/2 \otimes M\mathbb{Z}/2)$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ |
| 1 | 0 | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |

This implies that $H_q(\Sigma^2(M\mathbb{Z}/2 \otimes M\mathbb{Z}/2)) = \begin{cases} \mathbb{Z}/2 & q = 2,3 \\ 0 & q \neq 2,3 \end{cases}$.

Forming the direct sum of the homology of the all the summands in the direct sum decomposition of $C_*^{CW}(X_k) \otimes C_*^{CW}(X_\ell)$, we obtain

$$H_q(X_k \times X_\ell) \cong \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^{k+\ell-2} \oplus (\mathbb{Z}/2)^2 & q = 1 \\ \mathbb{Z}^{(k-1)(\ell-1)} \oplus (\mathbb{Z}/2)^{k+\ell-1} & q = 2 \\ \mathbb{Z}/2 & q = 3 \end{cases}$$

**Part (b).** According to the Künneth Theorem,

$$H_q(X_k \times X_\ell) \cong (H_*(X_k) \otimes H_*(X_\ell))_q \oplus \mathrm{Tor}_1(H_*(X_k), H_*(X_\ell))_{q-1},$$

where we use the shorthand notation

$$(H_*(X_k) \otimes H_*(X_\ell))_q := \bigoplus_{m+n=q} H_m(X_k) \otimes H_n(X_\ell)$$

$$\mathrm{Tor}_1(H_*(X_k), H_*(X_\ell))_{q-1} := \bigoplus_{m+n=q-1} \mathrm{Tor}_1(H_m(X_k), H_n(X_\ell)).$$

The following table shows all these graded $\mathbb{Z}$-modules.

| q | $H_q(X_k)$ | $(H_*(X_k) \otimes H_*(X_\ell))_q$ | $\mathrm{Tor}_1(H_*(X_k), H_*(X_\ell))_{q-1}$ | $H_q(X_k \times X_\ell)$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2$ | $\mathbb{Z}^{k+\ell-2} \oplus (\mathbb{Z}/2)^2$ | 0 | $\mathbb{Z}^{k+\ell-2} \oplus (\mathbb{Z}/2)^2$ |
| 2 | 0 | $\mathbb{Z}^{(k-1)(\ell-1)} \oplus (\mathbb{Z}/2)^{k+\ell-2}$ | 0 | $\mathbb{Z}^{(k-1)(\ell-1)} \oplus (\mathbb{Z}/2)^{k+\ell-1}$ |
| 3 | 0 | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |

The third and fourth columns are obtained from the second by using linearity in each slot of $- \otimes -$ resp. $\mathrm{Tor}_1(-,-)$ to write $H_*(X_k) \otimes H_*(X_\ell)$ and $\mathrm{Tor}_1(H_*(X_k), H_*(X_\ell))$ as a sum of terms of the form $M \otimes N$ (resp. $\mathrm{Tor}_1(M,N)$ where $M$, $N$ are $\mathbb{Z}$ or $\mathbb{Z}/2$. □

2. For a topological space $X$ with finitely generated homology groups its *Poincaré series* (resp. its Poincaré series with coefficients in a field $\mathbb{K}$) are defined to be the power series

$$P(X) := \sum_{q=0}^{\infty} \mathrm{rk}\, H_q(X) t^q \in \mathbb{Z}[[t]] \qquad \text{resp.} \qquad P(X;\mathbb{K}) := \sum_{q=0}^{\infty} \dim_{\mathbb{K}} H_q(X;\mathbb{K}) t^q \in \mathbb{Z}[[t]],$$

where $\dim_{\mathbb{K}} H_q(X;\mathbb{K})$ is the dimension of the vector space $H_q(X;\mathbb{K})$ over $\mathbb{K}$.
**Remark.** The Poincaré series $P(X;\mathbb{K})$ determines the homology groups $H_*(X;\mathbb{K})$ up to isomorphism, since two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.
(a) Show that

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p^k,\mathbb{K}) = \begin{cases} \mathbb{K} & \text{If } \mathrm{char}(\mathbb{K}) = p \text{ and } i = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Show that $P(X;\mathbb{K}) = P(X)$ if $\mathrm{char}(\mathbb{K}) = 0$.
(c) Show that if $X$ and $Y$ are spaces with finitely generated homology groups, then so is $X \times Y$ with Poincaré series given by

$$P(X \times Y;\mathbb{K}) = P(X;\mathbb{K}) \cdot P(Y:\mathbb{K})$$

(d) Calculate $H_*(\mathbb{CP};\mathbb{K})$ and write the Poincaré series $P(\mathbb{CP}^n;\mathbb{K})$ and $P(\mathbb{CP}^m \times \mathbb{CP}^n;\mathbb{K})$ as rational functions of $t$ (i.e., as quotients of polynomials).

*Proof.* **Part (a).** A free resolution of the $\mathbb{Z}$-module $\mathbb{Z}/p^k$ is given by the chain complex $M_* = \left( M_0 = \mathbb{Z} \xleftarrow{p^k} M_1 = \mathbb{Z} \right)$. Hence

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p^k,\mathbb{K}) = H_i\left( \mathbb{Z}\otimes\mathbb{K} \xleftarrow{p^k\otimes \mathrm{id}_{\mathbb{K}}} \mathbb{Z}\otimes\mathbb{K} \right).$$

Identifying $\mathbb{Z}\otimes\mathbb{K}$ with $\mathbb{K}$ via the map $n\otimes k \mapsto nk \in \mathbb{K}$, the map $p^k\otimes \mathrm{id}_{\mathbb{K}}$ corresponds to the map $\mathbb{K}\to\mathbb{K}$ given by multiplication by $p^k$, and hence

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p^k,\mathbb{K}) = \mathrm{coker}\left( \mathbb{K} \xleftarrow{p^k} \mathbb{K} \right) \qquad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p^k,\mathbb{K}) = \ker\left( \mathbb{K} \xleftarrow{p^k} \mathbb{K} \right).$$

If $\mathrm{char}(\mathbb{K}) = p$, then multiplication by $p^k$ is trivial. Hence kernel and cokernel are $\mathbb{K}$, which implies $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p^k,\mathbb{K}) = \mathbb{K}$ for $i = 0,1$. If $\mathrm{char}(\mathbb{K}) \neq p$, then multiplication by $p$, and hence multiplication by $p^k$ is an isomorphism. Hence kernel and cokernel are trivial and so $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p^k,\mathbb{K}) = 0$ for $i = 0,1$.

**Part (b).** According to the Universal Coefficient Theorem, we have

$$H_q(X;\mathbb{K}) \cong H_q(X)\otimes\mathbb{K} \,\oplus\, \mathrm{Tor}_1(H_{q-1}(X),\mathbb{K}).$$

Decomposing the finitely generated $\mathbb{Z}$-module $H_{q-1}(X)$ as a sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/p^k$ (for a prime $p$), we conclude that $\mathrm{Tor}_1(H_{q-1}(X),\mathbb{K}) = 0$, since $\mathrm{Tor}_1(\mathbb{Z},M) = 0$ for any $\mathbb{Z}$-module $M$, and $\mathrm{Tor}_1(\mathbb{Z}/p^k,\mathbb{K}) = 0$ for a field of characteristic 0 by part (a). Similarly, if we decompose $H_q$ in this way, we see by part (a) again that any summands of $H_q(X)$ do not contribute to $H_q(X;\mathbb{K})$. Hence if the rank of $H_q(X)$ is $r$, then

$$H_q(X)\otimes\mathbb{K} = (\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{r})\otimes\mathbb{K} \cong \underbrace{\mathbb{K}\oplus\cdots\oplus\mathbb{K}}_{r}.$$

If follows that $\dim_{\mathbb{K}} H_q(X;\mathbb{K}) = \mathrm{rk}\, H_q(X)$ and hence $P(X;\mathbb{K}) = P(X)$.

