Algebraic Topology

Stephan Stolz

January 18, 2023

These are incomplete notes based on a second semester basic topology course I taught in the Spring of 2016. A basic reference is Allen Hatcher's book [Ha].

Contents

1 Introduction

The following is an attempt at explaining what 'topology' is.

- Topology is the study of qualitative/global aspects of shapes, or – more generally – the study of qualitative/global aspects in mathematics.

A simple example of a 'shape' is a 2-dimensional surface in 3-space, like the surface of a ball, a football, or a donut. While a football is different from a ball (try kicking one...), it is *qualitatively* the same in the sense that you could squeeze a ball (say a balloon to make squeezing easier) into the shape of a football. While any surface is *locally homeomorphic* $\mathbb{R}^2$ (i.e., every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$) by definition of 'surface', the 'global shape' of two surfaces might be different meaning that they are not homeomorphic (e.g. the surface of a ball is not homeomorphic to the surface of a donut). The French mathematician Henry Poincaré (1854-1912) is regarded as one of the founders of topology, back then known as 'analysis situ'. He was interested in understanding qualitative aspects of the solutions of differential equation.

There are basically three flavors of topology:

1. Point set Topology: Study of general properties of topological spaces

2. Differential Topology: Study of manifolds (ideally: classification up to homeomorphism/diffeomorphism).

3. Algebraic topology: trying to distinguish topological spaces by assigning to them algebraic objects (e.g. a group, a ring, . . .).

Let us go in more detail concerning algebraic topology, since that is the topic of this course. Before mentioning two examples of algebraic objects associated to topological spaces, let us make the purpose of assigning these algebraic objects clear: if X and Y are *homeomorphic* objects, we insist that the associated algebraic objects $A(X)$, $A(Y)$ are isomorphic. That means in particular, that if we find that $A(X)$ and $A(Y)$ are not isomorphic, then we can conclude that the spaces X and Y are not homeomorphic. In other words, the algebraic objects help us to *distinguish homeomorphism classes of topological spaces*.

Here are two examples of algebraic objects we can assign to topological spaces, which satisfy this requirement. We will discuss them in more detail below:

Homotopy groups To any topological space X equipped with a distinguished point $x_0 \in X$ (called the *base point*), we can associate groups $\pi_n(X, x_0)$ for $n = 1, 2, \dots$ called *homotopy groups* of X. These are *abelian* groups for $n \geq 2$.

Homology groups To any topological space X we can associate abelian groups $H_n(X)$ for $n = 0, 1, \dots$, called *homology groups of* X.

The advantages/disadvantages of homotopy versus homology groups are

- homotopy groups are easy to define, but extremely hard to calculate;
- homology groups are harder to define, but comparatively easier to calculate (with the appropriate tools in place, which will take us about half the semester)

Let us illustrate these statements in a simple example. We will show (in about a month) that the homology group of spheres look as follows:

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

The homotopy groups of spheres are much more involved; for example:

k	1	2	3	4	5	6	7	8	9
$\pi_k(S^2, x_0)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$

It is perhaps surprising that these homotopy groups are *not known* for large k (not only in the sense that we don't have a 'closed formula' for these groups, but also in the sense that we don't have an algorithm that would crank out these groups one after another on a computer if we just give it enough time...). This holds not only for S^2, but for any sphere S^n (except $n = 1$). In fact, the calculation of the homotopy groups of spheres is something akin to the 'holy grail' of algebraic topology.

1.1 Homotopy groups

Suppose f and g are continuous maps from a topological space X to a topological space Y. Then true to the motto that in topology we are interested in 'qualitative' properties we shouldn't try to distinguish between f and g if they can be 'deformed' into each other in the sense that for each $t \in [0, 1]$ there is a map $f_t \colon X \to Y$ such that $f_0 = f$ and $f_1 = g$, and such that the family of maps f_t 'depends continuously on t'. The following definition makes precise what is meant by 'depending continuously on t' and introduces the technical terminology 'homotopic' for the informal 'can be deformed into each other'.

Definition 1.1. Two continuous maps $f, g\colon X \to Y$ between topological spaces X, Y are *homotopic* if there is a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a *homotopy between f and g*. We will denote by $[X,Y]$ the set of homotopy classes of maps from X to Y.

We note that if H is a homotopy, then we have a family of maps $f_t\colon X \to Y$ parametrized by $t \in [0,1]$ interpolating between f and g, given by $f_t(x) = H(t,x)$. Conversely, if $f_t\colon X \to Y$ is a family of maps parametrized by $t \in [0,1]$, then we can define a map $H\colon [0,1] \times X \to Y$ by the above formula. We note that the continuity requirement for H implies not only that each map f_t is continuous, but also implies that for fixed $x \in X$ the map $t \mapsto f_t(x)$ is continuous. In other words, our idea of requiring that f_t should 'depend continuously on t' is made precise by requiring continuity of H.

Examples of homotopic maps.

1. Any two maps $f, g\colon X \to \mathbb{R}$ are homotopic; in other words, $[X, \mathbb{R}]$ is a one point set. A homotopy $H\colon X \times [0,1] \to \mathbb{R}$ is given by $H(x,t) = (1-t)f(x) + tg(x)$. We note that for fixed x the map $[0,1] \to \mathbb{R}$ given by $t \mapsto (1-t)f(x) + tg(x)$ is the affine linear path (aka straight line) from $f(x)$ to $g(x)$. For this reason, the homotopy H is called a *linear homotopy*. The construction of a linear homotopy can be done more generally for maps $f, g\colon X \to Y$ if Y is a vector space, or a convex subspace of a vector space.

2. A map $S^1 \to Y$ is a loop in the space Y. Physically, we can think of it as the trajectory of a particle that moves in the topological space Y, returning to its original position after some time. In general, there are maps $f, g\colon S^1 \to Y$ that are not homotopic. For example, given an integer $k \in \mathbb{Z}$, let

   $$f_k\colon S^1 \to S^1 \qquad \text{be the map given by} \qquad f_k(z) = z^k.$$

   Physically that describes a particle that moves $|k|$ times around the circle, going counterclockwise for $k > 0$ and clockwise for $k < 0$. We will prove that f_k and f_ℓ are homotopic if and only if $k = \ell$. Moreover, we will show that *any* map $f\colon S^1 \to S^1$ is homotopic to f_k for some $k \in \mathbb{Z}$. In other words, we will prove that there is a bijection

   $$\mathbb{Z} \leftrightarrow [S^1, S^1] \qquad \text{given by} \qquad k \mapsto f_k$$

   This fact will be used to prove the fundamental theorem of algebra.

Sometimes it is useful to consider pairs (X, A) of topological spaces, meaning that X is a topological space and A is a subspace of X. If (Y, B) is another pair, we write

$$f\colon (X, A) \longrightarrow (Y, B)$$

if f is a continuous map from X to Y which sends A to B. Two such maps $f, g\colon (X, A) \to (Y, B)$ are *homotopic* if there is a map

$$H\colon (X \times I, A \times I) \longrightarrow (Y, B)$$

with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We will use the notation $[(X, A), (Y, B)]$ for the set of homotopy classes of maps from (X, A) to (Y, B).

Definition 1.2. Let X be a topological space, and let x_0 be a point of X. Then the *n-th homotopy group of* (X, x_0) is by definition

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)].$$

Here $I^n := \underbrace{I \times \cdots \times I}_{n} \subset \mathbb{R}^n$ is the n-dimensional cube, and ∂I^n is its boundary.

A map $f\colon (I, \partial) \to (X, x_0)$ is geometrically a path in X parametrized by the unit interval $I = [0, 1]$ with starting point $f(0) = x_0$ and endpoint $f(1) = x_0$. Such maps are also called *based loops*. Similarly, a map $f\colon (I^2, \partial I^2) \to (X, x_0)$ is geometrically a membrane in X parametrized by the square I^2, such that the boundary of the square maps to the base point x_0.

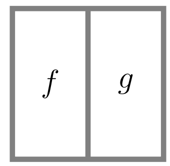
As suggested by the terminology of the above definition, the set $[(I^n, \partial I^n), (X, x_0)]$ has in fact the structure of a *group*. Given two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$, their product $f * g\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(f * g)(t_1, \dots, t_n) := \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{for } 0 \le t_1 \le \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{for } \frac{1}{2} \le t_1 \le 1 \end{cases}$$

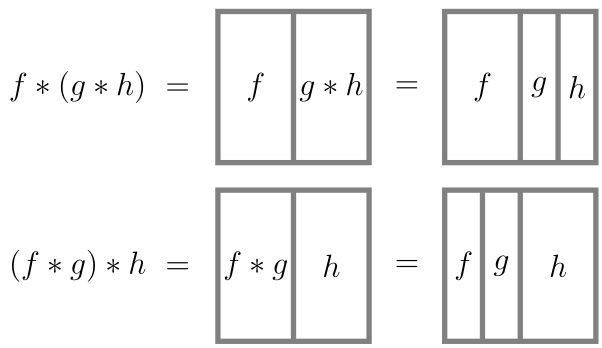
We note that this is a well-defined map, since for $t_1 = \frac{1}{2}$ the points $(2t_1, t_2, \dots, t_n)$ and $(2t_1 - 1, t_2, \dots, t_n)$ both belong to the boundary ∂I^n, and hence both map to x_0 via f and g. Moreover, $f * g$ is continuous since its restriction to the closed subsets consisting of the points $t = (t_1, \dots, t_n)$ with $t_1 \le \frac{1}{2}$ resp. $t_1 \ge \frac{1}{2}$ is continuous. We will refer to $f * g$ as the *concatenation* of the maps f and g, since for $n = 1$ the map $f * g\colon I \to X$ is usually referred to as the concatenation of the paths f and g.

The following picture shows where $f * g$ maps points in the square I^2: if $t = (t_1, t_2)$ belongs to the left half of the square, it is mapped via f; points in the right half map via g (here we implicitly identify the left and right halves of the square again with I^2). In particular the boundaries of the two halves map to the base point x_0; this subset of I^2 is indicated by the gray lines in the picture.

$f * g =$



Next we want to address the question whether given $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$ the maps $f * (g * h)$ and $(f * g) * h$ agree. Thinking in terms of pictures, we have



which shows that these two maps do not agree. However, they are homotopic to each other. We leave it to the reader to provide a proof of this. This implies the third of the following equalities in $\pi_n(X, x_0)$; the others hold by definition:

$$[f]([g][h]) = [f]([g * h]) = [f * (g * h)] = [(f * g) * h] = [f * g][h] = ([f][g])[h].$$

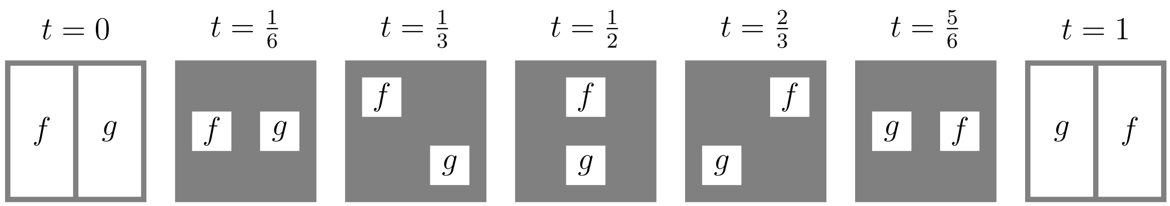
This shows that concatenation induces an associative product on $\pi_n(X, x_0)$. We leave it to the reader to show that this product gives $\pi_n(X, x_0)$ of a group where the unit element is represented by the constant map, and the inverse of an element $[f] \in \pi_n(X, x_0)$ is represented by $\bar{f}$, defined by $\bar{f}(t_1, \dots, t_n) := f(1 - t_1, t_2, \dots, t_n)$.

The group $\pi_1(X, x_0)$ is called the *fundamental group* of X, while the groups $\pi_n(X, x_0)$ for $n \geq 2$ are referred to as *higher homotopy groups*. Examples show that the fundamental group is in general not abelian. For example, the fundamental group of the "figure eight" is the free group generated by two elements. By contrast, for higher homotopy groups we have the following result.

Lemma 1.3. *For $n \geq 2$ the group $\pi_n(X, x_0)$ is abelian.*

Proof. We need to show that for maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ the concatenations $f * g$ and $g * f$ are homotopic to each other (as maps of pairs). Such a homotopy H is given by a continuous family of maps $H_t\colon (I^n, \partial I^n) \to (X, x_0)$ which agrees with $f * g$ for $t = 0$ and with

$g * f$ for $t = 1$. Thinking of each such maps as a picture, like the one for $f * g$ above, such a homotopy H_t is a family of pictures parametrized by $t \in [0, 1]$. Interpreting t as "time", this means that the homotopy H_t is a *movie*! Here it is:



Here all points in the gray areas of the square map to the base point. So shrinking the rectangles inside of the square labeled f resp. g allows us to rotate them past each other, a move which is not possible for $n = 1$, but for all $n \geq 2$. □

1.2 The Euler characteristic of closed surfaces

The goal of this section is to discuss the Euler characteristic of closed surfaces, that is, compact manifolds without boundary of dimension 2. We begin by recalling the definition of manifolds.

Definition 1.4. A *manifold of dimension n* or *n-manifold* is a topological space X which is locally homeomorphic to $\mathbb{R}^n$, that is, every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. Moreover, it is useful and customary to require that X is Hausdorff (see Definition 3.29) and second countable (see Definition 4.1). A *manifold with boundary of dimension n* is defined by replacing $\mathbb{R}^n$ in the definition above by the half-space $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. If X is an n-manifold with boundary, its *boundary* ∂X consists of those points of X which via some homeomorphism $U \approx V \subset \mathbb{R}^n_+$ correspond to points in the hyperplane given by the equation $x_1 = 0$. The complement $X \setminus \partial X$ is called the *interior of X*. A *closed n-manifold* is a compact n-manifold without boundary.

Examples of manifolds of dimension 1. An open interval (a, b) is a 1-manifold. A closed interval $[a, b]$ is a 1-manifold with boundary $\{a, b\}$. A half-open interval $(a, b]$ is a 1-manifold with boundary $\{b\}$.

A non-example. The subspace $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$ of $\mathbb{R}^2$ consisting of the x-axis and y-axis is not a 1-dimensional manifold, since X is not locally homeomorphic to $\mathbb{R}^1$ at the origin $x = (0, 0)$. To prove this intuitively obvious fact, suppose that U is an open neighborhood of $(0, 0)$ which is homeomorphic to an open subset $V \subset \mathbb{R}$. Replacing U by the connected component of U containing $(0, 0)$, and V by the image of that component, we can assume that U and V are connected. This implies that V is an open interval. Restricting the homeomorphism $f \colon U \to V$, we obtain a homeomorphism $U \setminus \{(0, 0)\} \approx V \setminus f(0, 0)$.

This is the desired contradiction, since $U \setminus \{(0,0)\}$ has four connected components, while $V \setminus f(0,0)$ has two.

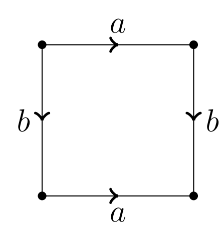
Examples of higher dimensional manifolds.

1. Any open subset $U \subset \mathbb{R}^n_+$ is an n-manifold whose boundary ∂U is the intersection of U with the hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$.

2. The n-sphere $S^n := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is an n-manifold.

3. The n-disk $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is an n-manifold with boundary $\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n \mid ||x|| = 1\}$.

4. The *torus* $T := S^1 \times S^1$ is a manifold of dimension 2. There at least two other ways to describe the torus. The usual picture we draw describes the torus as a subspace of $\mathbb{R}^3$. It can also be constructed as a quotient space of the square I^2: we identify the two horizontal edges of the square to obtain a cylinder, and then the two boundary circles to obtain the torus T. From a formal point of view, the last sentence describes an equivalence relation $\sim$ on I^2, and the claim is that the quotient space $I^2/\sim$ is homeomorphic to $S^1 \times S^1$. It will be convenient to use pictures for this and similar quotient spaces. Here is the picture for the quotient space $I^2/\sim$ described above:

$T =$ 

(1.5)

Question. Is the sphere S^2 homeomorphic to the torus T?

It seems intuitively clear that the answer is "no", but how do we prove that rigorously? The usual strategy for showing that two topological spaces aren't homeomorphic involves thinking of adjectives for topological spaces (e.g., compact, connected, Hausdorff, second countable, etc), and to show that one has some such property while the other doesn't. It turns out that the usual adjectives from point-set topology will not manage to distinguish these space. Instead, we will associate an integer to closed surfaces, called the Euler characteristic, that will distinguish S^2 and T. This is the most basic "algebraic topological invariant" for spaces.

The definition of the Euler characteristic of a closed 2-manifold Σ will involve choosing a "pattern of polygons" on Σ. By this we mean a graph Γ (consisting of vertices and edges) on Σ, such that all connected components of the complement $\Sigma \setminus \Gamma$ are homeomorphic to open discs. For example, the boundary of the 3-dimensional cube is a 2-dimensional manifold homeomorphic to S^2. The 8 vertices and 12 edges of the cube form a graph Γ on S^2; the complement $S^2 \setminus \Gamma$ consists of the 6 faces of the cube.

Given a pattern of polygons Γ on a surface Σ, we define the integer

$$\chi(\Sigma, \Gamma) := \#V - \#E + \#F,$$

where V is the set of vertices, E is the set of edges, and F is the set of faces.

Lemma 1.6. $\chi(\Sigma, \Gamma) = \chi(\Sigma, \Gamma')$ *for any two choices of graphs* Γ, Γ'.

Before proving this lemma, let us illustrate the statement in the example of two patterns on the 2-sphere S^2:

1. Let Γ be the graph described above obtained by identifying S^2 with the boundary of the cube. Then $\chi(S^2, \Gamma) = 8 - 12 + 6 = 2$.

2. Let Γ' be the graph obtained by identifying S^2 with the boundary of the tetrahedron. Then $\chi(S^2, \Gamma') = 4 - 6 + 4 = 2$.

Proof. We begin by proving the statement in the special case were the graph Γ' is obtained from Γ by adding an new edge. Then the number of vertices is the same for Γ and Γ', and the number of edges Γ' is the number of edges for Γ plus one. Similarly, the number of faces of Γ' is one larger than that for Γ, since the new edge subdivides one face for Γ into two faces for Γ'. Hence $\chi(\Gamma') = \chi(\Gamma)$.

Similarly, if Γ' is obtained from Γ by introducing a new vertex on one of the edges of Γ, then the number of vertices and edges goes up by one, while the number of faces doesn't change. Again, this implies that $\chi(\Gamma') = \chi(\Gamma)$ in this case as well. More generally, if Γ' is a *refinement* of Γ in the sense that Γ' is obtained from Γ by adding new edges and vertices, we see that $\chi(\Gamma') = \chi(\Gamma)$.

Finally, for general graphs Γ, Γ' we may assume without changing the number of vertices, edges or faces that Γ, Γ' are in general position to each other in the sense that the vertex sets $V(\Gamma)$, $V(\Gamma')$ are disjoint and that edges of Γ intersect those of Γ' in finitely many points. Then the union of the graphs Γ and Γ' can again be viewed as a graph Γ'' on Σ. For example, the vertices of Γ'' consist of the vertices of Γ, the vertices of Γ' and the intersection points of edges in Γ with edges in Γ'. The graph Γ'' is a refinement of both Γ and Γ', and hence $\chi(\Gamma') = \chi(\Gamma'') = \chi(\Gamma)$. □

Definition 1.7. The *Euler characteristic* of a closed surface Σ, denoted $\chi(\Sigma) \in \mathbb{Z}$ is defined to be $\chi(\Sigma, \Gamma)$ for any pattern of polygons on Σ.