**Part (c).** By the Künneth Theorem,

$$H_q(X \times Y) \cong \bigoplus_{k+\ell=q} H_k(X) \otimes H_\ell(Y) \;\oplus \bigoplus_{k+\ell=q-1} \mathrm{Tor}_1(H_k(X), H_\ell(Y)).$$

In particular, $H_q(X \times Y)$ is a *finite* direct sum of terms of tensor and tor-products of finitely generated $\mathbb{Z}$-modules and hence it is finitely generated.

By the Künneth Theorem for homology with coefficients in the field $\mathbb{K}$ we have

$$H_q(X \times Y; \mathbb{K}) \cong \bigoplus_{m+n=q} H_m(X; \mathbb{K}) \otimes H_n(Y; \mathbb{K})$$

Writing $P_k(X; \mathbb{K}) = \dim_{\mathbb{K}} H_k(X; \mathbb{K})$ for the coefficient of $t^k$ in the Poincaré series $P(X; \mathbb{K})$ we obtain

$$\begin{aligned} P_q(X \times Y; \mathbb{K}) &= \dim_{\mathbb{K}} H_q(X \times Y; \mathbb{K}) \\ &= \dim_{\mathbb{K}}\Big( \bigoplus_{m+n=q} H_k(X; \mathbb{K}) \otimes H_\ell(Y; \mathbb{K})\Big) \\ &= \sum_{m+n=q} \dim_{\mathbb{K}}(H_m(X; \mathbb{K}) \otimes H_n(Y; \mathbb{K})) \\ &= \sum_{m+n=q} \dim_{\mathbb{K}} H_m(X; \mathbb{K}) \cdot \dim_{\mathbb{K}} H_n(Y; \mathbb{K}) \\ &= \sum_{m+n=q} P_m(X; \mathbb{K}) \cdot P_n(Y; \mathbb{K}). \end{aligned}$$

We recognize the last sum as the expression giving the coefficient of $t^q$ of the power series $P(X; \mathbb{K}) \cdot P(Y; \mathbb{K})$, which shows that $P(X \times Y; \mathbb{K}) = P(X; \mathbb{K}) \cdot P(Y; \mathbb{K})$.

**Part (d).** We recall that

$$H_q(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & q = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

and hence $P(\mathbb{CP}^n) = 1 + t^2 + t^4 + \cdots + t^{2n} = \frac{1-t^{2n+2}}{1-t^2}$. By part (a) it follows that

$$P(\mathbb{CP}^m \times \mathbb{CP}^n) = P(\mathbb{CP}^m) \cdot P(\mathbb{CP}^n) = \frac{(1 - t^{2m+2})(1 - t^{2n+2})}{(1 - t^2)^2}.$$

□

3. The *lens space* $L^{2n-1}(\mathbb{Z}/k)$ is the quotient of the sphere $S^{2n-1} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ given by identifying any point $(z_1, \ldots, z_n) \in S^{2n-1}$ with $(\zeta z_1, \ldots, \zeta z_n)$ for any $k^{\text{th}}$ root of unity $\zeta \in S^1$. For $k = 2$, this is just the real projective space $\mathbb{RP}^{2n-1}$. Like this projective space, the lens space $L^{2n-1}(\mathbb{Z}/k)$ for any $k$ has a CW structure with exactly one cell $e^q$ for $0 \le q \le 2n - 1$. Hence its cellular chain complex is given by $C_q^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k)) = \mathbb{Z}e^q$ for $0 \le i \le 2n - 1$ (it is zero for all other $q$). It can be shown that the boundary map $d$ is given by $d(e^q) = ke^{q-1}$ for $0 < q \le 2n$, $q$ even, and $d(e^q) = 0$ otherwise.
(a) Calculate $H_*(L^{2n-1}(\mathbb{Z}/k))$.

(b) Calculate $H_*(L^{2n-1}(\mathbb{Z}/k);\mathbb{K})$ for a field $\mathbb{K}$ and write down the Poincaré series $P(L^{2n-1}(\mathbb{Z}/k);\mathbb{K})$.

*Proof.* **Part (a).** Writing out the cellular chain complex $C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k))$ explicitly, we get

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \dots & 2n-3 & 2n-2 & 2n-1 \\ \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \xleftarrow{k} & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \xleftarrow{k} & \mathbb{Z} & \dots \xleftarrow{0} & \mathbb{Z} \xleftarrow{k} & \mathbb{Z} \xleftarrow{0} & \mathbb{Z} \end{array}$$

This shows that the chain complex $C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k)$ has a direct sum decomposition of the form:

$$C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k)) = \mathbb{Z} \oplus \Sigma M\mathbb{Z}/k \oplus \Sigma^3 M\mathbb{Z}/k \oplus \cdots \oplus \Sigma^{2n-3} M\mathbb{Z}/k \oplus \Sigma^{2n-1}\mathbb{Z}.$$

Since we know that the homology groups of $\Sigma^m\mathbb{Z}$ and $\Sigma^m M\mathbb{Z}/k$ are trivial for degree $\neq m$ and isomorphic to $\mathbb{Z}$ resp. $\mathbb{Z}/k$ in degree $m$, we can read of the homology groups of $L^{2n-1}(\mathbb{Z}/k)$ from the decomposition above and obtain

$$H_q(L^{2n-1}(\mathbb{Z}/k)) = \begin{cases} \mathbb{Z} & q = 0, 2n-1 \\ \mathbb{Z}/k & 0 < q < 2n-1,\ q \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

**Part (b).** There are two ways to calculate $H_*(L^{2n-1}(\mathbb{Z}/k);\mathbb{K})$, by the Universal Coefficient Theorem, or as the homology of the cellular chain complex $C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) := C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k)) \otimes \mathbb{K}$ with coefficients in $\mathbb{K}$. Doing the latter, tensoring the chain complex $C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k))$ described explicitly in part (a) with $\mathbb{K}$, every summand $\mathbb{Z}$ gets replaced by $\mathbb{Z} \otimes \mathbb{K} = \mathbb{K}$ and hence we obtain the chain complex

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \dots & 2n-3 & 2n-2 & 2n-1 \\ \mathbb{K} \xleftarrow{0} & \mathbb{K} \xleftarrow{k} & \mathbb{K} \xleftarrow{0} & \mathbb{K} \xleftarrow{k} & \mathbb{K} & \dots \xleftarrow{0} & \mathbb{K} \xleftarrow{k} & \mathbb{K} \xleftarrow{0} & \mathbb{K} \end{array}$$

If $\mathrm{char}(\mathbb{K})$ divides $k$, then multiplication by $k$ is trivial in $\mathbb{K}$, hence all boundary maps in $C_*^{\mathrm{CW}}(L^{2n-1}(\mathbb{Z}/k);\mathbb{K})$ are trivial, and consequently

$$H_q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) = \begin{cases} \mathbb{K} & 0 \le q \le 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

If $\mathrm{char}(\mathbb{K})$ does not divide $k$, then multiplication by $k$ is an isomorphism $\mathbb{K} \to \mathbb{K}$. Hence only the copies of $\mathbb{K}$ in degree 0 and $2n-1$ contribute to the homology and hence

$$H_q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) = \begin{cases} \mathbb{K} & q = 0, 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

From this we can read off the Poincaré series:

$$P(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) = \begin{cases} 1 + t + t^2 + \cdots + t^{2n-2} + t^{2n-1} & \text{if } \mathrm{char}(\mathbb{K}) \text{ divides } k \\ 1 + t^{2n-1} & \text{otherwise} \end{cases}$$

□

4. We recall that a space $X$ is of *bounded finite type* if all its homology groups are finitely generated and $H_q(X) = 0$ for $q$ sufficiently large. For such a space $X$ its *Euler characteristic* is defined as the alternating sum

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rk} H_q(X)$$

of the ranks of the homology groups of $X$ (note that this is a *finite* sum). More generally, for a field $\mathbb{K}$ we define

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q \dim_{\mathbb{K}} H_q(X; \mathbb{K}).$$

(a) Show that $\chi(X; \mathbb{K}) = \chi(X)$. Hint: for a fixed prime $p$, decompose $H_q(X)$ as a direct sum of $\mathbb{Z}$-modules $A_q$, $B_q$, $C_q$, where $A_q$ is free, and $B_q$, $C_q$ are finite $\mathbb{Z}$-modules, with the order of $B_q$ a power of $p$, and the order of $C_q$ prime to $p$. Then use the universal coefficent theorem.
(b) How can $\chi(X; \mathbb{K})$ be expressed in terms of the Poincaré series $P(X; \mathbb{K})$?
(c) Use part (a) to show that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ for bounded spaces $X$, $Y$ of finite type.

*Proof.* If $\mathbb{K}$ has characteristic 0, then by the proof of problem 2(b), $\operatorname{rk} H_q(X) = \dim_k H_q(X; \mathbb{K})$, and consequently, $\chi(X) = \chi(X; \mathbb{K})$. From now on we assume that $\operatorname{char}(\mathbb{K}) = p$. By the Universal Coefficient Theorem,

$$H_q(X; \mathbb{K}) \;\cong\; H_q(X) \otimes \mathbb{K} \;\oplus\; \operatorname{Tor}_1(H_{q-1}(X), \mathbb{K}). \tag{11}$$

The finitely generated $\mathbb{Z}$-module $H_q(X)$ can be decomposed as a direct sum

$$H_q(X) \cong A_q \oplus B_q \oplus C_q,$$

where $A_q$ is a sum of $\mathbb{Z}$'s, $B_q$ is a sum of finite cyclic groups whose order is a power of $p$, and $C_q$ is a sum of finite cyclic groups of order prime to $p$. Let us write $a_q$ (resp. $b_q$) for the number of summands in $A_q$ (resp. $B_q$). We note that $a_q$ is the rank of $H_q(X)$.

The results of problem 1(a) then imply

$$\begin{aligned} \dim H_q(X) \otimes \mathbb{K} &= \dim A_q \otimes \mathbb{K} + \dim B_q \otimes \mathbb{K} + \dim C_q \otimes \mathbb{K} = a_q + b_q \\ \dim \operatorname{Tor}_1(H_{q-1}(X), \mathbb{K}) &= \dim \operatorname{Tor}(A_{q-1}, \mathbb{K}) + \dim \operatorname{Tor}(B_{q-1}, \mathbb{K}) + \dim \operatorname{Tor}(C_{q-1}, \mathbb{K}) = b_{q-1} \end{aligned}$$

Hence the Universal Coefficient Theorem (11) implies Hence by the Universal Coefficient

$$\dim H_q(X; \mathbb{K}) = \dim \left(H_q(X) \otimes \mathbb{K}\right) \;+\; \dim \operatorname{Tor}_1(H_{q-1}(X), \mathbb{K}) = (a_q + b_q) + b_{q-1}.$$

It follows that

$$\begin{aligned}\chi(X;\mathbb{K}) &= \sum_{q=0}^{\infty}(-1)^q(a_q+b_q+b_{q-1})\\ &= \sum_{q=0}^{\infty}(-1)^q a_q \;+\; \sum_{q=0}^{\infty}(-1)^q b_q \;+\; \sum_{q=0}^{\infty}(-1)^q b_{q-1}\\ &= \sum_{q=0}^{\infty}(-1)^q a_q = \sum_{q=0}^{\infty}(-1)^q \operatorname{rk} H_q(X) = \chi(X).\end{aligned}$$

**Part (b).** We note that the assumption that the homology group $H_q(X)$ is zero for sufficiently large $q$ means that the Poincaré series $P(X)$ is a *polynomial*. In particular, we can evaluate $P(X)$ for any complex number $t$. For $t=-1$ we obtain

$$P(X)(-1) = \sum_{q=0}^{\infty} \operatorname{rk} H_q(X)(-1)^q = \chi(X).$$

**Part (c).** Using problem 2(c) and part (b) of this problem, we obtain

$$\begin{aligned}\chi(X\times Y) &= P(X\times Y;\mathbb{K})(-1) = (P(X;\mathbb{K})\cdot P(Y;\mathbb{K}))(-1)\\ &= (P(X;\mathbb{K})(-1))\cdot(P(Y;\mathbb{K})(-1)) = \chi(X)\cdot\chi(Y).\end{aligned}$$

$\square$

## 9. Homework Assignment # 9

1. (a) Calculate $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k,\mathbb{Z}/\ell)$.
   (b) Calculate $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k,\mathbb{Z})$.

*Proof.* We recall that $\operatorname{Ext}^q_{\mathbb{Z}}(M,N)$ for $\mathbb{Z}$-modules $M$, $N$ is defined as

$$\operatorname{Ext}^q_{\mathbb{Z}}(M,N) = H^q(\operatorname{Hom}_{\mathbb{Z}}(M_*,N)),$$

where $M_*$ is a free resolution of $M$.
**Part (a).** For $M=\mathbb{Z}/k$, a free resolution is given by

$$\begin{array}{ccc} \text{degree} & 0 & 1 \\ & \mathbb{Z} \xleftarrow{\;\partial\;} & \mathbb{Z} \end{array},$$

where the map $\partial\colon \mathbb{Z}\to\mathbb{Z}$ is multiplication by $k$. Then $\operatorname{Hom}_{\mathbb{Z}}(M_*,\mathbb{Z}/\ell)$ is given by

$$\begin{array}{ccc} \text{degree} & 0 & 1 \\ & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}/\ell) \xrightarrow{\;\delta\;} & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}/\ell \end{array},$$

where $\delta$ maps $f \in \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/\ell)$ to $f \circ \partial \in \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/\ell)$. As discussed in class, the map $\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/\ell) \to \mathbb{Z}/\ell$, $f \mapsto f(1)$ is an isomorphism. We claim that the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/\ell) & \xrightarrow{\delta} & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/\ell) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z}/\ell & \xrightarrow{\quad k \quad} & \mathbb{Z}/\ell \end{array}$$

where $k$ denotes multiplication by $k$. It follows that $\operatorname{Ext}^1_Z(\mathbb{Z}/k, \mathbb{Z}/\ell)$ (defined as the cokernel of $\delta$, i.e., the codomain of $\delta$ modulo the image of $\delta$) is isomorphic to the cokernel of $k\colon \mathbb{Z}/\ell \to \mathbb{Z}/\ell$, i.e., the quotient of $\mathbb{Z}/\ell$ modulo the ideal generated by $k$, which is isomorphic to $\mathbb{Z}$ modulo the ideal generated by $\ell$ and $k$. The latter is the ideal in $\mathbb{Z}$ generated by $\gcd(k, \ell)$. It follows that

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\gcd(k, \ell).$$

**Part (b).** Following the same line of argument as in part (a), we have

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k, \mathbb{Z}) = H^1\Big( \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\delta} \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \Big) = \text{cokernel of} \Big( \mathbb{Z} \xrightarrow{k} \mathbb{Z} \Big) \cong \mathbb{Z}/k$$

□

2. Calculate $H^*(\mathbb{CP}^n; \mathbb{Z})$ (we recall that we calculated the cellular chain complex $C_*^{\mathrm{CW}}(\mathbb{CP}^n)$ and the homology groups $H_*(\mathbb{CP}^n)$ of the complex projective space $\mathbb{CP}^n$).

*Proof.* We recall that

$$H_q(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{for } q \text{ even, } 0 \le q \le 2n \\ 0 & \text{otherwise} \end{cases}$$

In particular, all homology groups of $\mathbb{CP}^n$ are *free*. Hence the terms $\operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(\mathbb{CP}^n), \mathbb{Z})$ and the cohomology UCT simplifies to the statement that the evaluation map

$$H^q(\mathbb{CP}^n; \mathbb{Z}) \xrightarrow{\text{ev}} \operatorname{Hom}(H_q(\mathbb{CP}^n), \mathbb{Z})$$

is an isomorphism. It follows that

$$H^q(\mathbb{CP}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } q \text{ even, } 0 \le q \le 2n \\ 0 & \text{otherwise} \end{cases}$$

□

3. Let $\Sigma_g$ be the surface of genus $g$ and let $X_k$ be the connected sum of $k$ copies of the real projective plane $\mathbb{RP}^2$. We recall that we calculated the homology groups $H_q(\Sigma_g)$, $H_q(X_k)$, and $H_q(X_k; \mathbb{Z}/2)$.