The following is a simple consequence of Lemma 1.6 and the definition of the Euler characteristic.

Corollary 1.8. *If Σ, Σ' are closed surfaces with $\chi(\Sigma) \neq \chi(\Sigma')$, then these surfaces are not homeomorphic.*

Proof. Suppose that there is a homeomorphism $f\colon \Sigma \xrightarrow{\approx} \Sigma'$. If Γ is a pattern of polygons on Σ, let Γ' be the pattern of polygons on Σ' whose vertices (resp. edges resp. faces) are the images of vertices (resp. edges resp. faces) of Γ under the map f. Then

$$\chi(\Sigma) = \chi(\Sigma, \Gamma) = \chi(\Sigma', \Gamma') = \chi(\Sigma')$$

is the desired contradiction. □

Corollary 1.9. *The sphere S^2 is not homeomorphic to the torus T.*

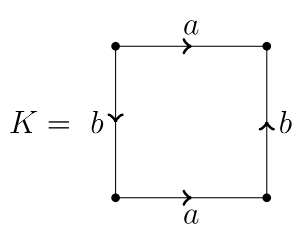
Proof. By the previous result it suffices to show $\chi(S^2) \neq \chi(T)$. By our calculations above we know $\chi(S^2) = 2$. We claim that $\chi(T) = 0$. To prove this, we use as "pattern of polygons" on T the picture (1.5), which has

- one face (the square);
- two edges (labeled a resp. b). There are only two rather than four edges since the two edges of the square labeled a lead to the *same edge* on the torus;
- one vertex; all four vertices of the square lead to the same vertex on the torus.

It follows that $\chi(T) = 1 - 2 + 1 = 0$. □

Other examples of closed surfaces and their Euler characteristic.

Klein bottle Like the torus, the Klein bottle K can be constructed as the quotient space of the square I^2 by identifying opposite edges of the square. Here is the picture:



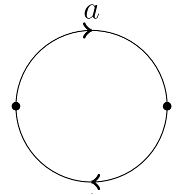
(1.10)

Like for the torus, the Euler characteristic of the Klein bottle can be calculated by using the "pattern of polygons" on K given by the above picture to obtain

$$\chi(K) = 1 - 2 + 1 = 0.$$

Real projective plane From an algebraic perspective, the real projective plane $\mathbb{RP}^2$ is the set of lines (= 1-dimensional subspaces of $\mathbb{R}^3$). To describe the usual topology on $\mathbb{RP}^2$, it is useful to identify this set of lines with the quotient $S^2/x \sim -x$ of the 2-sphere obtained by identifying antipodal points (the bijection is given by sending a point $x \in S^2$ to the line spanned by the unit vector x; since x and $-x$ span the same line, this gives a well-defined map $S^2/\sim \to \{\text{lines in } \mathbb{R}^3\}$ which is easily seen to be bijective). The usual topology of S^2 then induces the quotient topology on $\mathbb{RP}^2 = S^2/\sim$.

Another way to think of $\mathbb{RP}^2$ comes from noting that any equivalence class $[x] \in S^2/\sim$ is represented by a point in the upper hemisphere $S^2_+ = \{(x_1, x_2, x_3) \in S^2 \mid x_3 \geq 0\}$ which can be identified with the disk D^2 by sending $(x_1, x_2, x_3) \in S^2_+$ to $(x_1, x_2) \in D^2$. This shows that $\mathbb{RP}^2$ is homeomorphic to the quotient $D^2/\sim$, where the equivalence relation identifies antipodal points on the boundary of D^2. In other words, it identifies points of the upper semicircle with the corresponding points in the lower semicircle as indicated by the following picture.

$\mathbb{RP}^2 \approx$ 

(1.11)

Interpreting this picture as a pattern of polygons on $\mathbb{RP}^2$, we see that

$$\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$$

Another way to calculate the Euler characteristic of the real projective plane is to note that the projection map

$$S^2 \longrightarrow S^2/\sim \quad = \quad \mathbb{RP}^2$$

is a double covering. The following implies that $\chi(S^2) = 2\chi(\mathbb{RP}^2)$ and hence $\chi(\mathbb{RP}^2) = 1$ by our previous calculation of the Euler characteristic of the sphere.

Lemma 1.12. *If Σ is a closed surface and $p\colon \widetilde{\Sigma} \to \Sigma$ is a d-fold covering map, then $\chi(\widetilde{\Sigma}) = d\chi(\Sigma)$.*

The proof of this lemma is a homework problem.

1.3 Homology groups of surfaces

The goal in this section is to define homology groups for closed surfaces. These are abelian groups $H_n(\Sigma)$ associated to a closed surface Σ for $n \in \mathbb{Z}$. We begin by a quick review of abelian groups.

Digression on abelian groups. We will write abelian groups A additively, that is, we write $a + b \in A$ for the group operation applied to two elements $a, b \in A$, and $-a$ for the inverse of a. As usual, we write na as shorthand for the n-fold sum $a + \cdots + a$ of an element $a \in A$, and $-na$ for the n-fold sum $(-a) + \cdots + (-a)$. In particular, we can multiply an element $a \in A$ with any integer. This multiplication gives A the structure of a $\mathbb{Z}$-module. In fact, *abelian group* and *$\mathbb{Z}$-module* are just two different names for the same mathematical structure.

Here are some examples of abelian groups (aka $\mathbb{Z}$-modules).

- The infinite cyclic group $\mathbb{Z}$;
- The cyclic group $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$ of order n. Here $n\mathbb{Z} \subset \mathbb{Z}$ is the subgroup consisting of integers divisible by n. We write $[i] \in \mathbb{Z}/n$ for the coset represented by $i \in \mathbb{Z}$.
- If A, B are abelian groups, we can form their sum $A \oplus B$. The elements of this abelian group are all pairs (a, b) with $a \in A$, $b \in B$. The sum of two such pairs is given by $(a, b) + (a', b') = (a + a', b + b')$.
- If S is a set, the *free abelian group generated by S* or *free $\mathbb{Z}$-module generated by S*, denoted $\mathbb{Z}[S]$ is defined by

  $$\mathbb{Z}[S] := \left\{ \sum_{s \in S} n_s s \mid n_s \in \mathbb{Z},\ n_s \neq 0 \text{ for only finitely many } s \in S \right\}$$

  In other words, the elements of $\mathbb{Z}[S]$ consist of the finite linear combinations of elements of S with integer coefficients.

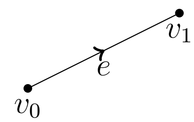
We recall the following important facts about abelian groups:

- The sum $\mathbb{Z}/m \oplus \mathbb{Z}/n$ is isomorphic to $\mathbb{Z}/mn$ if and only if m is prime to n.
- Any finitely generated group is isomorphic to a sum of $\mathbb{Z}$'s and $\mathbb{Z}/n$'s. Without loss of generality we can assume that the n's are powers of primes. We recall that an abelian group A is *finitely generated* if there are finitely many elements $a_i \in A$ such that every element $a \in A$ can be expressed as a linear combination of the a_i's.

Like for the definition of the Euler characteristic, the definition of the homology group $H_n(\Sigma)$ for a closed surface Σ requires us to first choose some additional structure on the surface. To construct the homology groups, we need to make the following choices:

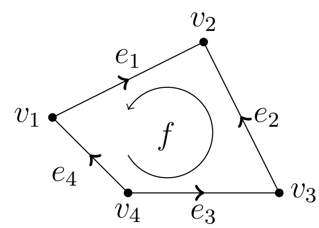
1. The choice of a pattern of polygons on Σ;
2. We need to choose an "orientation" for each edge and each face.

(a) For an edge, this means giving it a direction, which we indicate in pictorially by an arrow:



Thinking of the edge e as an arrow, we will refer to the vertices v_1 resp. v_0 as the *tip* resp. *tail* of the edge e, and write $\mathrm{tip}(e) = v_1$, $\mathrm{tail}(e) = v_0$.

(b) An orientation for a face means a direction for the boundary circle (clockwise or anti-clockwise), which we indicate in pictures as follows:



(1.13)

These choices allow us to construct the following abelian groups and homomorphisms:

$$\mathbb{Z}[V] \xleftarrow{\partial_1} \mathbb{Z}[E] \xleftarrow{\partial_2} \mathbb{Z}[F] \tag{1.14}$$

Here V (resp. E resp. F) is the set of vertices (resp. edges resp. faces) of the pattern of polygons that we picked, and $\mathbb{Z}[V]$ (resp. $\mathbb{Z}[E]$ resp. $\mathbb{Z}[F]$) is the free abelian group generated by these sets. Given an edge $e \in E$, then

$$\partial_1(e) = \mathrm{tip}(e) - \mathrm{tail}(e) \in \mathbb{Z}[V],$$

and this determines the map ∂_1 by linearity. If $f \in F$ is a face,

$$\partial_2(f) = \sum_{e \in \partial f} \pm e \in \mathbb{Z}[E].$$

Here the sum is over all edges e of the polygon f; the sign of an edge e is positive if the direction of e and the direction of f agree, and negative otherwise.

For example, for the face shown in (1.13) we have

$$\partial_2(f) = -e_1 + e_2 + e_3 - e_4.$$

We note that

$$\begin{aligned}\partial_1(\partial_2(f)) &= \partial_1(-e_1 + e_2 + e_3 - e_4)\\ &= (-\mathrm{tip}(e_1) + \mathrm{tail}(e_1)) + (\mathrm{tip}(e_2) - \mathrm{tail}(e_2)) + (\mathrm{tip}(e_3) - \mathrm{tail}(e_3)) + (-\mathrm{tip}(e_4) + \mathrm{tail}(e_4))\\ &= (-v_2 + v_1) + (v_2 - v_3) + (v_3 - v_4) + (-v_1 + v_4) = 0.\end{aligned}$$

This is true in general: for the maps ∂_1, ∂_2 in (1.14) associated for any surface Σ and the choice Γ of a pattern of polygons on Σ and orientations for all of its edges and faces (while in the section on the Euler characteristic we used Γ to denote the pattern of polygons, from now on Γ includes the choice of orientations, since they are required for the construction of the maps ∂_1 and ∂_2).

Lemma 1.15. *The maps ∂_1, ∂_2 of* (1.14) *have the property $\partial_1 \circ \partial_2 = 0$.*

Definition 1.16. A *chain complex* is a sequence

$$\xleftarrow{\partial_{k-1}} C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}}$$

of $\mathbb{Z}$-modules C_k and module maps $\partial_k \colon C_k \to C_{k-1}$ for $k \in \mathbb{Z}$, such that $\partial_k \circ \partial_{k+1} = 0$ for all $k \in \mathbb{Z}$. We typically abbreviate by writing (C_*, ∂_*) or just C_* for a chain complex, where $*$ is a placeholder for an index $n \in \mathbb{Z}$.

We note that (1.14) can be interpreted as a chain complex by setting

$$C_k := \begin{cases} \mathbb{Z}[V] & k = 0 \\ \mathbb{Z}[E] & k = 1 \\ \mathbb{Z}[F] & k = 2 \\ 0 & k \neq 0, 1, 2 \end{cases}.$$

We use the notation $C_*(\Sigma, \Gamma)$ for this chain complex, where Γ is a pattern of polygons on Σ equipped with orientations of edges and faces.

Terminology. Motivated by this example, the maps ∂_i are called *boundary maps*. An element $c \in C_k$ is a k-chain. If c is in the kernel of $\partial_k \colon C_k \to C_{k-1}$, it is a k-cycle, and if it is in the image of $\partial_{k+1} \colon C_{k+1} \to C_k$, it is a *$k$-boundary*. We note that the condition $\partial_k \circ \partial_{k+1} = 0$ implies that any k-boundary is a k-cycle. Furthermore, we write

$$Z_k := \{k\text{-cycles}\} = \ker(\partial_k \colon C_k \to C_{k-1})$$

for the $\mathbb{Z}$-module of k-cycles and

$$B_k := \{k\text{-boundaries}\} = \operatorname{im}(\partial_{k+1} \colon C_{k+1} \to C_k)$$

for the submodule of k-boundaries. The the *k-homology* is defined to be the quotient module

$$H_k := \frac{Z_k}{B_k} = \frac{\{k\text{-cycles}\}}{\{k\text{-boundaries}\}}.$$

If there is more than one chain complex around we use the notation

$$Z_k(C_*, \partial_*),\ B_k(C_*, \partial_*),\ H_k(C_*, \partial_*) \qquad \text{or} \qquad Z_k(C_*),\ B_k(C_*),\ H_k(C_*)$$

to indicate that we are talking about cycles, boundaries, or homology classes of the chain complex $C_* = (C_*, \partial_*)$.

Proposition 1.17. *The homology groups $H_k(C_*(\Sigma, \Gamma))$ of the chain complex $C_*(\Sigma, \Gamma)$ are independent of Γ. More precisely, if Γ and Γ' are two choices of a pattern of polygons on Σ and orientations of edges and faces in that pattern, then they determine an isomorphism*

$$\Phi_{\Gamma,\Gamma'} \colon H_k(C_*(\Sigma, \Gamma)) \xrightarrow{\simeq} H_k(C_*(\Sigma, \Gamma')).$$

This result allows us to define the k-th homology group $H_k(\Sigma)$ of a surface Σ by

$$H_k(\Sigma) := H_k(C_*(\Sigma, \Gamma)),$$

where Γ is a pattern of polygons on Σ equipped with orientations of its edges and faces.

Proposition 1.17 can be proved with the same strategy as in the proof that the Euler characteristic of a surface is independent of the choice of the pattern of polygons used in its definition: any two patterns have a common refinement, and hence it suffices to show that the homology group don't change if we refining a pattern Γ by

- subdividing an edge by putting an extra vertex on it, or by
- subdividing a face by connecting two of non-adjacent vertices by an extra edge.

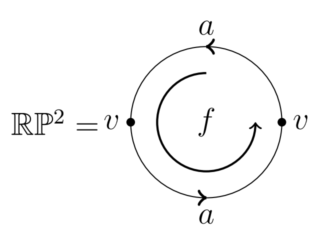
This is not hard to do, but we skip the proof, since we are much more ambitious: we would like to define homology groups $H_k(X)$

- for *any topological space*, rather than just for surfaces, and
- without the need to fix additional choices, like the the choice of a pattern of polygons above.

This will be done in the following section.

Calculation of the homology groups of some surfaces.

Real projective plane As in (1.11) we will use pattern of polygons on the real projective plane $\mathbb{RP}^2$ by thinking of it as a quotient of the 2-gon with edge-identifications given by the following picture.



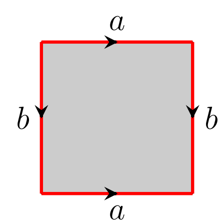
In other words, we have one vertex v, one edge a and one face f. The picture also indicates which orientation we pick for a and f. It follows that the associated chain complex $C_* = C_*(\mathbb{RP}^2, \Gamma)$ has the form:

$$C_0 = \mathbb{Z}v \xleftarrow{\partial_1} C_1 = \mathbb{Z}a \xleftarrow{\partial_1} C_2 = \mathbb{Z}f$$

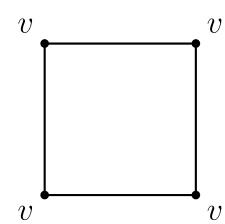
with $\partial_1(a) = \text{tip}(a) - \text{tail}(a) = v - v = 0$ and $\partial_2(f) = 2a$. It follows that

$$H_0 = \frac{Z_0}{B_0} = \frac{\mathbb{Z}v}{\{0\}} \cong \mathbb{Z}$$
$$H_1 = \frac{Z_1}{B_1} = \frac{\mathbb{Z}a}{\mathbb{Z}2a} \cong \mathbb{Z}/2\mathbb{Z}$$
$$H_2 = \frac{Z_2}{B_2} = \frac{\{0\}}{\{0\}} = \{0\}$$

Torus The torus T can be obtained as a quotient space of the square by gluing the edges with the same label as shown in the following picture.



$T =$



identifying the We will use the pattern Γ of polygons and orientations on the torus T

Klein bottle

Sphere

Surface of genus g

2 Singular homology

The goal of this section is to define homology groups $H_k(X)$ for any topological space X without the choice of auxiliary data, like the patterns of polygons in our discussion of homology groups of surfaces in section 1.3. An important role in that discussion was played by vertices, edges and faces on a surface X, and so the first step is to generalize those. We note that it suffices to restrict us to triangles on a surface X rather than work with more general polygons, since we can always subdivide polygons into triangles by introducing additional edges, which doesn't change the homology groups according to Proposition 1.17. We note that we can think of vertices (resp. edges resp. triangles) on X as given by continuous maps

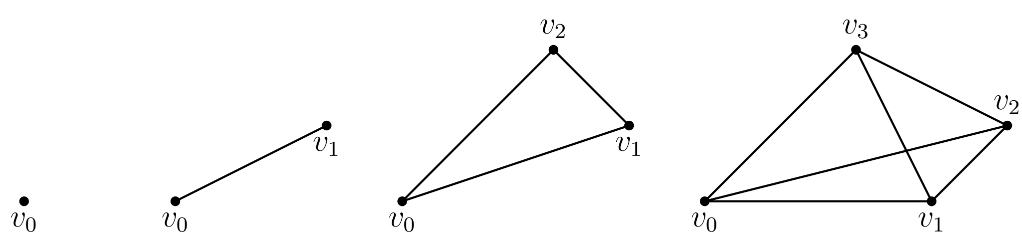
$$S \to X,$$

where $S \subset \mathbb{R}^m$ is a point (resp. a line segment resp. a triangular shaped region).

Definition 2.1. A subset $S \subset \mathbb{R}^m$ is a *k-simplex* if S is the convex hull of points $v_0, \ldots v_k \in \mathbb{R}^m$ such that the vectors $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent. We note that every element $s \in S$ can be written uniquely as an *affine linear combination of the v_is*, that is, in the form

$$s = \sum_{i=0}^{k} t_i v_i \qquad \text{with } t_i \in [0, \infty) \text{ and } \sum_{i=0}^{k} t_i = 1.$$

The *standard simplex of dimension k*, is the convex hull of $e_0, e_1, \ldots, e_k \in \mathbb{R}^{k+1}$, where $\{e_i\}_{i=0}^k$ is the standard basis of $\mathbb{R}^{k+1}$. We will use the notation $\Delta^k \subset \mathbb{R}^{k+1}$ for the standard k-simplex.



0-simplex 1-simplex 2-simplex 3-simplex

Definition 2.2. A *singular k-simplex* in a topological space X is a continuous map

$$\sigma \colon \Delta^k \longrightarrow X.$$

For example, a vertex (resp. edge resp. triangle) on a surface X is a singular k-simplex in X for $k = 0$ (resp. $k = 1$ resp. $k = 2$). The adjective 'singular' refers to the fact that a singular k-simplex is allowed to be quite degenerate, for example it could be the constant map.