(a) Calculate the cohomology groups $H^q(\Sigma_g; \mathbb{Z})$.
(b) Calculate the cohomology groups $H^q(X_k; \mathbb{Z}/2)$.

*Proof.* **Part (a).** We recall that

$$H_q(\Sigma_g;\mathbb{Z}) = \begin{cases} \mathbb{Z} & q=0,2 \\ \mathbb{Z}^{2g} & q=1 \\ 0 & \text{otherwise} \end{cases}$$

These homology groups are all free and hence $\operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(\Sigma),\mathbb{Z}) = 0$ for all $q$. The UCT for cohomology then implies that the evaluation map

$$\mathrm{ev}\colon H^q(\Sigma;\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_q(\Sigma),\mathbb{Z})$$

is an isomorphism, and hence

$$H^q(\Sigma_g;\mathbb{Z}) \cong \begin{cases} \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z} & q=0,2 \\ \operatorname{Hom}(\mathbb{Z}^{2g},\mathbb{Z}) \cong \mathbb{Z}^{2g} & q=1 \\ 0 & \text{otherwise} \end{cases}$$

**Part (b).** There are *two* versions of the UCT for cohomology that we could use to calculate $H^*(X_k;\mathbb{Z}/2)$, one expressing these cohomology groups in terms of $H_*(X_k;\mathbb{Z})$, the other in terms of $H_*(X_k;\mathbb{Z}/2)$. Using the latter is simpler, since the UCT for (co)homology with coefficients in a field $\mathbb{K}$ simply says that $H^q(X_k;\mathbb{K})$ is the vector space dual to $H_q(X_k;\mathbb{K})$.

We recall that

$$H_q(X_k;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & q=0,2 \\ (\mathbb{Z}/2)^k & q=1 \\ 0 & \text{otherwise} \end{cases}.$$

Applying the UCT for (co)homology with coefficients $\mathbb{Z}/2$ we obtain

$$H^q(X_k;\mathbb{Z}/2) \cong \begin{cases} \operatorname{Hom}_{\mathbb{Z}/2}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2 & q=0,2 \\ \operatorname{Hom}_{\mathbb{Z}/2}((\mathbb{Z}/2)^k,\mathbb{Z}/2) \cong (\mathbb{Z}/2)^k & q=1 \\ 0 & \text{otherwise} \end{cases}$$

□

4. Recall that there are *two* geometric versions of the UCT computing the cohomology $H^*(X;\mathbb{K})$ with coefficients in a field $\mathbb{K}$: the first version expresses $H^*(X;\mathbb{K})$ in terms of the homology $H_*(X;\mathbb{Z})$ with integer coefficients, the second version expresses it in terms of the homology $H_*(X;\mathbb{K})$ with coefficients in $\mathbb{K}$. We further recall the computations of the homology of the lens space $L^{2n-1}(\mathbb{Z}/k)$ from problem (3) of assignment # 8:

$$H_q(L^{2n-1}(\mathbb{Z}/k)) = \begin{cases} \mathbb{Z} & q=0,2n-1 \\ \mathbb{Z}/k & q=1,3,\dots 2n-3 \\ 0 & \text{otherwise} \end{cases}$$

$$H_q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) = \begin{cases} \mathbb{K} & q=0,2n-1 \\ \mathbb{K} & 1\le q\le 2n-2 \text{ and } \operatorname{char}(\mathbb{K}) \text{ divides } k \\ 0 & \text{otherwise} \end{cases}$$

(a) Use the first version to calculate $H^*(L^{2n-1}(\mathbb{Z}/k);\mathbb{K})$.
(b) Use the second version to calculate $H^*(L^{2n-1}(\mathbb{Z}/k);\mathbb{K})$.

*Proof.* **Part (a).** By the Universal Coefficient Theorem for cohomology we have

$$H^q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) \cong \mathrm{Hom}(H_q(L^{2n-1}(\mathbb{Z}/k)),\mathbb{Z}) \oplus \mathrm{Ext}^1(H_{q-1}(L^{2n-1}(\mathbb{Z}/k)),\mathbb{K}).$$

We recall from problem 1 on this assignment that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k,\mathbb{K}) = \mathbb{K}$ if $\mathrm{char}(\mathbb{K})$ is a divisor of $k$, and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/k,\mathbb{K}) = 0$ otherwise.

We first look at the case where $\mathrm{char}(\mathbb{K})$ divides $k$ (just writing $L$ instead of $L^{2n-1}(\mathbb{Z}/k)$ for typographical reasons)

| q | $H_q(L)$ | $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{K})$ | $\mathrm{Ext}^1_{\mathbb{Z}}(H_q(L),\mathbb{K})$ | $H^q(L;\mathbb{K})$ |
|---|---|---|---|---|
| $q = 0, 2n-1$ | $\mathbb{Z}$ | $\mathbb{K}$ | 0 | $\mathbb{K}$ |
| $0 < q < 2n-1$, $q$ odd | $\mathbb{Z}/k$ | $\mathbb{K}$ | $\mathbb{K}$ | $\mathbb{K}$ |
| $0 < q < 2n-1$, $q$ even | 0 | 0 | 0 | $\mathbb{K}$ |
| $q \geq 3$ | 0 | 0 | 0 | 0 |

If $\mathrm{char}(\mathbb{K})$ is not a divisor of $k$, then the Ext-term vanishes and the short exact sequence of the UCT reduces to the isomorphism

$$H^q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) \cong \mathrm{Hom}(H_q(L^{2n-1}(\mathbb{Z}/k)),\mathbb{K}).$$

Moreover, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/k,\mathbb{K}) = 0$, and hence

$$H^q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) \cong \begin{cases} \mathbb{K} & q = 0, 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

**Part (b).** The version of the UCT for coefficients in a field $\mathbb{K}$ gives an isomorphism

$$H^q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(H_q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}),\mathbb{K}).$$

Using $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K},\mathbb{K}) \cong \mathbb{K}$, we immediately obtain

$$H^q(L^{2n-1}(\mathbb{Z}/k);\mathbb{K}) \cong \begin{cases} \mathbb{K} & q = 0, 2n-1 \\ \mathbb{K} & 1 \leq q \leq 2n-2 \text{ and } \mathrm{char}(\mathbb{K}) \text{ divides } k \\ 0 & \text{otherwise} \end{cases}$$

∎

## 10. Homework Assignment # 10

1. Show that the map

$$\cup \colon C^*(X) \otimes C^*(X) \longrightarrow C^*(X) \qquad \text{given by} \qquad \phi \otimes \psi \mapsto \phi \cup \psi$$

is cochain map, that is, $\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^{|\phi|}\phi \cup \delta\psi$.

*Proof.* For $\sigma\colon \Delta^{k+l+1} \to X$ we have

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{k+1}])\psi(\sigma \circ [e_k, \dots, e_{k+\ell+1}])$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma \circ [e_0, \dots, e_k])\psi(\sigma \circ [e_k, \dots, \widehat{e_i}, \dots, e_{k+\ell+1}])$$

Adding these two expressions, the last term of the first sum cancels the first term of the second sum, and the remaining terms are exactly

$$\delta(\varphi \cup \psi) = (\varphi \cup \psi)(\partial\sigma)$$

since

$$\partial\sigma = \sum_{i=0}^{k+\ell+1} (-1)^i \sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{k+\ell+1}].$$

□

2. (a) For a continuous map $f\colon X \to Y$, construct an induced homomorphism

$$f^*\colon H^q(Y;M) \to H^q(X;M)$$

for cohomology with coefficients in a $\mathbb{Z}$-module $M$. Hint: first construct a cochain map $f^\#\colon C^*(Y;M) \to C^*(X;M)$ (make sure to check compatibility with the differential $\delta$).