Proposition 2.3. *Let* pt *be the topological space consisting of one point. Then*

$$H_k(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Proposition 2.4. *Let X be a space with path connected components X_α. Show that $H_d(X)$ is isomorphic to the direct sum $\bigoplus_\alpha H_d(X_\alpha)$.*

Contrasting $\bigoplus$ and $\prod$.

Proposition 2.5. *If X is a path connected space, then $H_0(X)$ is isomorphic to $\mathbb{Z}$.*

Corollary 2.6. *For any space X its homology group $H_0(X)$ is the direct sum of as many copies of $\mathbb{Z}$ as there are connected components of X.*

Proof. The desired isomorphism $H_0(X) = C_0(X)/B_0(X) \to \mathbb{Z}$ is induced by the homomorphism

$$\epsilon\colon C_0(X) \longrightarrow \mathbb{Z} \qquad \sum n_i x_i \mapsto \sum n_i,$$

which is called the *augmentation*. We note that the composition

$$C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

is trivial, since if $\sigma\colon \Delta^1 \to X$ is a 1-simplex, then $\partial\sigma = \partial_0\sigma - \partial_1\sigma = \sigma(v_1) - \sigma(v_0)$, and hence $\epsilon\partial\sigma = \epsilon(\sigma(v_1) - \sigma(v_0)) = +1 - 1 = 0$. This shows that ϵ induces a well-defined homomorphism $\bar\epsilon\colon H_0(X) = C_0(X)/B_0(X) \to \mathbb{Z}$.

It is clear that $\bar\epsilon$ is surjective; to prove injectivity, assume that $\bar\epsilon([\sum n_i x_i]) = \sum n_i$ is zero. To show that $[\sum n_i x_i]$ is the trivial homology class, we need to construct a 2-chain c with $\partial c = \sum n_i x_i$. To construct c, we pick a base point x_0 and paths $\lambda_i\colon I \to X$ from x_0 to x_i. If we consider λ_i as a 1-simplex, we have

$$\partial\lambda_i = \partial_0\lambda_i - \partial_1\lambda_i = x_i - x_0,$$

and hence for $c = \sum n_i x_i$, we get

$$\partial c = \partial(\sum n_i\lambda_i) = \sum n_i x_i - (\sum n_i)x_0 = \sum n_i x_i$$

as desired. □

2.1 Relating the first homology group and the fundamental group

We recall that elements of the fundamental group $\pi_1(X, x_0)$ of a topological space X with basepoint x_0 are homotopy classes of based loops, that is paths $\gamma\colon I \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_0$. Identifying the unit interval I with the standard 1-simplex Δ^1 via the affine linear map $\Delta \to I$ that sends e_0 to 0 and e_1 to 1, we can interpret the path γ as a 1-simplex. We note that γ is in fact a *cycle*, that is γ belongs to $Z_1(X) = \ker(\partial\colon C_1(X) \to C_0(X))$, since

$$\partial\gamma = \partial_0\gamma - \partial_1\gamma = \gamma(1) - \gamma(0) = x_0 - x_0 = 0 \in C_0(X).$$

Let us denote in this section by $[[\gamma]] \in H_1(X) = Z_1(X)/B_1(X)$ the homology class it represents. The map

$$h\colon \pi_1(X, x_0) \longrightarrow H_1(X) \qquad \text{given by} \qquad [\gamma] \mapsto [[\gamma]]$$

is called the *Hurewicz homomorphism.*

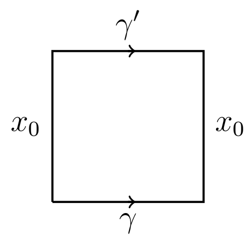
Lemma 2.7. *h is a well-defined group homomorphism.*

Proof. We will show that h is well-defined. The proof that h is a group homomorphism is left as a homework problem.

Let γ, γ' be two based loops such that $[\gamma] = [\gamma'] \in \pi_1(X, x_0)$. Let $H\colon I \times I \to X$ be a homotopy of based loops between γ and γ'. In other words,

$$H(s,0) = \gamma(s) \qquad H(s,1) = \gamma'(s) \qquad H(0,t) = H(1,t) = x_0 \qquad \text{for all } s \in I,\ t \in I.$$

More visually, this can be represented by the picture

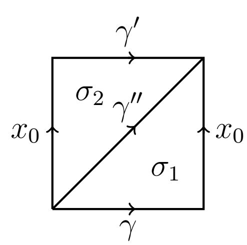


Here the labels next to the edges tell us where H maps these edges: on the bottom (resp. top) horizontal edge H restricts to the path γ (resp. γ') while the vertical edges both map to the basepoint x_0.

Showing that h is well-defined amounts to proving that $h([\gamma]) = h([\gamma']) \in H_1(X) = Z_1(X)/B_1(X)$. In other words, we need to show that the cycles $\gamma, \gamma' \in Z_1(X)$ represent the same homology class, that is

$$\gamma - \gamma' \in B_1(X) = \operatorname{im}(\partial_2\colon C_2(X) \to C_1).$$

So we need to construct 2-simplices such that ∂_2 of a linear combination of these is $\gamma - \gamma'$. The idea is to decompose the square I^2 into two triangles, and to define singular 2-simplices σ_1, σ_2 by restricting $H\colon I^2 \to X$ to these two triangles. Here is a picture:



□

The Hurewicz homomorphism $h\colon \pi_1(X, x_0) \to H_1(X)$ maps any commutator $[g, h] = ghg^{-1}h^{-1} \in \pi = \pi_1(X, x_0)$ to the trivial element in $H_1(X)$ since $H_1(X)$ is an abelian group. In particular, the commutator subgroup $[\pi, \pi] \subset \pi$, the normal subgroup of π generated by all commutators, maps to 0. Hence h induces a well-defined homomorphism

$$\bar{h}\colon \pi_1^{\mathrm{ab}}(X, x_0) \longrightarrow H_1(X)$$

from the *abelianized fundamental group* $\pi^{\mathrm{ab}} := \pi/[\pi, \pi]$ to $H_1(X)$.

Theorem 2.8. *For any path connected topological space X the map $\bar{h}\colon \pi_1^{\mathrm{ab}}(X, x_0) \to H_1(X)$ is a group isomorphism.*

Proof. The idea is to construct an inverse to $\bar{h}$ as follows. Choose for every point $x \in X$ a path λ_x from x_0 to x. Define the map

$$\Psi\colon C_1(X)/B_1(X) \longrightarrow \pi_1^{ab}(X, x_0) \qquad \text{by} \qquad [[\gamma]] \mapsto [\lambda_{\gamma(0)} * \gamma * \bar{\lambda}_{\gamma(1)}]$$

for any singular 1-simplex γ, also known as path $\gamma\colon I \to X$. Here $\lambda_{\gamma(0)} * \gamma * \bar{\lambda}_{\gamma(1)}$ is the concatenation of the path $\lambda_{\gamma(0)}$ (from x_0 to $\gamma(1)$), the path γ (from $\gamma(0)$ to $\gamma(1)$) and the path $\bar{\lambda}_{\gamma(1)}$ (from $\gamma(1)$ to x_0, obtained by running the path $\lambda_{\gamma(1)}$ from x_0 to $\gamma(1)$ backwards).

We show here that Ψ is a well-defined map, and will leave it to the reader to show that Ψ restricted to $H_1(X) = Z_1(X)/B_1(X) \subset C_1(X)/B_1(X)$ is in fact an inverse to $\bar{h}$.

We recall that $B_1(X)$ is the image of $\partial_2\colon C_2(X) \to C_1(X)$, and hence $B_1(X)$ is generated by the elements $\partial_2(\sigma)$, where $\sigma\colon \Delta^2 \to X$ is a singular 2-simplex.

We recall that the element of $\pi_1(X) = \pi_1(X, x_0)$ are homotopy class of a based loops, where a based loop is a path $\gamma\colon I \to X$ with $\gamma(0) = \gamma(1) = x_0$. Such a based loop can be regarded as a 1-simplex or 1-chain. In fact, γ is a *cycle*, since

$$\partial\gamma = \partial_0\gamma - \partial_1\gamma = \gamma(1) - \gamma(0) = x_0 - x_0 = 0.$$

We will show that the map

$$h\colon \pi_1(X; x_0) \longrightarrow H_1(X) \qquad [\gamma] \mapsto [[\gamma]],$$

known as the *Hurewicz homomorphism* is well-defined and a homomorphism. Here we write $[\gamma] \in \pi_1(X; x_0)$ for the *homotopy class* of the based loop γ and $[[\gamma]$
$inH_1(X)$ for the *homology class* of the 1-cycle γ. To distinguish the two different equivalence relations involved, we will write $\gamma \cong \delta$ if two based loops γ, δ are homotopic relative base point and we will write $c \sim d$ if two 1-chains are homologous.

well-defined So to prove that the Hurewicz map is well-defined amounts to showing that $\gamma \cong \delta$ implies $\gamma \sim \delta$ (we note that γ, δ can be interpreted as 1-chains).

So suppose that $H\colon I \times I \to X$ is a based homotopy between γ and δ. Then we need to show that γ is homologous to δ, i.e., that there is a 2-chain c with $\partial c = \gamma - \delta$. This chain can be manufactured out of the homotopy H by subdividing the square $I \times I$ into two triangles as shown in the picture below

homomorphism

To show that the Hurewicz homomorphism is an *isomorphism*, we will construct a map $\Psi_*\colon H_1(X) \to \pi_1^{ab}(X)$ which we will prove to be an inverse to h. As a first step towards constructing Ψ_*, we pick for every point $x \in X$ a path $\lambda_x\colon I \to X$ from the base point x_0 to x and define a homomorphism

$$\Psi\colon C_1(X) \longrightarrow \pi_1^{ab}(X, x_0) \qquad \text{by} \quad \gamma \mapsto [\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}].$$

Here $\gamma\colon I \to X$ is a 1-simplex; the concatenation $\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}$ of the three paths $\lambda_{\gamma(0)}$, γ, and $\bar{\lambda}_{\gamma(1)}$ then is a based loop and hence represents an element $[\lambda_{\gamma(0)}\gamma\bar{\lambda}_{\gamma(1)}]$ in $\pi_1^{ab}(X, x_0)$. We extend this map by linearity to the chain group $C_1(X)$ (which we recalll is the free abelian group generated by the 1-simplices in X; note that for this step we need to work with the *abelianized* fundamental group). We note that the above map is not canonical, but rather depends on our choices of the paths λ_x.

We claim that Ψ vanishes on the boundaries $B_1(X) \subset C_1(X)$. To show this, it suffices to prove $\Psi(\partial\sigma) = 0$ for any 2-simplex σ in X. Let $y_i \overset{\text{def}}{=} \sigma(e_i) \in X$ be the three vertices of σ, and let $\gamma_i \overset{\text{def}}{=} \partial_i\sigma\colon I \to X$ be the three 1-dimensional faces of σ. Then we have:

$$\begin{aligned}
\Psi(\partial\sigma) &= [\lambda_{y_1}\gamma_0\bar{\lambda}_{y_2}] - [\lambda_{y_0}\gamma_1\bar{\lambda}_{y_2}] + [\lambda_{y_0}\gamma_2\bar{\lambda}_{y_1}] \in \pi_1^{ab}(X)\\
&= [\lambda_{y_1}\gamma_0\bar{\lambda}_{y_2}\overline{\lambda_{y_0}\gamma_1\bar{\lambda}_{y_2}}\lambda_{y_0}\gamma_2\bar{\lambda}_{y_1}] = [\lambda_{y_1}\gamma_0\bar{\lambda}_{y_2}\lambda_{y_2}\bar{\gamma}_1\bar{\lambda}_{y_0}\lambda_{y_0}\gamma_2\bar{\lambda}_{y_1}]\\
&= [\lambda_{y_1}\gamma_0\bar{\gamma}_1\gamma_2\bar{\lambda}_{y_1}] = [\lambda_{y_1}c_{y_1}\bar{\lambda}_{y_1}] = 0
\end{aligned}$$

Here c_{y_1} is the *constant* loop based at y_1; it is homotopic (relative endpoints) to the closed loop $\gamma_0\bar{\gamma}_1\gamma_2$.

$\Psi_* \circ h = 1$

$h \circ \Psi_* = 1$

□

2.2 Properties of singular homology groups: the Eilenberg-Steenrod axioms

In this section we will state properties of singular homology groups, some of which we have already proved. These properties suffice to calculate the homology groups for large classes of topological spaces, for example CW complexes (see section **??**). For this reason these properties can be viewed as *axioms* for homology groups, the *Eilenberg-Steenrod axioms*, named after Samuel Eilenberg and Norman Steenrod.

2.2.1 Functoriality of homology

One of the fundamental properties of singular homology, like any good mathematical construction, is that it is *functorial.* More precisely, there is a functor

$$H_k\colon \mathsf{Top} \longrightarrow \mathbb{Z}\text{-}\mathsf{Mod}$$

from the category Top of topological spaces and continuous maps to the category $\mathbb{Z}\text{-}\mathsf{Mod}$ of $\mathbb{Z}$-modules (aka abelian groups) and homomorphisms (see Appendix 5 for the definition of categories and functors). On objects, this functor sends a topological space X to the singular homology group $H_k(X)$. On morphisms, it maps a continuous map $f\colon X \to Y$ to a homomorphism

$$f_* := H_k(f)\colon H_k(X) \longrightarrow H_k(Y)$$

that we will construct in this section.

Let $f\colon X \to Y$ to be a map (which later we will assume to be a homeomorphism). Then we can manufacture a map between the sets of k-simplices of X resp. Y

$$S_k(f)\colon S_k(X) \longrightarrow S_k(Y) \qquad \text{given by} \qquad \Delta^k \xrightarrow{\sigma} X \;\mapsto\; \Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$$

This in turn gives a homomorphism

$$C_k(f)\colon C_k(X) = \mathbb{Z}[S_k(X)] \longrightarrow C_k(Y) = \mathbb{Z}[S_k(Y)]$$

between the free $\mathbb{Z}$-modules generated by the sets $S_k(X)$ resp. $S_k(Y)$.

Lemma 2.9. *The diagram*

$$\begin{array}{ccc} C_k(X) & \xrightarrow{\partial} & C_{k-1}(X) \\ \downarrow{\scriptstyle C_k(f)} & & \downarrow{\scriptstyle C_{k-1}(f)} \\ C_k(Y) & \xrightarrow{\partial} & C_{k-1}(Y) \end{array}$$

is commutative.

This lemma means that the homomorphisms $C_k(f)$, $k \in \mathbb{Z}$, fit together to give a chain homomorphism $C_*(f)\colon C_*(X) \to C_*(Y)$ in the following sense.

Definition 2.10. Let C_*, D_* be chain complexes. A *chain map* $g_*\colon C_* \to D_*$ is a sequence of homomorphisms $g_k\colon C_k \to D_k$ such that the diagram

$$\begin{array}{ccccccc} \xleftarrow{\partial} & C_{k-1} & \xleftarrow{\partial} & C_k & \xleftarrow{\partial} & C_{k+1} & \longleftarrow \\ & \downarrow{\scriptstyle g_{k-1}} & & \downarrow{\scriptstyle g_k} & & \downarrow{\scriptstyle g_{k+1}} & \\ \xleftarrow{\partial} & D_{k-1} & \xleftarrow{\partial} & D_k & \xleftarrow{\partial} & D_{k+1} & \longleftarrow \end{array} \tag{2.11}$$

is commutative.

Lemma 2.12. *A chain map* $g_*\colon C_* \to D_*$ *induces a homomorphism of homology groups* $H_k(g_*)\colon H_k(C_*) \to H_k(D_*)$.

Proof. The commutative diagram (2.11) implies that $g_k\colon C_k \to D_k$ sends k-cycles (resp. boundaries) in C_* to k-cycles (resp. boundaries) in D_*. It follows that g_k induces a well-defined homomorphism

$$H_k(g_*)\colon H_k(C_*) = \frac{Z_k(C_*)}{B_k(C_*)} \longrightarrow \frac{Z_k(D_*)}{B_k(D_*)} = H_k(D_*).$$

□

Definition 2.13. Given a map $f\colon X \to Y$ between topological spaces, the homomorphism $H_k(C_*(f))\colon H_k(X) \to H_k(Y)$ is called the *homomorphism induced by f on homology.* We will simplify notation by writing $H_k(f)$ or simply f_* instead of $H_k(C_*(f))$.

Lemma 2.14. *Let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be maps between topological spaces, and* $f_*\colon H_k(X) \to H_k(Y)$, $g_*\colon H_k(Y) \to H_k(Z)$ *be the induced maps on homology. Then*

$$(g \circ f)_* = g_* \circ f_* \qquad \textit{and} \qquad (\mathrm{id}_X)_* = \mathrm{id}_{H_k(X)}$$

We leave the easy proof of these properties to the reader. We observe that the lemma implies that we in fact obtain a functor $H_k\colon \mathsf{Top} \to \mathbb{Z}\text{-}\mathsf{Mod}$ by mapping a topological space X to $H_k(X)$ and a continuous map $f\colon X \to Y$ to the induced map in homology $f_*\colon H_k(X) \to H_k(Y)$. The functoriality of H_k implies for example that if $f\colon X \to Y$ is an isomorphism in Top, that is, a homeomorphism, then $f_*\colon H_k(X) \to H_k(Y)$ is an isomorphism of $\mathbb{Z}$-modules.

Remark 2.15. The construction of the singular homology group $H_k(X)$ is a two-step process: we first construct the singular chain complex $C_*(X)$ and then define $H_k(X)$ as the k-homology group of $C_*(X)$. The first step of this construction can be viewed as a functor

$$C_*\colon \mathsf{Top} \longrightarrow \mathrm{Ch}$$

from the category of topological spaces to the category of chain complexes and chain maps. On objects, it sends a topological space X to its singular chain complex $C * (X)$, on morphisms, it sends a continuous map $f\colon X \to Y$ to the chain map $C_*(f)\colon C_*(X) \to C_*(Y)$.

The second step of this construction can be interpreted as a functor

$$H_k\colon \mathrm{Ch} \longrightarrow \mathbb{Z}\text{-}\mathsf{Mod}$$

that sends a chain complex M_* to its k-homology group $H_k(M_*)$ and a chain map $g_*\colon M_* \to N_*$ to the induced map $H_k(g_*)\colon H_k(M_*) \to H_k(N_*)$ on the k-th homology group. Here we abuse language by using the notation H_k for this functor as well as for the functor $H_k\colon \mathsf{Top} \to \mathbb{Z}\text{-}\mathsf{Mod}$.