(b) Show that the cup product is compatible with pull-back of cohomology classes in the sense that for a map $f\colon X \to Y$ and cohomology classes $\alpha \in H^k(Y;R)$, $\beta \in H^l(Y;R)$ we have

$$f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta).$$

Hint: Show first the analogous statement for cochains.

*Proof.* **Part (a).** We recall that $f\colon X \to Y$ induces a chain map $f_\#\colon C_q(X) \to C_q(Y)$. We define a map $f^\#\colon C^*(Y;M) \to C^*(X;M)$ by

$$(f^\#\varphi)(c) := \varphi(f_\# c) \qquad \text{for } \varphi \in C^q(Y;M) = \mathrm{Hom}(C_q(Y), M) \text{ and } c \in C_q(X).$$

We need to show that $f^\#$ is a cochain map, i.e., that the diagram

$$\begin{array}{ccc} C^q(Y;M) & \xrightarrow{\ \delta\ } & C^{q+1}(Y;M) \\ \big\downarrow{\scriptstyle f^\#} & & \big\downarrow{\scriptstyle f^\#} \\ C^q(X;M) & \xrightarrow{\ \delta\ } & C^{q+1}(X;M). \end{array}$$

is commutative. So let $\varphi \in C^q(Y;M)$, and let $c \in C_{q+1}(M)$. Then

$$(f^\#\delta\varphi)(c) \overset{(1)}{=} \delta\varphi(f_\# c) \overset{(2)}{=} \varphi(\partial f_\# c) \overset{(3)}{=} \varphi(f_\# \partial c) \overset{(4)}{=} (f^\#\varphi)(\partial c) \overset{(5)}{=} \delta f^\#\varphi(c)$$

Here equations (1)&(2) hold by definition of $f^\#$, equations (2)&(5) by definition of $\delta$, and equation (3) is a consequence of the fact that $f_\#$ is a chain map.

Since $f^\#\colon C^*(Y;M) \to C^*(X;M)$ is a cochain map it induces a well-defined homomorphism $f^*\colon H^q(Y;M) \to H^q(X;M)$ defined by $f^*([\varphi]) = [f^\#\varphi]$ for a cocycle $\varphi \in Z^q(Y;M)$.

**Part (b).** For $\varphi \in C^k(Y;R)$, $\psi \in C^\ell(Y;R)$ and $\sigma\colon \Delta^{k+\ell} \to X$ we have

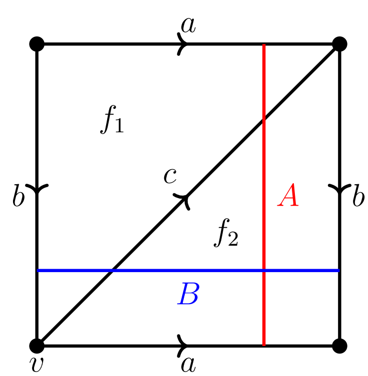
$$\begin{aligned}(f^\#(\varphi \cup \psi))(\sigma) &= (\varphi \cup \psi)(f \circ \sigma)\\ &= \varphi(f \circ \sigma \circ [e_0 \dots, e_k])\psi(f \circ \sigma \circ [e_k, \dots, e_{k+\ell}])\\ &= (f^\#\varphi)(\sigma \circ [e_0 \dots, e_k])(f^\#\psi)(\sigma \circ [e_k, \dots, e_{k+\ell}])\\ &= (f^\#\varphi \cup f^\#\psi)(\sigma)\end{aligned}$$

If the cohomology classes $\alpha$, $\beta$ are represented by the cocycles $\varphi$ resp. $\psi$, we obtain

$$\begin{aligned}f^*(\alpha \cup \beta) = f^*([\varphi \cup \psi]) &= [f^\#(\varphi \cup \psi)]\\ &= [f^\#\varphi \cup f^\#\psi] = [f^\#\varphi] \cup [f^\#\psi]\\ &= f^*\alpha \cup f^*\beta\end{aligned}$$

□

3. The goal of this and the next problem is to calculate the cup product in the cohomology of the torus $T$, using a refinement of the methods we used in class to calculate the cup product in $H^*(X_2;\mathbb{Z}/2)$ for $X_2 = \mathbb{RP}^2\#\mathbb{RP}^2$. We use our standard picture of the torus as the quotient space of the square by identifying the edges with the same label. Furthermore, we subdivide the square into two affine 2-simplices $f_1$ and $f_2$ and we pick two loops $A$, $B$ on $T$ as indicated in the following picture.




(a) Write down explicitly the sub chain complex $C_*$ of the singular chain complex $C_*(T)$ generated by the singular simplices in the picture above (i.e., $v, a, b, c, f_1, f_2$; use these names!). Identify a 2-cycle $t \in C_2$ that represents a generator of $H_2(T) \cong \mathbb{Z}$.

(b) Let $C^* := \operatorname{Hom}(C_*, \mathbb{Z})$ be the cochain complex corresponding to $C_*$. Let $\{\alpha, \beta\}$ be the basis of $H^1(T;\mathbb{Z})$ which is dual to the basis $\{[a],[b]\}$ of $H_1(T;\mathbb{Z})$ given by the cycles $a, b \in C_1$ via the evaluation pairing

$$\langle -,-\rangle\colon H^1(T;\mathbb{Z}) \times H_1(T;\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

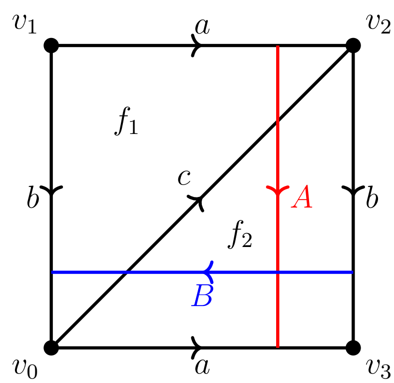
Construct cocycles $\phi, \psi \in C^1$ with $[\phi] = \alpha$ and $[\psi] = \beta$.

Hint: Use the loop $A$ to construct $\phi$, and the loop $B$ to construct $\psi$. Warning: Care is needed to adapt the "intersection number" construction used in class to this situation, since here we are talking about cochains with values in $\mathbb{Z}$ rather than $\mathbb{Z}/2$.

*Proof.* **Part (a)** The sub chain complex $C_* \subset C_*(T)$ consists of the $q$-simplices, $0 \leq q \leq 2$ which are visible in the picture, e.g., the point $v$ yields a 0-simplex in $T$. More care is needed to define the singular 1-simplices corresponding to the oriented edges $a$, $b$, $c$ and the 2-simplices corresponding to the triangles $f_1$, $f_2$, since a singular $q$-simplex is a map $\Delta^q \to T$, and we need to be precise about how that map is defined. To do so, let $\pi\colon S \twoheadrightarrow T$ be the projection map from the square $S$ to the torus $T = S/\sim$. We will describe each simplex as the composition

$$\Delta^q \xrightarrow{\sigma} S \xrightarrow{\pi} T \ ,$$

where the first map is an *affine linear* map, i.e., $\sigma = [x_0, \dots, x_q]$ for points $x_i \in S$. To do so, it is useful to label the vertices of the square; we will use the labeling indicated by the following picture:




(1) We interpret each oriented edge from a point $x$ to a point $y$ as a 1-simplex given by $\pi \circ [x, y]\colon \Delta^1 \to T$. For example, the top edge labeled $a$ yields the 1-simplex $\pi \circ [v_1, v_2]$. The lower edge labeled $a$ yields $\pi \circ [v_0, v_3]$ which is the *same* simplex in $T$, and use the notation $a$ for this 1-simplex. Similarly, $b := \pi \circ [v_1, v_0]$, and $c := \pi \circ [v_0, v_2]$.