Summarizing, the functor $H_k\colon \mathsf{Top} \to \mathbb{Z}\text{-}\mathsf{Mod}$ factors as

$$\begin{array}{ccc} \mathsf{Top} & \xrightarrow{H_k} & \mathbb{Z}\text{-}\mathsf{Mod}\,. \\ & {\scriptstyle C_*}\searrow \quad \nearrow{\scriptstyle H_k} & \\ & \mathrm{Ch} & \end{array}$$

2.2.2 The Eilenberg Steenrod axioms for homology

Theorem 2.16. (Properties of singular homology)

functoriality *Sending a topological space X to the k-th homology group $H_k(X)$ and a map $f\colon X \to Y$ to the induced homomorphism $f_*\colon H_k(X) \to H_k(Y)$ defines a functor $H_k\colon \mathsf{Top} \longrightarrow \mathbf{Ab}$ from the category of topological spaces to the category of abelian groups.*

dimension axiom *The homology groups of the one-point space* pt *are given by $H_0(\mathrm{pt}) \cong \mathbb{Z}$ and $H_k(\mathrm{pt}) = 0$ for $k \neq 0$.*

additivity *If X is a disjoint union of spaces X_α with inclusion maps $i_\alpha\colon X_\alpha \longrightarrow X$, then the direc sum map*

$$\bigoplus_\alpha (i_\alpha)_*\colon \bigoplus_\alpha H_k(X_\alpha) \longrightarrow H_k(X)$$

is an isomorphism for each k.

homotopy invariance *If $f, g\colon X \longrightarrow Y$ are homotopic maps, then $f_* = g_*\colon H_q(X) \longrightarrow H_q(Y)$.*

Mayer-Vietoris sequence *Let X be a topological space that is the union of open subspaces U, V. Then there is a long exact sequence*

$$\longrightarrow H_k(U \cap V) \xrightarrow{\Phi} H_k(U) \oplus H_k(V) \xrightarrow{\Psi} H_k(X) \xrightarrow{\partial} H_{k-1}(U \cap V) \longrightarrow$$

Here $\Phi(\alpha) = (i^U_*(\alpha), i^V_*(\alpha))$ and $\Psi(\beta, \gamma) = j^U_*(\beta) - j^V_*(\gamma)$, where $i^U: U \cap V \to U$, $i^V: U \cap V \to V$, $j^U: U \to X$ and $j^V: V \to X$ are inclusion maps. The homomorphism ∂ is called the boundary map. *It has the following* naturality property*: if X' is another space which is the union of open subspaces U', V' and $f\colon X \to X'$ is a map with $f(U) \subset U'$ and $f(V) \subset V'$, then the following diagram is commutative:*

$$\begin{array}{ccc} H_k(X) & \xrightarrow{\partial} & H_{k-1}(U \cap V) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f_{|U\cap V})_*} \\ H_k(X') & \xrightarrow{\partial} & H_{k-1}(U' \cap V') \end{array}$$

Comments on these properties.

Homotopy invariance We recall from Example **??** that any vector valued function $f\colon X \to V$ on a topological space X is homotopic to the zero map. So the homotopy invariance implies that the induced map in homology is completely useless for studying these functions. On the positive side, the goal of algebraic topology is to study “qualitative” phenomema, i.e., an algebraic topologist should not attempt to distinguish between homotopic maps, since the existence of a homotopy between them means that they can be deformed into one another, and so they should be thought of as “qualitatively the same”.

From that point of view the basic category an algebraic topologist is interested in is not so much the category Top of topological spaces and continuous maps, but rather the *homotopy category of topological spaces* HoTop whose objects are topological spaces, but whose morphisms $\mathsf{HoTop(X,Y)}$ from X to Y consists of the set $[X,Y]$ of *homotopy classes* of continuous maps from X to Y. More precisely, refining our philosophy as expressed in **??**, we can say the goal of an algebraic topologist is to study functors from the homotopy category HoTop to some algebraic category.

The Homotopy Invariance implies that singular homology can be thought of as such a functor, namely as the functor

$$H_k\colon \mathsf{HoTop} \longrightarrow \mathbf{A}\mathsf{b}$$

that sends a topological space X to the homology group $H_k(X)$ and the homotopy class of a map $f\colon X \to Y$ to the induced map $f_*\colon H_k(X) \to H_k(Y)$.

In particular, if topological spaces X, Y are isomorphic in the homotopy category, their homology groups are isomorphic. Let us unwind what it means for X, Y to be isomorphic in HoTop. It will be convenient to write $[f] \in \mathsf{HoTop(X,Y)}$ for the morphism in the homotopy category represented by a map $f\colon X \to Y$. If X and Y are isomorphic in HoTop if and only if there are $[f] \in \mathsf{HoTop(X,Y)}$ and $[g] \in \mathsf{HoTop(Y,X)}$ such that

$$[f] \circ [g] := [f \circ g] = [\mathrm{id}_Y] \qquad \text{and} \qquad [g] \circ [f] := [g \circ f] = [\mathrm{id}_X].$$

In other words, X and Y are isomorphic in HoTop if and only if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y, and $g \circ f$ is homotopic to id_X. In other words, two spaces are isomorphic in HoTop if and only if X and Y are homotopy equivalent.

At first glance the fact that $f_* = g_*$ whenever f is homotopic to g might be seems a disadvantage since it means that two homotopic maps cannot be distinguished by their induced map on homology. Keeping in mind the fact *any two* vector valued functions on a topological space X are homotopic

Keeping in mind the fact that algebraic topology aims to understand *qualitative*

2.2.3 Calculating with the Mayer-Vietoris sequence

In this section we will use the Eilenberg-Steenrod axioms to calculate the homology groups of spheres, the real projective plane and the Klein bottle. The Mayer-Vietoris sequence will be a central role here.

Theorem 2.17.

$$\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Corollary 2.18. *For* $n \geq 1$

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, k \end{cases}$$

The 0-sphere $S^0 \subset \mathbb{R}$ consists of the two points $\{\pm 1\}$ and hence we know by Proposition **??** that $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_k(S^0) = 0$ for $k \neq 0$. In particular, $\widetilde{H}_k(S^0) = 0$ for $k \neq 0$, and $\widetilde{H}_0(S^0) \cong \mathbb{Z}$. This shows that the *reduced* homology groups of the n-sphere have a simpler pattern that the homology groups, making it slightly easier to calculate their reduced homology groups by induction over n starting at $n = 0$.

Proof of Theorem. The idea is to write S^n as a union $S^n = U \cup V$ of subspaces U, V such that we know the homology groups of U, V and $U \cap V$. Then the Mayer-Vietoris sequence allows us to compute the homology groups of S^n in terms of those of U, V and $U \cap V$.

The first choice of U, V might be to take U to be the upper hemisphere and V to be the lower hemisphere of S^n. Then both are homeomorphic to the n-disk D^n, which implies that they are contractible and hence the reduced homology groups vanish. The intersection $U \cap V$ is the equator which is homeomorphic to S^{n-1} and whose homology groups we know by inductive assumption. However, this choice of U, V does not satisfy the assumption for the Mayer-Vietoris sequence that the interiors of U and V should cover S^n. Fortunately, we can make the hemispheres a little bigger to satisfy the assumption without changing the homology groups of U, V or $U \cap V$, a move that we will frequently use in other calculations.

So let us take $U = S^n \setminus \{(0, \ldots, 0, -1)\}$ and $V = S^n \setminus \{(0, \ldots, 0, 1)\}$. Then U and V are both homeomorphic to $\mathbb{R}^n$ via the stereographic projection and hence all reduced homology groups of U and V vanish. Restricting the homeomorphism $U \approx \mathbb{R}^n$ to $U \cap V$, we obtain a homeomorphism $U \cap V \approx \mathbb{R}^n \setminus \{0\}$, and hence $U \cap V$ is homotopy equivalent to S^{n-1}.

Now we consider the Mayer-Vietoris sequence for reduced homology:

$$\longrightarrow \widetilde{H}_k(U) \oplus \widetilde{H}_k(V) \longrightarrow \widetilde{H}_k(S^n) \xrightarrow{\partial} \widetilde{H}_{k-1}(U \cap V) \longrightarrow \widetilde{H}_{k-1}(U) \oplus \widetilde{H}_{k-1}(V) \longrightarrow$$

Since the reduced homology groups of U and V all vanish, this implies that the boundary homomorphism is an isomorphism. Since $U \cap V$ is homotopy equivalent to S^{n-1} we conclude that

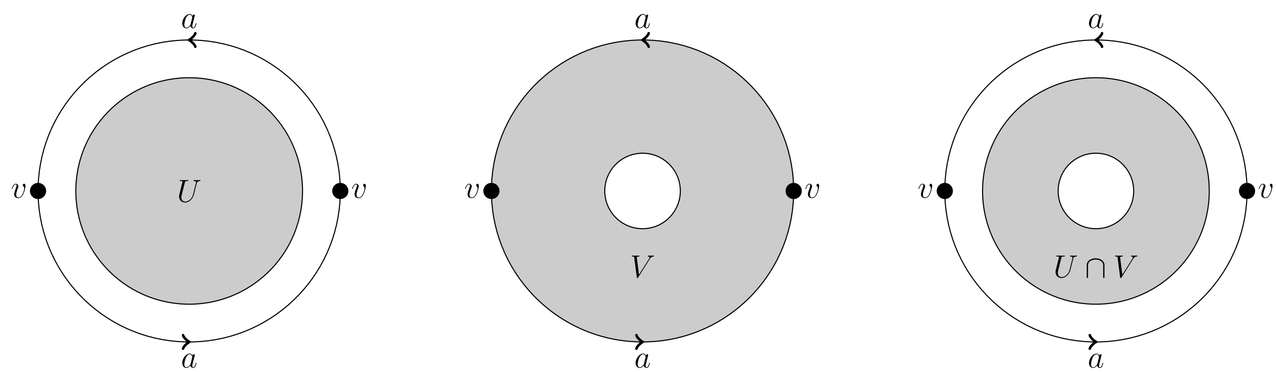
$$\widetilde{H}_k(S^n) \xrightarrow{\cong} \widetilde{H}_{k-1}(U \cap V) \xrightarrow{\cong} \widetilde{H}_{k-1}(S^{n-1})$$

This provides the inductive step of the argument, and finishes the proof. □

Proposition 2.19. *The reduced homology groups of the real projective plane* $\mathbb{RP}^2$ *are given by*

$$\widetilde{H}_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2 & k = 1 \\ 0 & k \neq 1 \end{cases}.$$

Proof. We will think of $\mathbb{RP}^2$ as the disk with antipodal points of the boundary identified. We define open subsets $U, V \subset \mathbb{RP}^2$ as indicated by the following pictures.



(2.20)

We see that U is contractible and hence the reduced homology groups $\widetilde{H}_k(U)$ are zero for all k. The subset V contains the subspace $(S^1/v \sim -v) = \mathbb{RP}^1 \approx S^1$ given by the boundary circle with antipodal points identified. Via radial projection, this subspace is a deformation retract of V and hence

$$\widetilde{H}_k(V) \cong \begin{cases} \mathbb{Z} & k = 1 \\ 0 & k \neq 1 \end{cases}.$$

The subspace $U \cap V$ is homotopy equivalent to S^1 and hence

$$\widetilde{H}_k(V) \cong \begin{cases} \mathbb{Z} & k = 1 \\ 0 & k \neq 1 \end{cases}.$$

This implies that the only portion of the Mayer-Vietoris sequence for reduced homology consisting of possibly non-trivial groups is the following:

$$0 \longrightarrow \widetilde{H}_2(\mathbb{RP}^2) \xrightarrow{\partial} \widetilde{H}_1(U \cap V) \xrightarrow{\Phi} \widetilde{H}_1(U) \oplus \widetilde{H}_1(V) \longrightarrow \widetilde{H}_1(\mathbb{RP}^2) \longrightarrow 0$$

The exactness of the sequence then implies

$$H_k(\mathbb{RP}^2) \cong \begin{cases} 0 & k \neq 1, 2 \\ \ker \Phi & k = 2 \\ \operatorname{coker} \Phi & k = 1 \end{cases}.$$

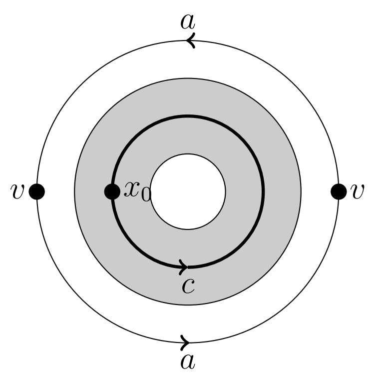
In other words, it remains to calculate the map $\Phi\colon H_1(U\cap V)\to H_1(U)\oplus H_1(V)$ that sends an element $\alpha\in H_1(U\cap V)$ to $(i^U_*(\alpha), i^V_*(\alpha)\in H_1(U)\oplus H_1(V)$, where i^U, i^V are the inclusion maps of $U\cap V$ to U (resp. V). Since $H_1(U)=0$, we can identify the homomorphism Φ with

$$\mathbb{Z}\cong H_1(U\cap V)\xrightarrow{i^V_*} H_1(V)\cong\mathbb{Z}.$$

Given the tools currently at our disposal, it is easier for us to determine an induced map on the fundamental group rather than on H_1. Therefore we use the Hurewicz homomorphism to identify the above map with the homomorphism

$$\mathbb{Z}\cong \pi_1(U\cap V,x_0)\xrightarrow{i^V_*} \pi_1(V,x_0)\cong\mathbb{Z}.$$

A generator for the fundamental group $\pi_1(U\cap V,x_0)$ is given by the loop shown in the following picture.



The spaces $U\cap V$ and V are both homotopy equivalent to S^1, and □

Proposition 2.21. *Let X, Y be topological spaces with basepoints $x_0\in X$, $y_0\in Y$ which*

Proposition 2.22. *The reduced homology groups of the Klein bottle K are given by*

$$\widetilde{H}_k(K)=\begin{cases}\mathbb{Z}\oplus\mathbb{Z}/2 & k=1\\ 0 & k\neq 1\end{cases}.$$

2.3 Classical applications

2.3.1 The Jordan curve theorem and its generalizations

A subset S of the plane $\mathbb{R}^2$ or its one-point compactification $\mathbb{R}^2\cup\{\infty\}\approx S^2$ is called a *Jordan curve* if S is homeomorphic to S^1.

Theorem 2.23. (The Jordan Curve Theorem) *If S is a Jordan curve in $\mathbb{R}^2$ or S^2, then the complement of S has two connected components.*

Theorem 2.24. (The Jordan-Brouwer Separation Theorem) *If S is an $(n-1)$-sphere in S^n, that is a subspace homeomorphic to S^{n-1}, then the complement of S has two connected components.*

We note that the complement $S^n \setminus S$ has two connected components if and only if its homology group $H_0(S^n \setminus S)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ or equivalently, if its reduced homology group $\widetilde{H}_0(S^n \setminus S)$ is isomorphic to $\mathbb{Z}$. Hence the theorem above is a consequence of the following much more general result.

Theorem 2.25. *If S is a k-sphere in S^n, that is a subspace homeomorphic to S^k, then $\widetilde{H}_q(S^n \setminus S) \cong \mathbb{Z}$ for $q = n-k-1$, and trivial for $q \neq n-k-1$.*

Examples of spheres in S^n.

1. For $S = \{0, \infty\} \subset S^2 = \mathbb{R}^2 \cup \{\infty\}$, the complement $S^2 \setminus S = \mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1, and hence the homology group $\widetilde{H}_q(S^2 \setminus S$ is isomorphic to $\mathbb{Z}$ for $q = 1$ and trivial for $q \neq 1$.

2. If S is the standard circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$, the complement $S^3 \setminus S$ is homotopy equivalent to the circle consisting of the z-axis in $\mathbb{R}^3$ and the point ∞. In particular, the $\widetilde{H}_q(S^3 \setminus S)$ is isomorphic to $\mathbb{Z}$ for $q = 1$ and trivial for $q \neq 1$.

3. If S is the trefoil knot, the knot complement $S^3 \setminus S$ is *not* homotopy equivalent to S^1; in fact, its fundamental group is not isomorphic to $\mathbb{Z}$. However, according to the theorem above the complement of *any knot* $S \subset S^3$, that is any subset homeomorphic to S^1, has the same reduced homology groups as the complement of the standard circle of example (2), often referred to as the *unknot*.

Proposition 2.26. *Let $D \subset S^n$ be a k-disk in S^n, that is a subset homeomorphic to the disk D^k. Then $\widetilde{H}_q(S^n \setminus D) = 0$ for all q.*

We will first prove the theorem assuming the proposition.

Proof of Theorem. We will prove the theorem by induction over k. For $k = 0$ the 0-sphere $S \subset S^n$ consists of two points and hence

$$S^n \setminus S \approx (\mathbb{R}^n \cup \{\infty\}) \setminus \{\infty, 0\} = \mathbb{R}^n \setminus \{0\} \sim S^{n-1}.$$

It follows that the reduced homology group $\widetilde{H}_q(S^n \setminus S)$ is isomorphic to $\mathbb{Z}$ for $q = 1$ and trivial for $q \neq 1$ as claimed.

For the inductive step we use the decomposition $S^k = D^k_+ \cup D^k_-$ of the sphere as the union of hemispheres. This leads to a corresponding decomposition

$$S = D_+ \cup D_- \subset S^n$$

where $D_\pm$ is the image of $D^k_\pm$ under the homeomorphism $h\colon S^k \to S$ which exists by the assumption that S is a k-sphere in S^n. We note that $D_\pm$ are k-disks in S^n, and that $S' := D_+ \cap D_- = h(D^k_+ \cap D^k_-) = h(S^{k-1})$ is a $(k-1)$-sphere in S^n. In particular, the reduced homology groups of the complement $D^c_\pm := S^n \setminus D_\pm$ vanish by the proposition, and we know the homology groups of the complement

$$(S')^c := S^n \setminus S' = (D_+ \cap D_-)^c = D^c_+ \cup D^c_- \tag{2.27}$$

by inductive assumption. The idea then is to obtain information about the homology groups of

$$S^c := S^n \setminus S = (D_+ \cup D_-)^c = D^c_+ \cap D^c_-$$

by using the Mayer-Vietoris sequence associated to the decomposition (2.27). We note that the subsets $D_\pm \subset S^n$ are closed, and hence their complements $D^c_\pm$ are open subsets of S^n and $(S')^c$. In other words, the openness assumption for the Mayer-Vietoris sequence is satisfied, and we get the exact sequence

$$\begin{array}{ccccccc}
\widetilde{H}_{q+1}(D^c_+) \oplus \widetilde{H}_{q+1}(D^c_-) & \longrightarrow & \widetilde{H}_{q+1}(D^c_+ \cup D^c_-) & \xrightarrow{\partial} & \widetilde{H}_q(D^c_+ \cap D^c_-) & \longrightarrow & \widetilde{H}_{q+1}(D^c_+) \oplus \widetilde{H}_{q+1}(D^c_-) \\
\| & & \| & & \| & & \| \\
0 & & \widetilde{H}_{q+1}((S')^c) & & \widetilde{H}_q(S^c) & & 0
\end{array}$$

The exactness of the sequence implies that the boundary homomorphism is an isomorphism. Hence by inductive assumption □

Proof of Proposition. Like the proof of the theorem, this proof proceeds by induction over k and makes use of the Mayer-Vietoris sequence. For $k = 0$ the disk $D \subset S^n$ consists of one point and hence the complement $D^c := S^n \setminus D$ is homeomorphic to $S^n \setminus \{\infty\} = \mathbb{R}^n$. Hence it is contractible and its reduced homology groups vanish.

For the inductive step we note that the disk D^k is homeomorphic to the k-dimensional cube $I^k = I \times \cdots \times I$, $I = [0, 1]$. We think of the k-disk $D \subset S^n$ as a k-dimensional cube and decompose it as a union $D = C_- \cup C_+$ of a "left" cube C_- and a "right" cube C_+. More precisely, let $h\colon I^k \to D$ be a homeomorphism, and define $C_- := h([0, 1/2] \times I^{k-1})$, $C_+ := h([1/2, 1] \times I^{k-1})$. We note that the intersection $C' := C_- \cap C_+ = h(\{1/2\} \times I^{k-1})$ is a $(k-1)$-cube, also known as a $(k-1)$-disk, and hence by inductive hypothesis the reduced homology groups of the complement $(C')^c = S^n \setminus C'$ vanish. It follows that the Mayer-Vietoris sequence for the decomposition

$$(C')^c = (C_- \cap C_+)^c = C^c_- \cup C^c_+ \qquad C^c_\pm := S^n \setminus C_\pm$$

takes the form:

$$\begin{array}{ccccccc} \widetilde{H}_{q+1}((C')^c) & \xrightarrow{\partial} & \widetilde{H}_q(C_-^c \cap C_+^c) & \longrightarrow & \widetilde{H}_q(C_-^c) \oplus \widetilde{H}_q(C_+^c) & \longrightarrow & \widetilde{H}_q((C')^c) \\ \| & & \| & & & & \| \\ 0 & & \widetilde{H}_q(D^c) & & & & 0 \end{array}$$

From the exactness of the Mayer-Vietoris sequence we obtain the isomorphism $\widetilde{H}_q(D^c) \cong \widetilde{H}_q(C_-^c) \oplus \widetilde{H}_q(C_+^c)$. To show that $\widetilde{H}_q(D^c)$ is trivial, we assume that there is some non-trivial homology class $a \in \widetilde{H}_q(D^c)$ and aim at deriving a contradiction. From the isomorphism above we conclude that a can't map to zero in the homology of both, C_-^c and C_+^c. So for a suitable choice of C_1 as the k-cube C_- or C_+ we can assume that a maps to a non-trivial element in $\widetilde{H}_1(C_1^c)$.

Repeating the process of decomposing cubes as the union of left cubes and right cubes, we obtain a sequence of cubes

$$D \supset C_1 \supset C_2 \supset \dots$$

such that $a \in \widetilde{H}_q(D^c)$ maps to a non-trivial element in all of the following homology groups

$$\widetilde{H}_q(D^c) \longrightarrow \widetilde{H}_q(C_1^c) \longrightarrow \widetilde{H}_q(C_2^c) \longrightarrow \dots$$

We note that the cube C_m is given by $h(I_m \times I^{k-1})$, where I_m is a subinterval of length $1/2^m$ of $I = [0,1]$. It follows that

$$\bigcap_{m=1}^{\infty} C_m = h(\bigcap_{m=1}^{\infty} I_m \times I^{k-1}) = h(\{t\} \times I^{k-1}),$$

where t is some element of I. In particular, this intersection is a $(k-1)$-disk in S^n, and hence the reduced homology of its complement $\bigcup_{m=1}^{\infty} C_m^c$ vanishes by the inductive hypothesis. In particular, if α is a q-cycle in D^c representing $a \in H_q(D^c)$, then α represents the trivial homology class in $H_q(\bigcup_m C_m^c)$, and hence there is a chain $\beta \in C_{q+1}(\bigcup_m C_m^c)$ with $\partial\beta = \alpha$. The chain β is a finite linear combination $\beta = \sum_i n_i \sigma_i$ of $(q+1)$-simplices $\sigma_i \colon \Delta^{q+1} \to \bigcup_m C_m^c$. Since $C_m^c \subset S^n$ is an open subset, the compactness of Δ^{q+1} implies that each simplex σ_i is contained in some C_m for sufficiently large m. Hence the β is an element of $\underset{\sim}{C_m^c}$ for a sufficiently large m, showing that $\alpha = \partial\beta$ represents the trivial homology class in $\widetilde{H}_q(C_m^c)$. This contradicts the statement above that a maps to a non-trivial element in $\widetilde{H}_q(C_m^c)$ for *every* m, and provides the desired contradiction. □

2.3.2 Local homology and invariance of dimension

Theorem 2.28. (Invariance of dimension) *If nonempty open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.*

Corollary 2.29. *Homeomorphic manifolds have the same dimension.*

This result might seem intuitively obvious, but the reader should be reminded that there are unexpected things like space filling curves, that is, continuous surjective maps $I \to I^2$.

Proof. Suppose that $f\colon U \to V$ is a homeomorphism. Then for $x \in U$ the map f also provides an isomorphism of pairs $f\colon (U, U \setminus \{x\}) \to (V, V \setminus \{y\})$, where $y = f(x)$. In particular, the induced map on homology groups

$$f_*\colon H_q(U, U \setminus \{x\}) \longrightarrow H_q(V, V \setminus \{y\}) \tag{2.30}$$

is an isomorphism. To calculate the homology group $H_q(U, U \setminus \{x\})$, we note that the inclusion map $(U, U \setminus \{x\}) \to (\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ is an isomorphism on homology by excision (we are excising $\mathbb{R}^m \setminus U \subset \mathbb{R}^m \setminus \{x\}$; this satisfies the assumption of excision since $\mathbb{R}^m \setminus U$ is closed and $\mathbb{R}^m \setminus \{x\}$ is open). To calculate the homology groups of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ we use the long exact homology sequence of that pair:

$$\begin{array}{ccccccc}
\widetilde{H}_q(\mathbb{R}^m) & \longrightarrow & H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) & \xrightarrow{\partial} & \widetilde{H}_{q-1}(\mathbb{R}^m \setminus \{x\}) & \longrightarrow & \widetilde{H}_{q-1}(\mathbb{R}^m) \\
\| & & & & \cong \big\uparrow i_* & & \| \\
0 & & & & \widetilde{H}_{q-1}(S^{m-1}) & & 0
\end{array}$$

The reduced homology groups of $\mathbb{R}^m$ vanish since $\mathbb{R}^m$ is contractible, and hence the boundary homomorphism is an isomorphism by exactness of the sequence. The inclusion map $i\colon S^{m-1} \to \mathbb{R}^m \setminus \{0\} \approx \mathbb{R}^m \setminus \{x\}$ is a homotopy equivalence and hence induces an isomorphism on homology groups as indicated in the diagram. It follows that

$$H_q(U, U \setminus \{x\}) \cong H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{q-1}(\mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{q-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z} & q = m \\ 0 & q \neq m \end{cases}$$

By the same argument, $H_q(V, V \setminus \{x\})$ is isomorphic to $\mathbb{Z}$ for $q = n$ and trivial for $q \neq n$, and hence the isomorphism (2.30) implies $m = n$. □

For any topological space X and $x \in X$, the homology groups $H_q(X, X \setminus \{x\})$ are called *local homology groups*. This is motivated by the following result.

Proposition 2.31. *Let $f\colon X \to Y$ be a continuous map which is a* local homeomorphism *near $x \in X$ in the sense that there is some open neighborhood $U \ni x$ such that the restriction of f to U is a homeomorphism onto its image. Then the local homology group $H_q(X, X \setminus \{x\})$ is isomorphic to $H_q(Y, Y \setminus \{y\})$ for $y = f(x)$.*

Proof. We would like to argue that the map f induces an isomorphism of the local homology groups, but caution is needed here, since f in general *does not* give a map of pairs $(X, X \setminus \{x\}) \to (Y, Y \setminus \{y\})$, since there might be points $x' \in X$ with $f(x') = y$. By the local homeomorphism assumption that this does not happen for $x' \in U$, and we obtain the desired isomorphism as the composition

$$H_q(X, X \setminus \{x\}) \xleftarrow[\cong]{i_*^U} H_q(U, U \setminus \{x\}) \xrightarrow[\cong]{f_*} H_q(V, V \setminus \{y\}) \xrightarrow[\cong]{i_*^V} H_q(Y, Y \setminus \{y\})$$

□

Homework 2.32. Compute the local homology groups $H_q(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus \{x\})$ for $x \in \mathbb{R}^{n-1} \subset \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

We recall that a manifold with boundary of dimension n is a topological space X which is locally homeomorphic to $\mathbb{R}^n_+$. How can we

2.4 The degree of a map

We recall from Theorem **??** that $\widetilde{H}_n(S^n)$ is isomorphic to $\mathbb{Z}$. we fix a generator $a \in \widetilde{H}_n(S^n)$. If $f\colon S^n \to S^n$ is a map, its induced map $f_*\colon \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ sends the generator a to da for some integer d.

Definition 2.33. The *degree* of a map $f\colon S^n \to S^n$ is the integer $\deg(f)$ characterized by $f_*(a) = \deg(f)a$.

We remark that the degree is independent of the choice of generator $a \in \widetilde{H}_n(S^n)$, since if $a' = -a$ is the other generator, then $f_*(a') = f_*(-a) = -\deg(f)a = \deg(f)a'$. The homotopy invariance of the induced map implies that two homotopic maps have the same degree. In other words, there is a well-defined map

$$[S^n, S^n] \longrightarrow \mathbb{Z} \qquad [f] \mapsto \deg(f),$$

where $[S^n, S^n]$ denotes the set of homotopy classes up maps from S^n to itself, and $[f] \in [S^n, S^n]$ denotes the homotopy class of a map $f\colon S^n \to S^n$. We want to mention, but not prove, that this map is a bijection. So from the point of view of an algebraic topologist, the important thing to know about a map $f\colon S^n \to S^n$ is its degree.

1. The degree of the identity is 1, since by the functor property, the identity map induces the identity homomorphism on homology.

2. $\deg(fg) = \deg(f)\deg(g)$, since again by the functor property, $(fg)_* = f_* g_*$.

3. If f is a homeomorphism, or more generally a homotopy equivalence, then $\deg(f) \in \{\pm 1\}$: if f is a homotopy equivalence, the induced map f_* is an isomorphism (with inverse map given by g_* for a homotopy inverse g), and hence f_* must map the generator $a \in \widetilde{H}_n(S^n)$ to $\pm a$.

4. $\deg(f) = 0$ of f is not surjective: If $x \in S^n$ is not in the image of f, then f can be factored in the form $S^n \to S^n \setminus \{x\} \hookrightarrow S^n$; applying homology, the induced homomorphism f_* then factors in the form $\widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n \setminus \{x\}) \to \widetilde{H}_n(S^n)$; hence $f_* = 0$, since the homology group $\widetilde{H}_n(S^n \setminus \{x\})$ is trivial.

The last property can be rephrased that the degree of f is zero if there is some $y \in S^n$ whose preimage $f^{-1}(y)$ is empty. This suggests that the degree might be

Theorem 2.34. *Let $f\colon S^n \to S^n$ be a map, and suppose that for some $y \in S^n$ the preimage $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m \in S^n$. Then*

$$\deg(f) = \sum_{i=1}^{m} \deg(f, x_i),$$

where $\deg(f, x_i)$ is the local degree of f at x_i defined below.

Let $x \in S^n$ and let $f\colon U \to S^n$ be a map defined in an open neighborhood U of x. The local degree of f at x is defined in terms of local homology groups. We would like to think of f as a map of pairs

$$f\colon (U, U \setminus x) \longrightarrow (S^n, S^n \setminus y)$$

where $y = f(x)$. For that reason we make the following
Assumption. The preimage $f^{-1}(y)$ consists only of the point x.

Definition 2.35. Let $f\colon U \to S^n$ be a map satisfying the assumption above, possibly after replacing U by a smaller neighborhood of $x \in S^n$. Let

$$f_*\colon H_n(U, U \setminus x) \longrightarrow H_n(S^n, S^n \setminus y)$$

be the induced map of local homology groups. The *local degree of f at x*, denoted $\deg(f, x)$, is the integer determined by the equation

$$f_*(a) = \deg(f, x) a.$$

Abusing notation here we denote by a the generator of $H_n(U, U \setminus x)$ as well as the generator of $H_n(S^n, S^n \setminus y)$ that correspond to our chosen generator $a \in \widetilde{H}_n(S^n)$ via the isomorphisms

$$H_n(S^n, S^n \setminus y) \xleftarrow{\cong} \widetilde{H}_n(S^n) \xrightarrow{\cong} H_n(S^n, S^n \setminus x) \xleftarrow{\cong} H_n(U, U \setminus x)$$

induced by the obvious inclusion maps.

Proof. Let U_i, $i = 1, \dots, m$, be disjoint open neighborhoods of the points $x_1, \dots, x_m$. We note that disjointness assumption implies in particular that $f^{-1}(y) \cap U_i = \{x_i\}$; in other words, $f_{|U_i}$ satisfies the assumption for the definition of the local degree $\deg(f, x_i)$. To compare the degree of f with the local degrees $\deg(f, x_i)$ we consider the following commutative diagram.

$$\begin{array}{ccc}
\widetilde{H}_n(S^n) & \xrightarrow{\quad f_* \quad} & \widetilde{H}_n(S^n)\\
\downarrow & & \downarrow \cong\\
H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) & \xrightarrow{\quad f_* \quad} & H_n(S^n, S^n \setminus y)\\
\cong \uparrow & & \|\\
H_n(\coprod_{i=1}^m (U_i, U_i \setminus x_i)) & \xrightarrow{\quad f_* \quad} & H_n(S^n, S^n \setminus y)\\
\cong \uparrow & & \|\\
\bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i) & \xrightarrow[\bigoplus (f_{|U_i})_*]{} & H_n(S^n, S^n \setminus y)
\end{array}$$

All unlabeled maps in this diagram are induced by inclusions. The top horizontal map encodes the degree of f, while the bottom horizontal map encodes the local degrees of f at x_i in the sense that the generator $a \in H_n(U_i, U_i \setminus x_i)$ maps to

$$(f_{|U_i})_*(a) = \deg(f, x_i)a \in H_n(S^n, S^n \setminus y).$$

It follows that the diagonal element $(a, \dots, a) \in \bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i)$ maps to

$$\sum_{i=1}^m (f_{|U_i})_*(a) = \sum_{i=1}^m \deg(f, x_i)a = \left(\sum_{i=1}^m \deg(f, x_i)\right) a.$$

So to prove the theorem it remains to show that the composition of the vertical maps on the left map the generator $a \in \widetilde{H}_n(S^n)$ to the diagonal element $(a, \dots, a)$. To see this, suppose that the image of a under the vertical maps is $(a_1, \dots, , a_m) \in \bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i)$. To show that $a_i = a \in H_n(U_i, U_i \setminus x_i)$ we look at the diagram

$$\begin{array}{ccc}
 & & \widetilde{H}_n(S^n)\\
 & \swarrow \cong & \downarrow\\
H_n(S^n, S^n \setminus x_i) & \longleftarrow & H_n(S^n, S^n \setminus \{x_1, \dots, x_m\})\\
 & \nwarrow \cong & \uparrow\\
 & & H_n(U_i, U_i \setminus x_i)
\end{array}$$

Its commutativity implies $a_i = a$. □

The following result provides an effective way to calculate local degrees.

Theorem 2.36. 1. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^n \subset \widehat{\mathbb{R}}^n = S^n$ *be a linear isomorphism. Then* $\deg(f,0) = \operatorname{sign}\det(f)$.

2. *Let* $f\colon U \to \mathbb{R}^n$ *be a smooth map defined on an open subset* $U \subset \mathbb{R}^n$*. If the derivative* $Df_{x_0} \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ *is invertible at a point* $x_0 \in U$*, then* $\deg(f, x_0) = \operatorname{sign}\det(Df_{x_0})$.

We note that in part (1) the condition that f is an isomorphism guarantees that that $f^{-1}(0)$ consists of only one point, the assumption necessary to define the local index $\deg(f,0)$. Similarly, in part (2) the invertibility of Df_{x_0} guarantees by the inverse function theorem that f is a diffeomorphism from an open neighborhood of x_0 to its image; in particular, the preimage of $y = f(x_0)$ consists only of the point x_0, making it possible to define the local index $\deg(f, x_0)$.

For any non-constant polynomial $p(z)$ the map $p\colon \mathbb{C} \to \mathbb{C}$ is proper and hence extends to a map $\widehat{p}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Theorem 2.37. 1. *Let* $p(z)$ *be a polynomial of degree* $k > 0$*. Then* $\deg(\widehat{p}) = k$.

2. *Let* z_0 *be a zero of* $p(z)$*. Then* $\deg(\widehat{p}, z_0)$ *is the multiplicity of the zero* z_0.

Corollary 2.38. (The Fundamental Theorem of Algebra) *Any non-constant polynomial* $p(z)$ *has a zero.*

Proof. Assuming that $p(z)$ has no zeroes, the preimage $\widehat{p}^{-1}(0)$ is empty, which implies $\deg(\widehat{p}) = 0$, contradicting the first part of the theorem above. □

In particular, for the polynomial $p(z) = z^k$ the preimage of $1 \in \mathbb{C}$ consists of the set $\{1, \zeta, \zeta^2, \ldots, \zeta^{k-1}\}$ of the k-th roots of unity ($\zeta = e^{2\pi i/k}$). To calculate the local degree $\deg(p, z)$ for $z = \zeta^i$, $i = 0, \ldots, k-1$, we note that the derivative $Dp_z \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$ is *complex linear*, namely the image of $p'(z) \in \mathbb{C}$ via the linear embedding

$$\mathbb{C} = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \hookrightarrow \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2) \qquad a + ib \mapsto \begin{matrix} a & -b \\ b & a \end{matrix}.$$

We note that $\det \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) = a^2 + b^2 = ||a + ib||^2$. In particular, $\det Dp_z \geq 0$, and $\det Dp_z > 0$ if and only if $p'(z) \neq 0$.

2.5 Homology of CW complexes

2.5.1 CW complexes

Definition 2.39. Attaching an n-cell.

Example 2.40. 1. $K \cong (S^1 \vee S^1) \cup e^2$

2. $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \cup_\varphi e^n$ with attaching map $\varphi\colon S^{n-1} \to \mathbb{RP}^{n-1} = S^{n-1}/\sim$ the projection map.

3. $\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \cup_\varphi e^{2n}$ with attaching map $\varphi\colon S^{2n-1} \to \mathbb{CP}^{n-1} S^{2n-1}/\sim$ the projection map.

Definition 2.41. CW complex

Example 2.42. Examples of CW complexes, identifying the skeleta.