(2) To identify each triangle $f_i$ with a 2-simplex $f_i\colon \Delta^2 \to T$, we need to identify the $0^{\text{th}}$, first and second vertex of each triangle. This is done using the orientation of the boundary edges by looking for the two consecutive edges in each triangle that are consistently oriented, e.g., the edges $b$ and $c$ for the triangle $f_1$. This determines an ordering of the vertices of $f_1$ as $v_1$, $v_0$, $v_2$, and hence a 2-simplex $f_1 := \pi \circ [v_1, v_0, v_2]$. Similarly, $f_2 := \pi \circ [v_0, v_2, v_3]$.

It is clear that the boundary map $\partial$ applied to $a$, $b$, $c$ is trivial.

$$\partial f_1 = \partial(\pi \circ [v_1, v_0, v_2] = \pi \circ [v_0, v_2] - \pi \circ [v_1, v_2] + \pi \circ [v_1, v_0] = c - a + b$$
$$\partial f_2 = \partial(\pi \circ [v_0, v_2, v_3] = \pi \circ [v_2, v_3] - \pi \circ [v_0, v_3] + \pi \circ [v_0, v_2] = b - a + c$$

This shows that the following chain complex $C_*$

| degree | 0 | 1 | 2 |
|---|---|---|---|

$$\mathbb{Z}v \xleftarrow{\partial \equiv 0} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xleftarrow{\partial} \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$$

is a subcomplex of $C_*(T)$. We see that $t := f_2 - f_1$ is a generator of $Z_2 \subset C_2$.

**Part (b)** For any edge $e = a, b, c$, we define cochains $\phi, \psi \in C^1 := \mathrm{Hom}(C_1, \mathbb{Z})$ via the *signed intersection number*

$$\phi(e) := A \cap e \in \mathbb{Z} \qquad \psi(e) := B \cap e \in \mathbb{Z}.$$

This requires us to choose orientations on $A$ and $B$, and we do so as indicated in the picture (these orientations are chosen so that the cocycles $\phi$, $\psi$ we construct this way indeed represent a basis of $H^1(T)$ dual to the basis $H_1(T)$ with respect to the evaluation pairing). The integer $A \cap e$ is defined as the sum

$$A \cap e := \sum_{x \in A \cap e} \epsilon_x(\vec{A}, \vec{e})$$

where $\vec{A}$ is a tangent vector to the curve $A$ at $x$ pointing in the direction determined by the orientation of $A$, and analogously for $\vec{e}$. Then $\epsilon_x(\vec{A}, \vec{e}) \in \{\pm 1\}$ is $+1$ if and only if the basis $\{\vec{A}, \vec{e}\}$ of $T_x S = \mathbb{R}^2$ represents the standard orientation of $\mathbb{R}^2$.

Consulting the picture above we calculate:

$$\begin{aligned} \phi(a) &= A \cap a = +1 & \phi(b) &= A \cap b = 0 & \phi(c) &= A \cap c = +1 \\ \psi(a) &= B \cap a = 0 & \psi(b) &= B \cap b = +1 & \psi(c) &= B \cap c = -1 \end{aligned} \tag{12}$$

Next we check that $\phi$ and $\psi$ are cocycles:

$$\begin{aligned} \delta\phi(f_1) &= \phi(\partial f_1) = \phi(c - a + b) = 0 & \delta\phi(f_2) &= \phi(\partial f_2) = \phi(b - a + c) = 0 \\ \delta\psi(f_1) &= \psi(\partial f_1) = \psi(c - a + b) = 0 & \delta\psi(f_2) &= \psi(\partial f_2) = \psi(b - a + c) = 0 \end{aligned}$$

Finally, we calculate the evaluation pairings of the cohomology classes $\alpha := [\phi]$ and $\beta := [\psi]$ in $H^1(T)$ with the homology classes $[a], [b] \in H_1(T)$ that form a basis of $H_1(T)$.

$$\begin{aligned} \langle \alpha, [a] \rangle &= \phi(a) = A \cap a = 1 & \langle \alpha, [b] \rangle &= \phi(b) = A \cap b = 0 \\ \langle \beta, [a] \rangle &= \psi(a) = B \cap a = 0 & \langle \beta, [b] \rangle &= \psi(b) = B \cap b = 1 \end{aligned}$$

□

4. This is a continuation of the previous problem using the same notation.
   (a) Calculate the numbers

$$\langle \alpha \cup \beta, [t] \rangle \qquad \langle \alpha \cup \alpha, [t] \rangle \qquad \langle \beta \cup \beta, [t] \rangle.$$

   (b) Calculate the cup products $\alpha \cup \beta$, $\alpha \cup \alpha$ and $\beta \cup \beta$ in $H^2(T; \mathbb{Z})$, by expressing them as multiples of the generator $\gamma \in H^2(T; \mathbb{Z})$ which is dual to $[t] \in H_2(T; \mathbb{Z})$ with respect to the evaluation pairing.

*Proof.* **Part (a)** The cochain level cup product is defined in terms of front $k$-faces and back $\ell$-faces of simplices that we evaluate the cup product on. So we first record $\mathrm{ff}(f_i), \mathrm{bf}(f_i) \in C_1$, the front (resp. back) 1-face of our 2-simplex $f_i$:

$$\mathrm{ff}(f_1) = b \qquad \mathrm{bf}(f_1) = c \qquad \mathrm{ff}(f_2) = c \qquad \mathrm{bf}(f_2) = b$$

Using (12) we obtain

$$\begin{aligned}
\langle \alpha \cup \beta, [t] \rangle =& \langle \phi \cup \psi, f_2 - f_1 \rangle = \phi(\mathrm{ff}(f_2))\psi(\mathrm{bf}(f_2)) - \phi(\mathrm{ff}(f_1))\psi(\mathrm{bf}(f_1)) \\
=& \phi(c)\psi(b) - \phi(b)\psi(c) = 1 \cdot 1 - 0 \cdot 1 = 1 \\
\langle \alpha \cup \alpha, [t] \rangle =& \langle \phi \cup \phi, f_2 - f_1 \rangle = \phi(\mathrm{ff}(f_2))\phi(\mathrm{bf}(f_2)) - \phi(\mathrm{ff}(f_1))\phi(\mathrm{bf}(f_1)) \\
=& \phi(c)\phi(b) - \phi(b)\phi(c) = 1 \cdot 0 - 0 \cdot 1 = 0 \\
\langle \beta \cup \alpha, [t] \rangle =& \langle \psi \cup \phi, f_2 - f_1 \rangle = \psi(\mathrm{ff}(f_2))\phi(\mathrm{bf}(f_2)) - \psi(\mathrm{ff}(f_1))\phi(\mathrm{bf}(f_1)) \\
=& \psi(c)\phi(b) - \psi(b)\phi(c) = -1 \cdot 0 - 1 \cdot 1 = -1 \\
\langle \beta \cup \beta, [t] \rangle =& \langle \psi \cup \psi, f_2 - f_1 \rangle = \psi(\mathrm{ff}(f_2))\psi(\mathrm{bf}(f_2)) - \psi(\mathrm{ff}(f_1))\psi(\mathrm{bf}(f_1)) \\
=& \psi(c)\psi(b) - \psi(b)\psi(c) = -1 \cdot 1 - 1 \cdot (-1) = 0
\end{aligned}$$

**Part (b)** The generator $\gamma \in H^2(T) \cong \mathbb{Z}$ is characterized by the property $\langle \gamma, [t] \rangle = 1$. Hence the calculation of $\langle \alpha \cup \beta, [t] \rangle$ and the evaluation of the other cup products on $[t]$ from part (a) immediately implies

$$\alpha \cup \beta = \gamma \qquad \alpha \cup \alpha = 0 \qquad \beta \cup \beta = -\gamma \qquad \beta \cup \beta = 0$$

□

## 11. Homework assignment # 11

1. The *Kummer surface* is the submanifold of $\mathbb{CP}^3$ of (real) dimension 4 given by

$$K := \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

It can be shown that

- $K$ is simply connected (i.e., $K$ is connected and its fundamental group $\pi_1(K)$ is trivial)
- its Euler characteristic is given by $\chi(K) = 24$.