1. graphs

2. surfaces: Σ_g and $\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$;

3. $\mathbb{RP}^n$, $\mathbb{CP}^n$, lens spaces

2.5.2 The cellular chain complex of a CW complex

Construction. Let X be a CW complex. Let $\Phi_\alpha\colon D^q \to X^{(n)}$ be the characteristic map of the q-cell e^q_α. The restriction of Φ_α to $S^{q-1} \subset D^q$ is the attaching map $\phi_\alpha\colon S^{q-1} \to X^{(q-1)}$. In particular, we have a map of pairs

$$(D^q, S^{q-1}) \xrightarrow{\Phi_\alpha} (X^{(q)}, X^{(q-1)})$$

which gives a map $\Phi_\alpha\colon S^q = D^q/S^{q-1} \longrightarrow X^{(q)}/X^{(q-1)}$ of quotient spaces. Putting these maps together as α ranges through the indexing set for the q-cells of X, we obtain a map

$$\bigvee_\alpha S^q_\alpha \xrightarrow{\approx} X^{(q)}/X^{(q-1)}$$

which is a continuous bijection and in fact a homeomorphism. Fixing

Proposition 2.43. *Let A be a closed subspace of a topological space X and assume that there is a neighborhood V of A which deformation retracts to A. Then the quotient map $q\colon (X, A) \to (X/A, A/A)$ induces an isomorphism*

$$q_*\colon H_q(X, A) \xrightarrow{\cong} H_q(X/A, A/A) = \widetilde{H}_q(X/A).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xleftarrow{\cong} & H_n(X \setminus A, V \setminus A) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \Vert \\
H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X \setminus A, V \setminus A)
\end{array}$$

The top left arrow is an isomorphism by considering the long exact sequence of the triple (X, V, A), and noting that the homology groups (V, A) vanish due to the long exact sequence

$$H_{n+1}(V,A) \xrightarrow{\partial} H_n(A) \xrightarrow[\cong]{i_*} H_n(V) \longrightarrow H_n(V,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow$$

The same argument implies that the bottom left map is an isomorphism. The two horizontal maps on the right are isomorphisms by excision. This implies that the left vertical map is an isomorphism. □

Corollary 2.44. $H_q(X^{(n)}, X^{(n-1)})$ *is trivial for* $q \neq 0$ *and is the free* $\mathbb{Z}$*-module on generators* e^n_α *corresponding to the* n*-cells of* X *for* $q = n$*.*

Proof. The quotient $X^{(n)}/X^{(n-1)}$ is a wedge $\bigvee_\gamma S^n_\gamma$ of spheres parametrized by the n-cells of X. Hence the proposition implies the statement. □

For later use, we need an *explicit* construction of the generator $e^n_\alpha \in H_n(X^{(n)}, X^{(n-1)})$ determined by an n-cell e^n_α. Let $\Phi_\alpha \colon D^n \to X^{(n)}$ be the characteristic map of the n-cell e^n_α. Its restriction to $S^{n-1} = \partial D^n$ is the attaching map $\varphi_\alpha \colon S^{n-1} \to X^{(n-1)}$.

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \xrightarrow[\cong]{q_*} & \widetilde{H}_n(S^n) \\ \Phi_\alpha \downarrow & & \downarrow \Phi_\alpha \\ H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow[\cong]{q_*} & \widetilde{H}_n(X^{(n)}/X^{(n-1)}) \end{array} \quad .$$

Let a_n be a fixed generator of $\widetilde{H}_n(S^n) \cong \mathbb{Z}$, and let $[D^n] \in H_n(D^n, S^{n-1})$ the generator determined by $q_*([D^n]) = a_n$, and define

$$e^n_\alpha := \Phi_\alpha([D^n]) \in H_n(X^{(n)}, X^{(n-1)}). \tag{2.45}$$

3 Appendix: Pointset Topology

3.1 Metric spaces

We recall that a map $f \colon \mathbb{R}^m \to \mathbb{R}^n$ between Euclidean spaces is *continuous* if and only if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \in X \quad d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon, \tag{3.1}$$

where

$$d(x,y) = ||x - y|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \in \mathbb{R}_{\geq 0}$$

is the *Euclidean distance* between two points x, y in $\mathbb{R}^n$.

Example 3.2. (Examples of continuous maps.)

1. The addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 + x_2$;
2. The multiplication map $m\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 x_2$;

The proofs that these maps are continuous are simple estimates that you probably remember from calculus. Since the continuity of *all* the maps we'll look at in these notes is proved by expressing them in terms of the maps a and m, we include the proofs of continuity of a and m for completeness.

Proof. To prove that the addition map a is continuous, suppose $x = (x_1, x_2) \in \mathbb{R}^2$ and $\epsilon > 0$ are given. We claim that for $\delta := \epsilon/2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(a(x), a(y)) < \epsilon$ and hence a is a continuous function. To prove the claim, we note that

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

and hence $|x_1 - y_1| \leq d(x, y)$, $|x_1 - y_1| \leq d(x, y)$. It follows that

$$d(a(x), a(y)) = |a(x) - a(y)| = |x_1 + x_2 - y_1 - y_2| \leq |x_1 - y_1| + |x_2 - y_2| \leq 2d(x, y) < 2\delta = \epsilon.$$

To prove that the multiplication map m is continuous, we claim that for

$$\delta := \min\{1, \epsilon/(|x_1| + |x_2| + 1)\}$$

and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(m(x), m(y)) < \epsilon$ and hence m is a continuous function. The claim follows from the following estimates:

$$\begin{aligned} d(m(y), m(x)) &= |y_1 y_2 - x_1 x_2| = |y_1 y_2 - x_1 y_2 + x_1 y_2 - x_1 x_2| \\ &\leq |y_1 y_2 - x_1 y_2| + |x_1 y_2 - x_1 x_2| = |y_1 - x_1||y_2| + |x_1||y_2 - x_2| \\ &\leq d(x, y)(|y_2| + |x_1|) \leq d(x, y)(|x_2| + |y_2 - x_2| + |x_1|) \\ &\leq d(x, y)(|x_1| + |x_2| + 1) < \delta(|x_1| + |x_2| + 1) \leq \epsilon \end{aligned}$$

□

Lemma 3.3. *The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ has the following properties:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ (symmetry);*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)*

Definition 3.4. A *metric space* is a set X equipped with a map

$$d\colon X \times X \to \mathbb{R}_{\geq 0}$$

with properties (1)-(3) above. A map $f\colon X \to Y$ between metric spaces X, Y is

continuous if condition (3.1) is satisfied.

an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$;

Two metric spaces X, Y are *homeomorphic* (resp. *isometric*) if there are continuous maps (resp. isometries) $f\colon X \to Y$ and $g\colon Y \to X$ which are inverses of each other.

Example 3.5. An important class of examples of metric spaces are subsets of $\mathbb{R}^n$. Here are particular examples we will be talking about during the semester:

1. The *n-disk* $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\} \subset \mathbb{R}^n$, and $D^n_r := \{x \in \mathbb{R}^n \mid |x| \leq r\}$, the *n-disk of radius* $r > 0$.

   The dilation map
   $$D^n \longrightarrow D^n_r \qquad x \mapsto rx$$
   is a homeomorphism between D^n and D^n_r with inverse given by multiplication by $1/r$. However, these two metric spaces are *not* isometric for $r \neq 1$. To see this, define the *diameter* $\operatorname{diam}(X)$ of a metric space X by
   $$\operatorname{diam}(X) := \sup\{d(x, y) \mid x, y \in X\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$
   For example, $\operatorname{diam}(D^n_r) = 2r$. It is easy to see that if two metric spaces X, Y are isometric, then their diameters agree. In particular, the disks D^n_r and $D^n_{r'}$ are not isometric unless $r = r'$.

2. The *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \subset \mathbb{R}^{n+1}$.

3. The *torus* $T = \{v \in \mathbb{R}^3 \mid d(v, C) = r\}$ for $0 < r < 1$. Here
   $$C = \{(x, y, 0) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$$
   is the unit circle in the xy-plane, and $d(v, C) = \inf_{w \in C} d(v, w)$ is the distance between v and C.

4. The *general linear group*
   $$\begin{aligned} GL_n(\mathbb{R}) &= \{\text{vector space isomorphisms } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &\longleftrightarrow \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) \neq 0\} \\ &= \{\text{invertible } n \times n\text{-matrices}\} \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} = \mathbb{R}^{n^2} \end{aligned}$$

Here we think of $(v_1, \dots, v_n)$ as an $n \times n$-matrix with column vectors v_i, and the bijection is the usual one in linear algebra that sends a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ to the matrix $(f(e_1), \dots, f(e_n))$ whose column vectors are the images of the standard basis elements $e_i \in \mathbb{R}^n$.

5. The *special linear group*

$$SL_n(\mathbb{R}) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

6. The *orthogonal group*

$$\begin{aligned} O(n) &= \{\text{linear isometries } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{n^2} \end{aligned}$$

We recall that a collection of vectors $v_i \in \mathbb{R}^n$ is *orthonormal* if $|v_i| = 1$ for all i, and v_i is perpendicular to v_j for $i \neq j$.

7. The *special orthogonal group*

$$SO(n) = \{(v_1, \dots, v_n) \in O(n) \mid \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

8. The *Stiefel manifold*

$$\begin{aligned} V_k(\mathbb{R}^n) &= \{\text{linear isometries } f\colon \mathbb{R}^k \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{kn} \end{aligned}$$

Example 3.6. The following maps between metric spaces are continuous. While it is possible to prove their continuity using the definition of continuity, it will be much simpler to prove their continuity by 'building' these maps using compositions and products from the continuous maps a and m of Example 3.2. We will do this below in Lemma 3.22.

1. Every polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous. We recall that a polynomial function is of the form $f(x_1, \dots, x_n) = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$ for $a_{i_1,\dots,i_n} \in \mathbb{R}$.

2. Let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices. Then the map

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

given by matrix multiplication is continuous. Here we use the fact that a map to the product $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R} \times \dots \times \mathbb{R}$ is continuous if and only if each component map is continuous (see Lemma 3.21), and each matrix entry of AB is a polynomial and hence a continuous function of the matrix entries of A and B. Restricting to the invertible matrices $GL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$, we see that the multiplication map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

is continuous. The same holds for the subgroups $SO(n) \subset O(n) \subset GL_n(\mathbb{R})$.

3. The map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous (this is a homework problem). The same statement follows for the subgroups of $GL_n(\mathbb{R})$.

The Euclidean metric on $\mathbb{R}^n$ given by $d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ for $x, y \in \mathbb{R}^n$ is not the only reasonable metric on $\mathbb{R}^n$. Another metric on $\mathbb{R}^n$ is given by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|. \tag{3.7}$$

The question arises whether it can happen that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to one of these metrics, but not with respect to the other. To see that this doen't happen, it is useful to characterize continuity of a map $f\colon X \to Y$ between metric spaces X, Y in a way that involves the metrics on X and Y less directly than Definition 3.4 does. This alternative characterization will be based on the following notion of "open subsets" of a metric space.

Definition 3.8. Let X be a metric space. A subset $U \subset X$ is *open* if for every point $x \in U$ there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Here $B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\}$ is the *ball of radius ϵ around x.*

To illustrate this, lets look at examples of subsets of $\mathbb{R}^n$ equipped with the Euclidean metric. The subset $D^n_r = \{v \in \mathbb{R}^n \mid ||v|| \le r\} \subset \mathbb{R}^n$ is not open, since for for a point $v \in D^n_r$ with $||v|| = r$ any open ball $B_\epsilon(v)$ with center v will contain points not in D^n_r. By contrast, the subset $B_r(0) \subset \mathbb{R}^n$ is open, since for any $x \in B_r(0)$ the ball $B_\delta(x)$ of radius $\delta = r - ||x||$ is contained in $B_r(0)$, since for $y \in B_\delta(x)$ by the triangle inequality we have

$$d(y,0) \le d(y,x) + d(x,0) < \delta + ||x|| = (r - ||x||) + ||x|| = r.$$

Lemma 3.9. *A map $f\colon X \to Y$ between metric spaces is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset $V \subset Y$.*

Corollary 3.10. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then so it their composition $g \circ f\colon X \to Z$.*

Exercise 3.11. (a) Prove Lemma 3.9

(b) Assume that d, d' are two metrics on a set X which are equivalent in the sense that there are constants $C, C' > 0$ such that $d(x,y) \le C d_1(x,y)$ and $d_1(x,y) \le C' d(x,y)$ for all $x, y \in X$. Show that a subset $U \subset X$ is open with respect to d if and only if it is open with respect to d'.

(c) Show that the Euclidean metric d and the metric (3.7) on $\mathbb{R}^n$ are equivalent. This shows in particular that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous w.r.t. d if and only if it is continuous w.r.t. d_1.

3.2 Topological spaces

Lemma 3.9 and Exercise (b) above shows that it is better to *define* continuity of maps between metric spaces in terms of the *open subsets* of these metric space instead of the original ϵ-δ-definition. In fact, we can go one step further, forget about the metric on a set X altogether, and just consider a collection $\mathcal{T}$ of subsets of X that we declare to be "open". The next result summarizes the basic properties of open subsets of a metric space X, which then motivates the restrictions that we wish to put on such collections $\mathcal{T}$.

Lemma 3.12. *Open subsets of a metric space X have the following properties.*

(i) X and $\emptyset$ are open.

(ii) Any union of open sets is open.

(iii) The intersection of any finite number of open sets is open.

Definition 3.13. A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X, called *open sets* which are required to satisfy conditions (i), (ii) and (iii) of the lemma above. The collection $\mathcal{T}$ is called a *topology* on X. The sets in $\mathcal{T}$ are called the *open sets*, and their complements in X are called *closed sets.* A subset of X may be neither closed nor open, either closed or open, or both.

A map $f\colon X \to Y$ between topological spaces X, Y is *continuous* if the inverse image $f^{-1}(V)$ of every open subset $V \subset Y$ is an open subset of X.

It is easy to see that the composition of continuous maps is again continuous.

Examples of topological spaces.

1. Let X be a metric space, and $\mathcal{T}$ the collection of those subsets of X that are unions of balls $B_\epsilon(x)$ in X (i.e., the subsets which are open in the sense of Definition 3.8). Then $\mathcal{T}$ is a topology on X, the *metric topology.*

2. Let X be a set. Then $\mathcal{T} = \{\text{all subsets of } X\}$ is a topology, the *discrete topology.* We note that *any* map $f\colon X \to Y$ to a topological space Y is continuous. We will see later that the only continuous maps $\mathbb{R}^n \to X$ are the constant maps.

3. Let X be a set. Then $\mathcal{T} = \{\emptyset, X\}$ is a topology, the *indiscrete topology.*

Sometimes it is convenient to define a topology $\mathcal{U}$ on a set X by first describing a smaller collection $\mathcal{B}$ of subsets of X, and then defining $\mathcal{U}$ to be those subsets of X that can be written as *unions* of subsets belonging to $\mathcal{B}$. We've done this already when defining the metric topology: Let X be a metric space and let $\mathcal{B}$ be the collection of subsets of X of the form $B_\epsilon(x) := \{y \in X \mid d(y,x) < \epsilon\}$ (the balls in X). Then the metric topology $\mathcal{U}$ on X consists of those subsets U which are unions of subsets belonging to $\mathcal{B}$.

Lemma 3.14. *Let $\mathcal{B}$ be a collection of subsets of a set X satisfying the following conditions*

1. *Every point $x \in X$ belongs to some subset $B \in \mathcal{B}$.*
2. *If $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$ there is some $B \in \mathcal{B}$ with $x \in B$ and $B \subset B_1 \cap B_2$.*

Then $\mathcal{T} := \{\text{unions of subsets belonging to } \mathcal{B}\}$ is a topology on X.

Definition 3.15. If the above conditions are satisfied, we call the collection $\mathcal{B}$ is called a *basis for the topology* $\mathcal{T}$ or we say that $\mathcal{B}$ *generates the topology* $\mathcal{T}$.

It is easy to check that the collection of balls in a metric space satisfies the above conditions and hence the collection of open subsets is a topology as claimed by Lemma 3.12.

3.3 Constructions with topological spaces

3.3.1 Subspace topology

Definition 3.16. Let X be a topological space, and $A \subset X$ a subset. Then

$$\mathcal{T} = \{A \cap U \mid U \underset{open}{\subset} X\}$$

is a topology on A called the *subspace topology*.

Lemma 3.17. *Let X be a metric space and $A \subset X$. Then the metric topology on A agrees with the subspace topology on A (as a subset of X equipped with the metric topology).*

Lemma 3.18. *Let X, Y be topological spaces and let A be a subset of X equipped with the subspace topology. Then the inclusion map $i\colon A \to X$ is continuous and a map $f\colon Y \to A$ is continuous if and only if the composition $i \circ f\colon Y \to X$ is continuous.*

3.3.2 Product topology

Definition 3.19. The *product topology* on the Cartesian product $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ of topological spaces X, Y is the topology with basis

$$\mathcal{B} = \{U \times V \mid U \underset{open}{\subset} X, V \underset{open}{\subset} Y\}$$

The collection $\mathcal{B}$ obviously satisfies property (1) of a basis; property (2) holds since $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$. We note that the collection $\mathcal{B}$ is *not* a topology since the union of $U \times V$ and $U' \times V'$ is typically not a Cartesian product (e.g., draw a picture for the case where $X = Y = \mathbb{R}$ and U, U', V, V' are open intervals).

Lemma 3.20. *The product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (with each factor equipped with the metric topology) agrees with the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.*

Proof: homework.

Lemma 3.21. *Let X, Y_1, Y_2 be topological spaces. Then the projection maps $p_i\colon Y_1 \times Y_2 \to Y_i$ is continuous and a map $f\colon X \to Y_1 \times Y_2$ is continuous if and only if the* component maps

$$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{p_i} Y_i$$

are continuous for $i = 1, 2$.

Proof: homework

Lemma 3.22. 1. *Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous maps. Then $f + g$ and $f \cdot g$ continuous maps from X to $\mathbb{R}$. If $g(x) \neq 0$ for all $x \in X$, then also f/g is continuous.*

2. *Any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.*

3. *The multiplication map $\mu\colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is continuous.*

Proof. To prove part (1) we note that the map $f + g\colon X \to \mathbb{R}$ can be factored in the form

$$X \xrightarrow{f\times g} \mathbb{R} \times \mathbb{R} \xrightarrow{a} \mathbb{R}$$

The map $f \times g$ is continuous by Lemma 3.21 since its component maps f, g are continuous; the map a is continuous by Example 3.2, and hence the composition $f + g$ is continuous. The argument for $f \cdot g$ is the same, with a replaced by m. To prove that f/g is continuous, we factor it in the form

$$X \xrightarrow{f\times g} \mathbb{R} \times \mathbb{R}^\times \xrightarrow{p_1\times(I\circ p_2)} \mathbb{R} \times \mathbb{R}^\times \xrightarrow{m} \mathbb{R},$$

where $\mathbb{R}^\times = \{t \in \mathbb{R} \mid t \neq 0\}$, p_1 (resp. p_2) is the projection to the first (resp. second) factor of $\mathbb{R} \times \mathbb{R}^\times$, and $I\colon \mathbb{R}^\times \to \mathbb{R}^\times$ is the inversion map $t \mapsto t^{-1}$. By Lemma 3.21 the p_i's are continuous, in calculus we learned that I is continuous, and hence again by Lemma 3.21 the map $p_1 \times (I \circ p_2)$ is continuous.

To prove part (2), we note that the constant map $\mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto a$ is obviously continuous, and that the projection map $p_i\colon \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_i$ is continuous by Lemma 3.21. Hence by part (1) of this lemma, the monomial function $x \mapsto ax_1^{i_1} \cdots x_n^{i_n}$ is continuous. Any polynomial function is a sum of monomial functions and hence continuous.

For the proof of (3), let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices and let

$$\mu\colon M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

be the map given by matrix multiplication. By Lemma 3.21 the map μ is continuous if and only if the composition

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \xrightarrow{\mu} M_{n\times n}(\mathbb{R}) \xrightarrow{p_{ij}} \mathbb{R}$$

is continuous for all $1 \le i, j \le n$, where p_{ij} is the projection map that sends a matrix A to its entry $A_{ij} \in \mathbb{R}$. Since the $p_{ij}(\mu(A, B)) = (A \cdot B)_{ij}$ is a *polynomial* in the entries of the matrices A and B, this is a continuous map by part (2) and hence μ is continuous.

Restricting μ to invertible matrices, we obtain the multiplication map

$$\mu_{|}\colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

that we want to show is continuous. We will argue that in general if $f\colon X \to Y$ is a continuous map with $f(A) \subset B$ for subsets $A \subset X$, $B \subset Y$, then the *restriction* $f_{|A}\colon A \to B$ is continuous. To prove this, consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_{|A}} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

where i, j are the obvious inclusion maps. These inclusion maps are continuous w.r.t. the subspace topology on A, B by Lemma 3.18. The continuity of f and i implies the continuity of $f \circ i = j \circ f_{|A}$ which again by Lemma 3.18 implies the continuity of $f_{|A}$. □

3.3.3 Quotient topology.

Definition 3.23. Let X be a topological space and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ be the set of equivalence classes and by

$$p\colon X \to X/\sim \qquad x \mapsto [x]$$

be the projection map that sends a point $x \in X$ to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is given by the collection of subsets

$$\mathcal{U} = \{U \subset X/\sim \mid p^{-1}(U) \text{ is an open subset of } X\}.$$

The set $X/\sim$ equipped with the quotient topology is called the *quotient space*.

The quotient topology is often used to construct a topology on a set Y which is not a subset of some Euclidean space $\mathbb{R}^n$, or for which it is not clear how to construct a metric. If there is a surjective map

$$p\colon X \longrightarrow Y$$

from a topological space X, then Y can be identified with the quotient space $X/\sim$, where the equivalence relation is given by $x \sim x'$ if and only if $p(x) = p(x')$. In particular, $Y = X/\sim$ can be equipped with the quotient topology. Here are important examples.

Example 3.24. 1. The *real projective space of dimension* n is the set

$$\mathbb{RP}^n := \{1\text{-dimensional subspaces of } \mathbb{R}^{n+1}\}.$$

The map

$$S^n \longrightarrow \mathbb{RP}^n \qquad \mathbb{R}^{n+1} \ni v \mapsto \text{subspace generated by } v$$

is surjective, leading to the identification

$$\mathbb{RP}^n = S^n/(v \sim \pm v),$$

and the quotient topology on $\mathbb{RP}^n$.

2. Similarly, working with complex vector spaces, we obtain a quotient topology on the the *complex projective space*

$$\mathbb{CP}^n := \{1\text{-dimensional subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/(v \sim zv), \qquad z \in S^1$$

3. Generalizing, we can consider the *Grassmann manifold*

$$G_k(\mathbb{R}^{n+k}) := \{k\text{-dimensional subspaces of } \mathbb{R}^{n+k}\}.$$

There is a surjective map

$$V_k(\mathbb{R}^{n+k}) = \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^{n+k},\ v_i\text{'s are orthonormal}\} \quad \twoheadrightarrow \quad G_k(\mathbb{R}^{n+k})$$

given by sending $(v_1, \dots, v_k) \in V_k(\mathbb{R}^{n+k})$ to the k-dimensional subspace of $\mathbb{R}^{n+k}$ spanned by the v_i's. Hence the subspace topology on the Stiefel manifold $V_k(\mathbb{R}^{n+k}) \subset \mathbb{R}^{(n+k)k}$ gives a quotient topology on the Grassmann manifold $G_k(\mathbb{R}^{n+k}) = V_k(\mathbb{R}^{n+k})/\sim$. The same construction works for the complex Grassmann manifold $G_k(\mathbb{C}^{n+k})$.

As the examples below will show, sometimes a quotient space $X/\sim$ is homeomorphic to a topological space Z constructed in a different way. To establish the homeomorphism between $X/\sim$ and Z, we need to construct continuous maps

$$f\colon X/\sim \longrightarrow Z \qquad g\colon Z \to X/\sim$$

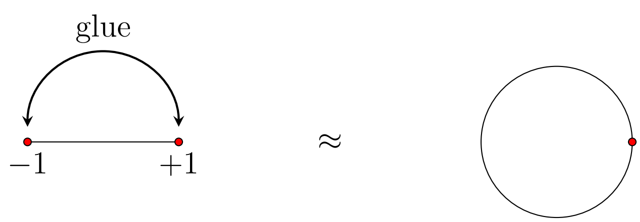
that are inverse to each other. The next lemma shows that it is easy to check continuity of the map f, the map *out of the quotient space.*

Lemma 3.25. *The projection map* $p\colon X \to X/\sim$ *is continuous and a map* $f\colon X/\sim \to Z$ *to a topological space* Z *is continuous if and only if the composition* $f \circ p\colon X \to Z$ *is continuous.*

As we will see in the next section, there are many situations where the continuity of the inverse map for a continuous bijection f is automatic. So in the examples below, and for the exercises in this section, we will defer checking the continuity of f^{-1} to that section.

Notation. Let A be a subset of a topological space X. Define a equivalence relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. We use the notation X/A for the quotient space $X/\sim$.

Example 3.26. (1) We claim that the quotient space $[-1, +1]/\{\pm 1\}$ is homeomorphic to S^1 via the map $f\colon [-1, +1]/\{\pm 1\} \to S^1$ given by $[t] \mapsto e^{\pi i t}$. Geometrically speaking, the map f wraps the interval $[-1, +1]$ once around the circle. Here is a picture.



It is easy to check that the map f is a bijection. To see that f is continuous, consider the composition

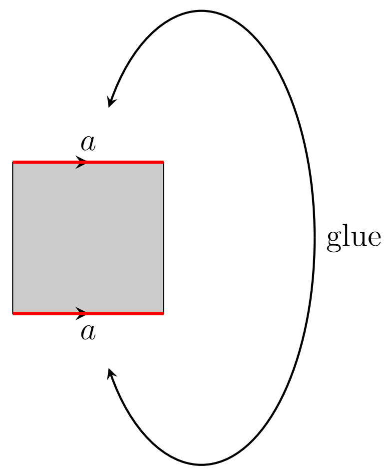
$$[-1, +1] \xrightarrow{p} [-1, +1]/\{\pm 1\} \xrightarrow{f} S^1 \xrightarrow{i} \mathbb{C} = \mathbb{R}^2,$$

where p is the projection map and i the inclusion map. This composition sends $t \in [-1, +1]$ to $e^{\pi i t} = (\sin \pi t, \cos \pi t) \in \mathbb{R}^2$. By Lemma 3.21 it is a continuous function, since its component functions $\sin \pi t$ and $\cos \pi t$ are continuous functions. By Lemma 3.25 the continuity of $i \circ f \circ p$ implies the continuity of $i \circ f$, which by Lemma 3.18 implies the continuity of f. As mentioned above, we'll postpone the proof of the continuity of the inverse map f^{-1} to the next section.

(2) More generally, D^n/S^{n-1} is homeomorphic to S^n. (proof: homework)

(3) Consider the quotient space of the square $[-1, +1] \times [-1, +1]$ given by identifying $(s, -1)$ with $(s, 1)$ for all $s \in [-1, 1]$. It can be visualized as a square whose top edge is to be glued with its bottom edge. In the picture below we indicate that identification by

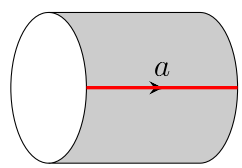
labeling those two edges by the same letter.



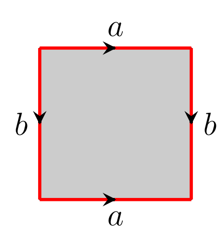
The quotient $([-1,+1]\times[-1,+1])\,/(s,-1)\sim(s,+1)$ is homeomorphic to the cylinder

$$C=\{(x,y,z)\in\mathbb{R}^3 \mid x\in[-1,+1], y^2+z^2=1\}.$$

The proof is essentially the same as in (1). A homeomorphism from the quotient space to C is given by $f([s,t])=(s,\sin\pi t,\cos\pi t)$. The picture below shows the cylinder C with the image of the edge a indicated.



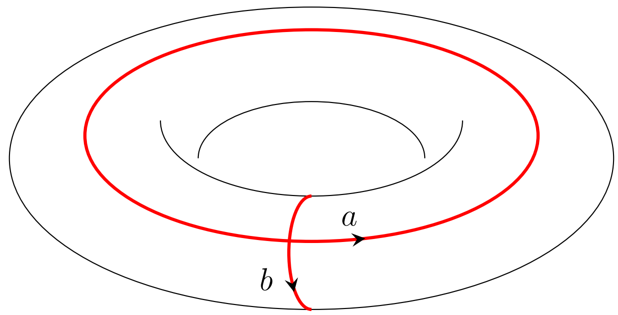
(4) Consider again the square, but this time using an equivalence relations that identifies more points than the one in the previous example. As before we identify $(s,-1)$ and $(s,1)$ for $s\in[-1,1]$, and in addition we identify $(-1,t)$ with $(1,t)$ for $t\in[-1,1]$. Here is the picture, where again corresponding points of edges labeled by the same letter are to be identified.



We claim that the quotient space is homeomorphic to the torus

$$T:=\{x\in\mathbb{R}^3 \mid d(x,K)=d\},$$

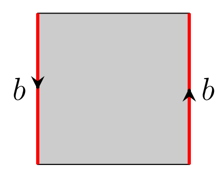
where $K = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1\}$ is the unit circle in the xy-plane and $0 < d < 1$ is a real number (see) via a homeomorphism that maps the edges of the square to the loops in T indicated in the following picture below.



Exercise: prove this by writing down an explicity map from the quotient space to T, and arguing that this map is a continuous bijection (as always in this section, we defer the proof of the continuity of the inverse to the next section).

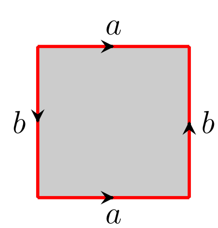
(5) We claim that the quotient space $D^n/\sim$ with equivalence relation generated by $v \sim -v$ for $v \in S^{n-1} \subset D^n$ is homeomorphic to the real projective space $\mathbb{RP}^n$. Proof: exercise. In particular, $\mathbb{RP}^1 = S^1/v \sim -v$ is homeomorphic to $D^1/\sim = [-1, 1]/-1 \sim 1$, which by example (1) is homeomorphic to S^1.

(6) The quotient space $[-1, 1] \times [-1, 1]/\sim$ with the equivalence relation generated by $(-1, t) \sim (1, -t)$ is represented graphically by the following picture.



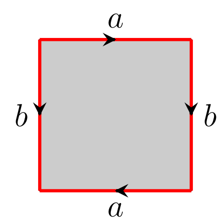
This topological space is called the *Möbius band*. It is homeomorphic to a subspace of $\mathbb{R}^3$ shown by the following picture

(7) The quotient space of the square by edge identifications given by the picture



is the *Klein bottle*. It is harder to visualize, since it is not homeomorphic to a subspace of $\mathbb{R}^3$ (which can be proved by the methods of algebraic topology).

(8) The quotient space of the square given by the picture



is homeomorphic to the real projective plane $\mathbb{RP}^2$. Exercise: prove this (hint: use the statement of example (5)). Like the Klein bottle, it is challenging to visualize the real projective plane, since it is not homeomorphic to a subspace of $\mathbb{R}^3$.

3.4 Properties of topological spaces

In the previous subsection we described a number of examples of topological spaces X, Y that we claimed to be homeomorphic. We typically constructed a bijection $f\colon X \to Y$ and argued that f is continuous. However, we did not finish the proof that f is a homeomorphism, since we defered the argument that the inverse map $f^{-1}\colon Y \to X$ is continuous. We note that not every continuous bijection is a homeomorphism. For example if X is a set, X_δ (resp. X_{ind}) is the topological space given by equipping the set X with the discrete (resp. indiscrete) topology, then the identity map is a continuous bijection from X_δ to X_{ind}. However its inverse, the identity map $X_{\text{ind}} \to X_\delta$ is not continuous if X contains at least two points.

Fortunately, there are situations where the continuity of the inverse map is automatic as the following proposition shows.

Proposition 3.27. *Let $f\colon X \to Y$ be a continuous bijection. Then f is a homeomorphism provided X is compact and Y is Hausdorff.*

The goal of this section is to define these notions, prove the proposition above, and to give a tools to recognize that a topological space is compact and/or Hausdorff.

3.4.1 Hausdorff spaces

Definition 3.28. Let X be a topological space, $x_i \in X$, $i = 1, 2, \dots$ a sequence in X and $x \in X$. Then *x is the limit of the x_i's* if for any open subset $U \subset X$ containing x there is some N such that $x_i \in U$ for all $i \geq N$.

Caveat: If X is a topological space with the indiscrete topology, *every point* is the limit of every sequence. The limit is *unique* if the topological space has the following property:

Definition 3.29. A topological space X is *Hausdorff* if for every $x, y \in X$, $x \neq y$, there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$.

Note: if X is a metric space, then the metric topology on X is Hausdorff (since for $x \neq y$ and $\epsilon = d(x,y)/2$, the balls $B_\epsilon(x)$, $B_\epsilon(y)$ are disjoint open subsets). In particular, any subset of $\mathbb{R}^n$, equipped with the subspace topology, is Hausdorff.

Warning: The notion of *Cauchy sequences* can be defined in metric spaces, but not in general for topological spaces (even when they are Hausdorff).

Lemma 3.30. *Let X be a topological space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.*

Proof. Assume $x \notin A$. Then x is a point in the open subset $X \setminus A$ and hence by the definition of limit, all but finitely many elements x_n must belong to $X \setminus A$, contradicting our assumptions. □

3.4.2 Compact spaces

Definition 3.31. An *open cover* of a topological space X is a collection of open subsets of X whose union is X. If for every open cover of X there is a finite subcollection which also covers X, then X is called *compact*.

Some books (like Munkres' *Topology*) refer to open covers as *open coverings*, while newer books (and wikipedia) seem to prefer to above terminology, probably for the same reasons as me: to avoid confusions with *covering spaces*, a notion we'll introduce soon.

Now we'll prove some useful properties of compact spaces and maps between them, which will lead to the important Corollaries 3.36 and 3.34.

Lemma 3.32. *If $f\colon X \to Y$ is a continuous map and X is compact, then the image $f(X)$ is compact.*

In particular, if X is compact, then any quotient space $X/\sim$ is compact, since the projection map $X \to X/\sim$ is continuous with image $X/\sim$.

Proof. To show that $f(X)$ is compact assume that $\{U_a\}$, $a \in A$ is an open cover of the subspace $f(X)$. Then each U_a is of the form $U_a = V_a \cap f(X)$ for some open subset $V_a \in Y$. Then $\{f^{-1}(V_a)\}$, $a \in A$ is an open cover of X. Since X is compact, there is a finite subset A' of A such that $\{f^{-1}(V_a)\}$, $a \in A'$ is a cover of X. This implies that $\{U_a\}$, $a \in A'$ is a finite cover of $f(X)$, and hence $f(X)$ is compact. □

Lemma 3.33.

1. *If K is a closed subspace of a compact space X, then K is compact.*
2. *If K is compact subspace of a Hausdorff space X, then K is closed.*

Proof. To prove (1), assume that $\{U_a\}$, $a \in A$ is an open covering of K. Since the U_a's are open w.r.t. the subspace topology of K, there are open subsets V_a of X such that $U_a = V_a \cap K$. Then the V_a's together with the open subset $X \setminus K$ form an open covering of

X. The compactness of X implies that there is a finite subset $A' \subset A$ such that the subsets V_a for $a \in A'$, together with $X \setminus K$ still cover X. It follows that U_a, $a \in A'$ is a finite cover of K, showing that K is compact.

The proof of part (2) is a homework problem. □

Corollary 3.34. *If $f\colon X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. We need to show that the map $g\colon Y \to X$ inverse to f is continuous, i.e., that $g^{-1}(U) = f(U)$ is an open subset of Y for any open subset U of X. Equivalently (by passing to complements), it suffices to show that $g^{-1}(C) = f(C)$ is a closed subset of Y for any closed subset C of C.

Now the assumption that X is compact implies that the closed subset $C \subset X$ is compact by part (1) of Lemma 3.33 and hence $f(C) \subset Y$ is compact by Lemma 3.32. The assumption that Y is Hausdorff then implies by part (2) of Lemma 3.33 that $f(C)$ is closed. □

Lemma 3.35. *Let K be a compact subset of $\mathbb{R}^n$. Then K is* bounded*, meaning that there is some $r > 0$ such that K is contained in the open ball $B_r(0) := \{x \in \mathbb{R}^n \mid d(x, 0) < r\}$.*

Proof. The collection $B_r(0) \cap K$, $r \in (0, \infty)$, is an open cover of K. By compactness, K is covered by a *finite* number of these balls; if R is the maximum of the radii of these finitely many balls, this implies $K \subset B_R(0)$ as desired. □

Corollary 3.36. *If $f\colon X \to \mathbb{R}$ is a continuous function on a compact space X, then f has a maximum and a minimum.*

Proof. $K = f(X)$ is a compact subset of $\mathbb{R}$. Hence K is bounded, and thus K has an infimum $a := \inf K \in \mathbb{R}$ and a supremum $b := \sup K \in \mathbb{R}$. The infimum (resp. supremum) of K is the limit of a sequence of elements in K; since K is closed (by Lemma 3.33 (2)), the limit points a and b belong to K by Lemma 3.30. In other words, there are elements $x_{min}, x_{max} \in X$ with $f(x_{min}) = a \leq f(x)$ for all $x \in X$ and $f(x_{max}) = b \geq f(x)$ for all $x \in X$. □

In order to use Corollaries 3.34 and 3.36, we need to be able to show that topological spaces we are interested in, are in fact compact. Note that this is *quite difficult* just working from the definition of compactness: you need to ensure that *every* open cover has a finite subcover. That sounds like a lot of work...

Fortunately, there is a very simple classical characterization of compact subspaces of Euclidean spaces:

Theorem 3.37. (Heine-Borel Theorem) *A subspace $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

We note that we've already proved that if $K \subset \mathbb{R}^n$ is compact, then K is a closed subset of $\mathbb{R}^n$ (Lemma 3.33(2)), and K is bounded (Lemma 3.35).

There two important ingredients to the proof of the converse, namely the following two results:

Lemma 3.38. *A closed interval* $[a, b]$ *is compact.*

This lemma has a short proof that can be found in any pointset topology book, e.g., [Mu].

Theorem 3.39. *If* $X_1, \ldots, X_n$ *are compact topological spaces, then their product* $X_1 \times \cdots \times X_n$ *is compact.*

For a proof see e.g. [Mu, Ch. 3, Thm. 5.7]. The statement is true more generally for a product of *infinitely many* compact space (as discussed in [Mu, p. 113], the correct definition of the product topology for infinite products requires some care), and this result is called *Tychonoff's Theorem*, see [Mu, Ch. 5, Thm. 1.1].

Proof of the Heine-Borel Theorem. Let $K \subset \mathbb{R}^n$ be closed and bounded, say $K \subset B_r(0)$. We note that $B_r(0)$ is contained in the n-fold product

$$P := [-r, r] \times \cdots \times [-r, r] \subset \mathbb{R}^n$$

which is compact by Theorem 3.39. So K is a closed subset of P and hence compact by Lemma 3.33(1). □

3.4.3 Connected spaces

Definition 3.40. A topological space X is *connected* if it can't be written as decomposed in the form $X = U \cup V$, where U, V are two non-empty disjoint open subsets of X.