Use these facts to calculate the homology groups $H_q(K)$ and the cohomology groups $H^q(K)$ for all $q$. Hint: use the Universal Coefficient Theorem *and* Poincaré duality to relate homology and cohomology groups. Make sure to provide an argument for why the assumptions of the Poincaré Duality Theorem are satisfied.

*Proof.* The fact that $K$ is simply connected implies that $H_0(K) \cong \mathbb{Z}$ (since $K$ is path connected), and that $H_1(K) = 0$ (since $H_1(K) \cong \pi_1(K)^{\mathrm{ab}} = 0$ by the Hurewicz isomorphism).

In order to apply Poincaré duality we need to make sure that the manifold $K$ is compact and oriented.

$K$ **is compact:** The subspace

$$\widehat{K} := \{(z_0, z_1, z_2, z_3) \in S^7 \subset \mathbb{C}^4 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset S^7$$

is closed since $\widehat{K} = f^{-0}$ for the continuous map $f \colon S^7 \to \mathbb{C}$ given by $f(z_0, z_1, z_2, z_3) = z_0^4 + z_1^4 + z_2^4 + z_3^4$. Since $S^7 \subset \mathbb{C}^4 = \mathbb{R}^8$ is compact by Heine-Borel, the closed subspace $\widehat{K} \subset S^7$ is compact. Since $K$ is the image of $\widehat{K}$ under the projection map $\pi \colon S^7 \to \mathbb{CP}^3$, it is compact.

**$K$ is orientable:** We recall that an orientation for a manifold $M$ is a section of the double covering $\widetilde{M} \to M$ whose fiber $\widetilde{M}_x$ for a point $x \in M$ is the set consisting of the two orientations of the tangent space $T_xM$. If $M$ is simply connected, like the Kummer surface, this double covering is isomorphic to the trivial double covering. In particular, $M$ is orientable.

Next we calculate the cohomology group $H^q(K)$ in *two* ways in terms of the homology groups of $K$:

- by Poincaré duality, $H^q(K) \cong H_{4-q}(K)$;
- by the Universal Coefficient Theorem, $H^q(K) \cong \mathrm{Hom}(H_q(K), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(K), \mathbb{Z})$.

The homology group $H_2(K)$ can be written in the form $H_2(K) = \mathbb{Z}^r \oplus T$, where $T$ is a torsion group. It will be convenient to collect our knowledge about the various (co)homology groups in the following table.

| q | $H_q(K)$ | $H^q(K) \cong H_{4-q}(K)$ | $\mathrm{Hom}(H_q(K), \mathbb{Z})$ | $\mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(K), \mathbb{Z})$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | $\mathbb{Z}^r \oplus T$ | $\mathbb{Z}^r$ | $\mathbb{Z}^r$ | 0 |
| 3 | 0 | | | |
| 4 | $\mathbb{Z}$ | | | |

Here the colors of the entries indicate in which logical order we calculate these groups – black entries first, then blue entries, and finally red entries. The information we start with is $H_0(K) = \mathbb{Z}$, $H_1(K) = 0$ and $H_2(K) = \mathbb{Z}^r \oplus T$ (with $r$ a to be determined integer, and $T$ a to be determined torsion group) in the second column of the table. These black entries in the second column immediately determine the black entries in the other columns of the table. By the UCT, the sum of the last two rows is isomorphic to the third row. In particular, we obtain the three blue entries in third row that way.

The Poincaré duality isomorphism $H^q(K) \cong H_{4-q}(K)$ for $q = 0, 1$ results in the two red entries in the second column. For $q = 2$, this isomorphism becomes an isomoprhism $\mathbb{Z}^r \cong \mathbb{Z}^r \oplus T$, which implies that the torsion group $T$ must be trivial. Finally, to determine $r = \mathrm{rk}\, H_2(K)$, we use the fact that the Euler characteristic of $K$ is 24, which implies the equation

$$24 = \chi(K) = \sum_{q=0}^{4} (-1)^q \,\mathrm{rk}\, H_q(K) = 1 - 0 + r - 0 + 1 = 2 + r,$$

and hence $r = 22$. Summarizing our results, we have

$$H_q(K) = \begin{cases} \mathbb{Z} & q = 0, 4 \\ \mathbb{Z}^{22} & q = 2 \\ 0 & q \neq 0, 2, 4 \end{cases}$$

$\square$

2. (a) Using the cup product structure, show there is no map $g\colon \mathbb{RP}^n \to \mathbb{RP}^m$ inducing a nontrivial map $H^1(\mathbb{RP}^m; \mathbb{Z}/2) \to H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ if $n > m$.

(b) Prove the Borsuk-Ulam theorem according to which for every map $f\colon S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$ by the following argument. Hint: suppose on the contrary that $f\colon S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all $x$. Then define

$$g\colon S^n \to S^{n-1} \qquad \text{by} \qquad g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|},$$

so $g(-x) = -g(x)$ and $g$ induces a map $\bar{g}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$. Show that this yields a contradiction, discussing separately the cases $n = 1$ (easy), $n = 2$ (consider the map induced by $\bar{g}$ on the fundamental group) and $n \geq 3$ (apply part (a) to the map $\bar{g}$).

Note that the Borsuk-Ulam Theorem implies for example that at each point in time there are two places on earth where the temperature and the barometric pressure are exactly the same.

*Proof.* **Part (a)** We will use proof by contradiction. Assume $g\colon \mathbb{RP}^n \to \mathbb{RP}^m$ is a map for some $n > m$ which induces a nontrivial map $H^1(\mathbb{RP}^m; \mathbb{Z}/2) \to H^1(\mathbb{RP}^n; \mathbb{Z}/2)$.

We recall that the cohomology algebra $H^*(\mathbb{RP}^m; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2[\alpha]/(\alpha^{m+1})$, where $|\alpha| = 1$. As shown in a previous homework problem the induced map in cohomology

$$g^*\colon H^*(\mathbb{RP}^m; \mathbb{Z}/2) \longrightarrow H^*(\mathbb{RP}^n; \mathbb{Z}/2)$$

is compatible with cup products. In other words, it is a homomorphism of graded algebras. Let $\beta \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ be the non-trivial element. The assumption that the induced map $g^*$ on the first cohomology is non-trivial implies that $g^*(\alpha) = \beta$, and hence $g^*(\alpha^n) = \beta^n$. This is the desired contradiction, since $\beta^n \neq 0 \in H^n(\mathbb{RP}^n; \mathbb{Z}/2)$, while $\alpha^n \in H^n(\mathbb{RP}^m; \mathbb{Z}/2) = 0$ due to $n > m$.

**Part (b)** As suggested in the hint, assume that $f(x) \neq f(-x)$ for all $x \in S^n$, and consider the associated map $g\colon S^n \to S^{n-1}$ with the property $g(-x) = -g(x)$; in other words, $g$ is equivariant with respect to the antipodal action of $\mathbb{Z}/2$ on domain and codomain. Let $\bar{g}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ be the induced map between the quotient spaces of this action.