For example, if a, b, c, d are real numbers with $a < b < c < d$, consider the subspace $X = (a, b) \amalg (c, d) \subset \mathbb{R}$. The topological space X is not connected, since $U = (a, b)$, $V = (c, d)$ are open disjoint subsets of X whose union is X. This remains true if we replace the open intervals by closed intervals. The space $X' = [a, b] \amalg [c, d]$ is not connected, since it is the disjoint union of the subsets $U' = [a, b]$, $V' = [c, d]$. We want to emphasize that while U' and V' are not open as subsets of $\mathbb{R}$, they are *open subsets of* X', since they can be written as

$$U' = (-\infty, c) \cap X' \qquad V' = (b, \infty) \cap X',$$

showing that they are open subsets for the subspace topology of $X' \subset \mathbb{R}$.

Lemma 3.41. *Any interval* I *in* $\mathbb{R}$ *(open, closed, half-open, bounded or not) is connected.*

Proof. Using proof by contradiction, let us assume that I has a decomposition $I = U \cup V$ as the union of two non-empty disjoint open subsets. Pick points $u \in U$ and $v \in V$, and let us assume $u < v$ without loss of generality. Then

$$[u, v] = U' \cup V' \qquad \text{with} \qquad U' := U \cap [u, v] \quad V' := U \cap [u, v]$$

is a decomposition of $[u, v]$ as the disjoint union of non-empty disjoint open subsets U', V' of $[u, v]$. We claim that the supremum $c := \sup U'$ belongs to both, U' and V', thus leading to the desired contradiction. Here is the argument.

- Assuming that c doesn't belong to U', for any $\epsilon > 0$, there must be some element of U' belonging to the interval $(c - \epsilon, c)$, allowing us to construct a sequence of elements $u_i \in U'$ converging to c. This implies $c \in U'$ by Lemma 3.30, since U' is a *closed* subspace of $[u, v]$ (its complement V' is open).
- By construction, every $x \in [u, v]$ with $x > c = \sup U'$ belongs to V'. So we can construct a sequence $v_i \in V'$ converging to c. Since V' is a closed subset of $[u, v]$, we conclude $c \in V'$.

□

Theorem 3.42. (Intermediate Value Theorem) *Let X be a connected topological space, and $f \colon X \to \mathbb{R}$ a continuous map. If elements $a, b \in \mathbb{R}$ belong to the image of f, then also any real number c between a and b belongs to the image of f.*

Proof. Assume that c is not in the image of f. Then $X = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty)$ is a decomposion of X as a union of non-empty disjoint open subsets. □

There is another notion, closely related to the notion of connected topological space, which might be easier to think of geometrically.

Definition 3.43. A topological space X is *path connected* if for any points $x, y \in X$ there is a path connecting them. In other words, there is a continuous map $\gamma \colon [a, b] \to X$ from some interval to X with $\gamma(a) = x$, $\gamma(b) = y$.

Lemma 3.44. *Any path connected topological space is connected.*

Proof. Using proof by contradiction, let us assume that the topological space X is path connected, but not connected. So there is a decomposition $X = U \cup V$ of X as the union of non-empty open subsets $U, V \subset X$. The assumption that X is path connected allows us to find a path $\gamma \colon [a, b] \to X$ with $\gamma(a) \in U$ and $\gamma(b) \in V$. Then we obtain the decomposition

$$[a, b] = f^{-1}(U) \cup f^{-1}(V)$$

of the interval $[a, b]$ as the disjoint union of open subsets. These are non-empty since $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. This implies that $[a, b]$ is not connected, the desired contradiction.

□

For typical topological spaces we will consider, the properties “connected” and “path connected” are equivalent. But here is an example known as the *topologist’s sine curve* which is connected, but not path connected, see [Mu, Example 7, p. 156]. It is the following subspace of $\mathbb{R}^2$:

$$X = \{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < 1\} \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}.$$

4 Appendix: Manifolds

The purpose of this section is to provide interesting examples of topological spaces and homeomorphisms between them. There are many examples of “weird” topological spaces. There are non-Hausdorff spaces (they don’t have well-defined limits) or the topologist’s sine curve, which is connected, but not path connected. While there is a huge literature concering pathological topological spaces, I must admit that I find those examples most interesting that “show up in nature”. For example, topological spaces that appear as “configuration spaces” or “phase spaces” of physical systems. Often these are a particularly nice kind of topological space known as *manifold.*

There is much to say about manifolds. For example, you can find the text books *Introduction to topological manifolds* and *Introduction to smooth manifolds* by John Lee. For this section, our focus is to discuss manifolds of dimension 2. Unlike higher dimensional manifolds, we can represent manifolds of dimension 2 by pictures, which greatly helps the intuition about these objects.

Definition 4.1. A *manifold of dimension* n or n*-manifold* is a topological space X which is locally homeomorphic to $\mathbb{R}^n$, that is, every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. Moreover, it is useful and customary to require that X is Hausdorff (see Definition 3.29) and *second countable*, which means that the topology of X has a countable basis.

In most examples, the technical conditions of being Hausdorff and second countable are easy to check, since these properties are inherited by subspaces.

Homework 4.2. Show that a subspace of a Hausdorff space is Hausdorff. Show that a subspace of a second countable space is second countable.

Examples of manifolds.

1. Any open subset $U \subset \mathbb{R}^n$ is an n-manifold. The technical condition of being a second countable Hausdorff space is satisfied for U as a subspace of the second countable Hausdorff space $\mathbb{R}^n$.

2. The n-sphere $S^n := \{x \in \mathbb{R}^n \mid ||x|| = 1\}$ is an n-manifold. To prove this, let us look at the subsets

$$U_i^+ := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i > 0\} \subset S^n$$
$$U_i^- := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i < 0\} \subset S^n$$

We want to argue that the map

$$\phi_i^\pm \colon U_i^\pm \longrightarrow \mathring{D}^n \qquad \text{given by} \qquad \phi_i^\pm(x_0, \dots, x_n) := (x_0, \dots, x_{i-1}, \widehat{x}_i, x_{i+1}, \dots, x_n)$$

is a homeomorphism, where $\mathring{D}^n := \{(v_1, \dots, v_n) \in D^n \mid v_1^2 + \cdots + v_n^2 < 1\}$ is the open n-disk. It is easy to verify that the map

$$\mathring{D}^n \longrightarrow U_i^\pm \qquad v = (v_1, \dots, v_n) \mapsto (v_1, \dots, v_i, \pm\sqrt{1 - ||v||^2}, v_{i+1}, \dots, v_n)$$

is in fact the inverse to $\phi_i^\pm$. Here $||v||^2 = v_1^2 + \cdots + v_n^2$ is norm squared of $v \in \mathring{D}^n$. Both maps, $\phi_i^\pm$ and its inverse, are continuous since all their components are continuous. This shows that $\phi_i^\pm$ is in fact a homeomorphism, and hence the n-sphere S^n is a manifold of dimension n.

Homework 4.3. Show that the product $X \times Y$ of manifold X of dimension m and a manifold Y of dimension n is a manifold of dimension $m+n$. Make sure to prove that $X \times Y$ is second countable and Hausdorff.

Homework 4.4. Show that the real projective space $\mathbb{RP}^n$ is manifold of dimension n. Make sure to prove that $\mathbb{RP}^2$ is second countable and Hausdorff.

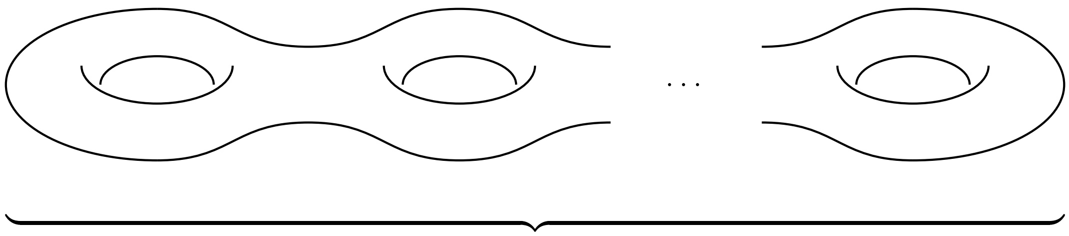
Examples of manifolds of dimension 2.

1. The 2-torus T. We recall that there are various ways of defining the torus, one being as the product $S^1 \times S^1$ which is a manifold of dimension 2 by Excercise 4.3, since S^1 is a manifold of dimension 1.

2. The real projective plane $\mathbb{RP}^2$.

3. The Klein bottle K. It is not hard to verify directly that K is a manifold of dimension 2. Alternatively, we will see in Lemma **??** that the Klein bottle is homeomorphic to the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ of two copies of the projective plane $\mathbb{RP}^2$, which implies in particular that K is a 2-manifold.

4. The surface Σ_g of genus g is the subspace of $\mathbb{R}^3$ given by the following picture:



Here g is the number of "holes" of Σ_g. In particular Σ_1, the surface of genus 1, is the torus. By convention, the surface Σ_0 of genus 0 is the 2-sphere S^2. Since we have described the surface of genus g as a subspace of $\mathbb{R}^3$ given by a picture rather than a formula, it is impossible to give a precise argument that this subspace is locally homeomorphic to $\mathbb{R}^2$, but hopefully the picture makes this obvious at a heuristic level.

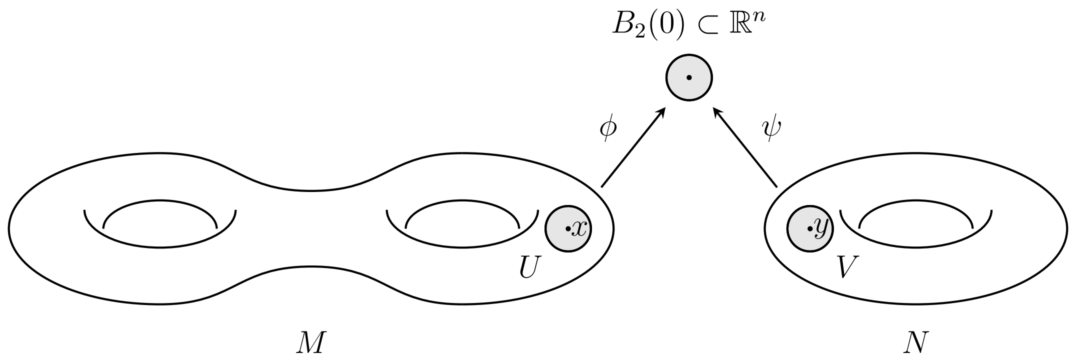
The connected sum construction. This construction produces a new manifold $M\#N$ of dimension n from two given manifolds M and N of dimension n. The manifold $M\#N$ is called the *connected sum* of M and N. The construction proceeds as follows. First we make some choices:

- We pick points $x \in M$ and $y \in N$.
- We pick a homeomorphism ϕ between an open neighborhood U of x and the open ball $B_2(0)$ of radius 2 around the origin $0 \in \mathbb{R}^n$. Similarly, we pick a homeomorphism $\psi\colon V \xrightarrow{\approx} B_2(0)$ where $V \subset N$ is an open neighborhood of $y \in N$.

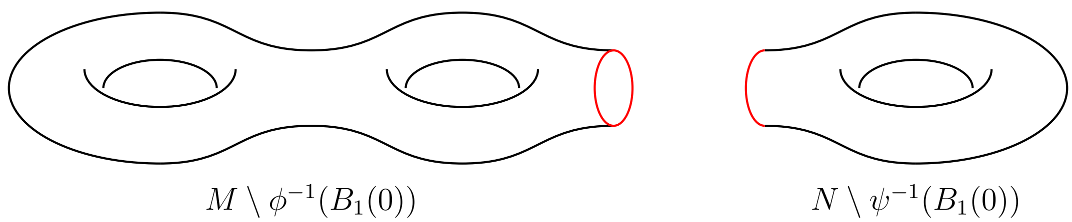
The existence of homeomorphisms ϕ, ψ with these properties follows from the assumption that M, N are manifolds of dimension n. This implies that there is an open neighborhood $U' \subset M$ of x and a homeomorphism ϕ' between U' and an open subset $V' \subset \mathbb{R}^n$. Composing ϕ by a translation in $\mathbb{R}^n$ we can assume that $\phi(x) = 0 \in \mathbb{R}^n$. Since V' is open, there is some $\epsilon > 0$ such that the open ball $B_\epsilon(0)$ of radius ϵ around $0 \in \mathbb{R}^n$ is contained in V'. Then restricting ϕ' to $U := (\phi')^{-1}(B_\epsilon(0)) \subset M$ gives a homeomorphism between U and $B_\epsilon(0)$. Then the composition

$$U \xrightarrow[\approx]{\phi'_U} B_\epsilon(0) \xrightarrow[\approx]{\text{multiplication by } 2/\epsilon} B_2(0)$$

is the desired homeomorphism ϕ between a neighborhood U of $x \in M$ and $B_2(0) \subset \mathbb{R}^n$. Analogously, we construct the homeomorphism ψ. Here is a picture illustrating the situation.



The next step is to remove the open disc $\phi^{-1}(B_1(0))$ from the manifold M and the open disc $\psi^{-1}(B_1(0))$ from the manifold N. The following picture shows the resulting topological spaces $M \backslash \phi^{-1}(B_1(0))$ and $N \backslash \psi^{-1}(B_1(0))$. Here the red circles mark the points corresponding to the sphere $S^{n-1} \subset B_2(0)$ via the homeomorphisms ϕ and ψ, respectively.



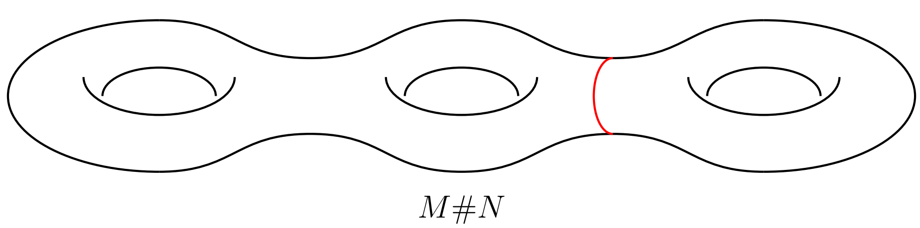
The final step is to pass to a quotient space of the union

$$M \setminus \phi^{-1}(B_1(0)) \quad \cup \quad N \setminus \psi^{-1}(B_1(0))$$

given by identifying points in $\phi^{-1}(S^{n-1})$ with their images under the homeomorphism

$$\phi^{-1}(S^{n-1}) \xrightarrow{\approx} \psi^{-1}(S^{n-1}) \qquad z \mapsto \psi^{-1}(\phi(z)).$$

The connected sum $M\#N$ is this quotient space. In terms of our pictures, the manifold $M\#N$ is obtained by gluing the two red circles, and is given by the following picture.



Theorem 4.5. (Classification Theorem for compact connected 2-manifolds.) *Every compact connected manifold of dimension 2 is homeomorphic to exactly one of the following manifolds:*

- *The connected sum* $\underbrace{T\#\dots\#T}_{g}$ *of* g *copies of the torus* T, $g \geq 0$;
- *The connected sum* $\underbrace{\mathbb{RP}^2\#\dots\#\mathbb{RP}^2}_{k}$ *of* k *copies of the real projective plane* $\mathbb{RP}^2$, $k \geq 1$;

5 Appendix: Categories and functors

Before giving the formal definition of categories, let us recall examples of mathematical objects that are quite familiar.

mathematical objects	appropriate maps between these objects
sets	maps
groups	group homomorphisms
vector spaces over a fixed field	linear maps
topological spaces	continuous maps

There are obvious similarities between these four cases of mathematical objects, suggesting to destill their commonality into a definition.

Definition 5.1. A *category* C consists of the following data:

- A class $\operatorname{ob}\mathsf{C}$ of *objects of* C.
- For any two objects $A, B \in \operatorname{ob}\mathsf{C}$ a set $\mathsf{C}(A,B)$ of *morphisms from* A *to* B. It is common to use the notation $A \xrightarrow{f} B$ to indicate that f is a morphism from A to B, and to call A the *domain* or *source* of f, and B its *codomain* or *target*.
- Morphisms $f \in \mathsf{C}(A,B)$ and $g \in \mathsf{C}(B,C)$ can be composed to obtain a morphism $g \circ f \in \mathsf{C}(A,C)$. In other words, there is a *composition map*

$$\circ\colon \mathsf{C}(B,C) \times \mathsf{C}(A,B) \longrightarrow \mathsf{C}(A,C)$$
$$(g,f) \mapsto g \circ f$$

These are required to satisfy the following properties:

(associativity) If $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ are morphisms, then $(h \circ g) \circ f = h \circ (g \circ f)$.

(identity) For every object B there exists a morphism $\mathrm{id}_B\colon B \to B$, called *identity morphism* such that for all morphism $f\colon A \to B$ and $g\colon B \to C$ we have $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$.

Next we want to define what a *functor* is. As usual, before giving the formal definition, we want to give at least one example of the to be defined notion as a motivation for the definition. Our motivating example of a functor is the fundamental group:

- For each topological space X equipped with a base point $x_0 \in X$, we have its fundamental group $\pi_1(X, x_0)$.
- A continuous map $f\colon X \to Y$ with $f(x_0) = y_0 \in Y$ leads to a group homomorphism $f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$.

From an abstract point of view, the fundamental group takes an object (X, x_0) of one category (the category of pointed topological spaces) and produces an object $\pi_1(X, x_0)$ of another category (the category of groups). Moreover, it takes a morphism $f\colon (X, x_0) \to (Y, y_0)$ in the category of pointed topological spaces and produces a morphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ in the category of groups.

Definition 5.2. A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ from a category C to a category D consists of the following data:

- An assignment that maps each object $A \in \operatorname{ob}\mathsf{C}$ to an object $F(A) \in \operatorname{ob}\mathsf{D}$.
- An assignment that maps each morphism $g\colon A \to B$ in C to a morphism $F(g)\colon F(A) \to F(B)$ in D.

We require:

(Compatibility with composition) For morphisms $f\colon A \to B$ and $g\colon B \to C$ in C

$$F(g \circ f) = F(g) \circ F(f) \in \mathsf{D}(A, C).$$

(Compatibility with identities) For any object $A \in \mathsf{C}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Examples of functors.

functors	on objects	on morphisms
$\pi_1\colon \mathsf{Top}_* \to \mathcal{G}\mathrm{p}$	$(X, x_0) \mapsto \pi_1(X, x_0)$	$(X, x_0) \xrightarrow{f} (Y, y_0) \mapsto \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$
$-\otimes W\colon \mathsf{Vect}_k \to \mathsf{Vect}_k$	$V \mapsto V \otimes W$	$V \xrightarrow{f} V' \mapsto V \otimes W \xrightarrow{f\otimes \mathrm{id}} V' \otimes W$
$F\colon \mathsf{Set} \longrightarrow \mathsf{Vect}_k$	$S \mapsto k[S]$	$S \xrightarrow{f} T \mapsto k[S] \xrightarrow{k[f]} k[T]$

Here $k[S]$ is the k-vector space of finite linear combinations $\sum_{s\in S} k_s s$ of elements of s with coefficients $k_s \in k$. The adjective *finite* means that we require $k_s = 0$ for all but finitely many $s \in S$. The map $k[f]\colon k[S] \to k[T]$ sends $\sum_{s\in S} k_s s$ to $\sum_{s\in S} k_s f(s)$, which is a finite linear combination of elements of T.

References

[Ha] Hatcher, Allen, *Algebraic topology*. Cambridge University Press, Cambridge, 2002. xii+544 pp.

[Mu] Munkres, James R. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975. xvi+413 pp.