(1) For $n = 1$ we have a map $g\colon S^1 \to S^0$. The property $g(-x) = -g(x)$ implies that $g\colon S^1 \to S^0$ is surjective, but the image $g(S^1)$ of the connected space $S^1$ must be connected subset of $S^0$. This is the desired contradiction.
(2) For $n = 2$, we have a map $\bar{g}\colon \mathbb{RP}^2 \to \mathbb{RP}^1$ and we consider the induced map $\bar{g}_*\colon \pi_1(\mathbb{RP}^2) \to \pi_1(\mathbb{RP}^1)$ on fundamental groups. By covering space theory, the generator of $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$ is given by a loop $\bar{\gamma}\colon [0,1] \to \mathbb{RP}^2$ that lifts to a path $\gamma\colon [0,1] \to S^2$ from a point $x_0 \in S^2$ to its antipodal point $-x_0$. The equivariance of $g$ implies that $g \circ \gamma$ is a path in $S^1$ from $g(x_0)$ to $-g(x_0)$; in particular, the loop $\bar{g} \circ \pi \circ \gamma$ in $\mathbb{RP}^1$ is not homotopic to the constant loop since its lift $g \circ \gamma\colon [0,1] \to S^1$ to the double covering $S^1 \to \mathbb{RP}^1$ is not a loop. In other words, $\bar{g}_*([\pi \circ \gamma])$ is a non-trivial element in $\pi_1(\mathbb{RP}^1)$, and the induced homomorphism $\bar{g}_*\colon \pi_1(\mathbb{RP}^2) \to \pi_1(\mathbb{RP}^1)$ is non-trivial. This is a contradiction since $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$, $\pi_1(\mathbb{RP}^1) \cong \pi_1(S^1) \cong \mathbb{Z}$, and there are no non-trivial homomorphisms $\mathbb{Z}/2 \to \mathbb{Z}$.
(3) For $n \geq 3$, the argument in (2) shows that the induced homomorphism $\bar{g}_*\colon \pi_1\mathbb{RP}^n \to \pi_1\mathbb{RP}^{n-1}$ is non-trivial, and hence an isomorphism, since both fundamental groups have order 2.

Consider the commutative diagram

$$\begin{tikzcd}
\pi_1\mathbb{RP}^n \arrow[r, "h"] \arrow[d, "\bar g_*"] & H_1(\mathbb{RP}^n;\mathbb{Z}) \arrow[r] \arrow[d, "\bar g_*"] & H_1(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[d, "\bar g_*"] \\
\pi_1\mathbb{RP}^{n-1} \arrow[r, "h"] & H_1(\mathbb{RP}^{n-1};\mathbb{Z}) \arrow[r] & H_1(\mathbb{RP}^{n-1};\mathbb{Z}/2),
\end{tikzcd}$$

where $h$ is the Hurewicz isomorphism and the unlabeled maps are induced by the homomorphism $\mathbb{Z} \to \mathbb{Z}/2$. The Hurewicz Theorem resp. the UCT imply that the horizontal maps are isomorphisms, and hence the induced map $\bar g_*$ on $H_1(-;\mathbb{Z}/2)$ is an isomorphism. By the UCT the induced map $\bar g^*$ on $H^1(-;\mathbb{Z}/2)$ is just the vector space dual to $\bar g_*$ and hence it is also an isomorphism. Then applying part (a) to the map $\bar g\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ yields the desired contradiction.

□

3. (a) Let $X$, $Y$ be connected topological spaces equipped with basepoints $x_0 \in X$, $y_0 \in Y$. Let $X \vee Y$ be their wedge product, and let $\pi_1\colon X \vee Y \twoheadrightarrow X$, $\pi_2\colon X \vee Y \twoheadrightarrow Y$ be the natural projection maps. Show that if $\alpha \in H^*(X;R)$, $\beta \in H^*(Y;R)$ with $|\alpha| \geq 1$ and $|\beta| \geq 1$, then $\pi_1^*\alpha \cup \pi_2^*\beta = 0$.
(b) Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$. Hint: use problem 2 (a).

*Proof.* **Part (a).** The additivity property for cohomology implies that the map

$$H^q(X \vee Y;R) \longrightarrow H^q(X;R) \oplus H^q(Y;R) \qquad \gamma \mapsto (i_1^*(\gamma), i_2^*(\gamma)) \tag{13}$$

is an isomorphism for $q \geq 1$. We note that

$$i_1^*(\pi_1^*\alpha \cup \pi_2^*\beta) = (i_1^*\pi_1^*\alpha) \cup (i_1^*\pi_2^*\beta) = (\pi_1 \circ i_1)^*\alpha \cup (\pi_2 \circ i_1)^*\beta$$

by compatibility between the cup product and induced maps. Moreover, the map

$$X \xrightarrow{i_1} X \vee Y \xrightarrow{\pi_2} Y$$

factors through a point, and hence $(\pi_2 \circ i_1)^*\beta = 0$, since $\beta \in H^q(Y;R)$ for $q > 0$. It follows that $i_1^*(\pi_1^*\alpha \cup \pi_2^*\beta) = 0$. An analogous argument shows $i_2^*(\pi_1^*\alpha \cup \pi_2^*\beta) = 0$ and hence (13) implies $\pi_1^*\alpha \cup \pi_2^*\beta = 0$.

**Part (b).** Assume on the contrary that there is a homotopy equivalence $f\colon \mathbb{RP}^3 \to \mathbb{RP}^2 \vee S^3$. Then $f$ induces in particular an isomorphism on cohomology with $\mathbb{Z}/2$-coefficients. It follows that the composition

$$\mathbb{RP}^3 \xrightarrow{f} \mathbb{RP}^2 \vee S^3 \xrightarrow{\pi_1} \mathbb{RP}^2$$

induces an isomorphism on $H^1(-;\mathbb{Z}/2)$, since $H^1(S^3;\mathbb{Z}/2) = 0$. By problem 2 (a) this is impossible. □

4. The construction of the fundamental class of a closed oriented manifold is based on the following

**Proposition 2.** *Let $M$ be a manifold of dimension $n$, and $K \subset M$ a compact subset. Then the homology $H_i(M, M-K)$ is zero for $i > n$ and an element $\alpha \in H_n(M, M-K)$ is trivial if and only if it is in the kernel of the map*

$$H_n(M, M-K) \longrightarrow H_n(M, M-x)$$

*induced by the inclusion $(M, M-K) \to (M, M-x)$ for every $x \in K$.*

A crucial step in the proof of this proposition is to show that if this statement holds for compact subsets $K_1$, $K_2$ and their intersection $K_1 \cap K_2$, then is also holds for $K = K_1 \cup K_2$. Prove this step. Hint: note that

$$\begin{aligned}(M, M-K) &= (M, M-K_1) \cap (M, M-K_2)\\ (M, M-(K_1 \cap K_2)) &= (M, M-K_1) \cup (M, M-K_2),\end{aligned}$$

and use without proof that there is a version of the Meyer-Vietoris sequence for pairs of spaces.

*Proof.* We will use the Meyer-Vietoris sequence for the suggested pairs of spaces which takes the form

$$\longrightarrow H_{i+1}(M, M-K_1 \cap K_2) \xrightarrow{\partial} H_i(M, M-K)$$
$$\longrightarrow H_i(M, M-K_1) \oplus H_i(M, M-K_2) \longrightarrow$$

This exact sequence implies that if the statement holds for $K_1$, $K_2$, and $K_1 \cap K_2$, then $H_i(M, M-K)$ vanishes for $i > 0$, since the adjacent terms in the long exact sequence both vanish. Now suppose that $\alpha \in H_n(M, M-K)$ which is in the kernel of the map $H_n(M, M-K) \longrightarrow H_n(M, M-x)$ for every $x \in K$. We need to show $\alpha = 0$.

Since the term to the left of $H_n(M, M-K)$ in the exact sequence above vanishes, it follows that the map $\Phi$ in the sequence is injective (for $i = n$), and hence it suffices to show that $\Phi(\alpha) = 0$. Now by construction of the Meyer-Vietoris sequence, $\Phi(\alpha) = (j^1_*(\alpha), j^2_*(\alpha))$, where

$$j^k \colon (M, M-K) \longrightarrow (M, M-K_k)$$

is the inclusion map of pairs. To show that $j^k_*(\alpha) = 0$, suppose $x \in K_k$, and consider the following commutative diagram of pairs

$$\begin{tikzcd}
(M, M-K) \arrow[r, "j^k"] \arrow[dr] & (M, M-K_k) \arrow[d] \\
 & (M, M-x)
\end{tikzcd}$$

The corresponding diagram of homology groups of degree $n$ shows that $j^k_*(\alpha)$ is in the kernel of the map $H_n(M, M-K_k) \to H_n(M, M-x)$. Since this holds for *every* point $x \in K_k$, this implies that $j^k_*(\alpha) = 0$. This holds for both, $k = 1$ and $k = 2$, and hence $\Phi(\alpha) = 0$. The exact sequence then implies $\alpha = 0$ as desired. ∎