Solutions to homework problems

November 22, 2023

Contents

1 Homework Assignment # 1

1. (10 points) Let $GL_n(\mathbb{R})$ be the set of invertible $n \times n$ matrices.

(a) Show that $GL_n(\mathbb{R})$ is an open subset of the topological space $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of all $n \times n$ matrices.

(b) Show that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is a continuous map.

Proof. **Part (a).** To show that $GL_n(\mathbb{R})$ is an open subset of $M_{n\times n}(\mathbb{R})$, we note that $GL_n(\mathbb{R})$ is the preimage of $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ under the determinant map

$$\det\colon M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R} \qquad A \mapsto \det(A).$$

Since $\det(A)$ is a polynomial in the matrix coefficients of A, this is a polynomial function and hence continuous. Therefore $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}^\times)$ is an open subset of $M_{n\times n}(\mathbb{R})$, provided that $\mathbb{R}^\times$ is an open subset of $\mathbb{R}$. This is the case, since $\mathbb{R}^\times$ is the union of the balls $B_1(k) = (k-1, k+1)$ for $k = \pm1, \pm2, \dots$.

Part (b). To prove continuity of the map f from open subset $\mathrm{GL}_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ to itself, it suffices by a result in class that all components of the map $GL_n(\mathbb{R}) \xrightarrow{f} GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ are continuous functions. In other words, we need to show that for $1 \le i, j \le n$ the map $GL_n(\mathbb{R}) \to \mathbb{R}$, $A \mapsto (A^{-1})_{ij}$ is continuous. Here $(A^{-1})_{ij}$ denotes the ij-entry of the matrix A^{-1}.

We recall from linear algebra that the inverse of A can be calculated by the formula

$$A^{-1} = \frac{C^t}{\det(A)},$$

where $\det(A)$ is the determinant of A, and C^t is the transpose of the $n \times n$ matrix C whose entry C_{ij} is $(-1)^{i+j}$ times the (i,j)-minor of A (the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A).

This shows that each matrix entry $(A^{-1})_{ij}$ is of the form $\frac{p(A)}{q(A)}$, where $p(A)$ and $q(A)$ are *polynomial functions* of the matrix entries of A. In particular the functions $p, q\colon GL_n(\mathbb{R}) \to \mathbb{R}$ are *continuous*. Since $q(A) = \det(A)$ in non-zero for all $A \in GL_n(\mathbb{R})$, we can regard q as a map $q\colon GL_n(\mathbb{R}) \to \mathbb{R}^\times$ ($\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$). Hence we can form the composition

$$GL_n(\mathbb{R}) \xrightarrow{q} \mathbb{R}^\times \xrightarrow{I} \mathbb{R},$$

where I is the inversion map $x \mapsto x^{-1}$. From calculus we know that I is continuous and hence so is $I \circ q$. This implies that the function

$$\frac{p}{q} = p \cdot (I \circ q)$$

is continuous. □

2. (10 points) The point of this problem is to show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology (where each factor is equipped with the metric topology). Since both, the metric topology and the product topology, are defined via a basis, it is good to know how to compare two topologies given in terms of bases. This is provided by the statement of part (a).

(a) Let X be a set, and let $\mathcal{T}$, $\mathcal{T}'$ be topologies generated by a basis $\mathcal{B}$ resp. $\mathcal{B}'$. Show that $\mathcal{T} \subset \mathcal{T}'$ if and only if for each $B \in \mathcal{B}$ and $x \in B$ there is some $B' \in \mathcal{B}'$ with $x \in B'$ and $B' \subset B$.

(b) Show that the products of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^m \times \mathbb{R}^n$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $s, r > 0$ generate the product topology on $\mathbb{R}^m \times \mathbb{R}^n$.

(c) Show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology. Hint: it might be helpful to draw pictures of a ball around $(x, y) \in \mathbb{R}^{m+n}$ and a product of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^{m+n}$ for $m = n = 1$.

Proof. **Part (a).** Assume that for each $B \in \mathcal{B}$ and $x \in B$ there is some $B' \in \mathcal{B}'$ with $x \in B'$ and $B' \subset B$. To show $\mathcal{T} \subset \mathcal{T}'$, let $U \in \mathcal{T}$, i.e., U is an open subset of X w.r.t. the topology $\mathcal{T}$. Then U is the union of subsets B_i belonging to $\mathcal{B}$. By assumption, each B_i is the union of subsets belonging to $\mathcal{B}'$, and hence U is the union of subsets belonging to $\mathcal{B}'$ and consequently $U \in \mathcal{T}'$.

Conversely, if $\mathcal{T} \subset \mathcal{T}'$, let $B \in \mathcal{B}$ and $x \in B$. Then $B \in \mathcal{T}$, and hence $B \in \mathcal{T}'$. It follows that B is a union of a collection of subsets B'_i belonging to $\mathcal{B}'$. In particular, there is some i such that B'_i contains the point x.

Part (b). We need to show that every subset $W \subset \mathbb{R}^m \times \mathbb{R}^n$ which is open with respect to the product topology can be written as a union of products of balls; in other words, if $(x, y) \in W$, we need to find $r > 0$, $s > 0$ such that $B_r(x) \times B_s(y) \subset W$. Since the product topology is generated by products $U \times V$ of open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, the openness of W guarantees that there is such a product $U \times V$ with $(x, y) \in U \times V \subset W$. The openness of U resp. V in the metric topology guarantees that there are balls $B_r(x) \subset U$ and $B_s(y) \subset V$. In particular,

$$(x, y) \in B_r(x) \times B_s(y) \subset U \times V \subset W$$

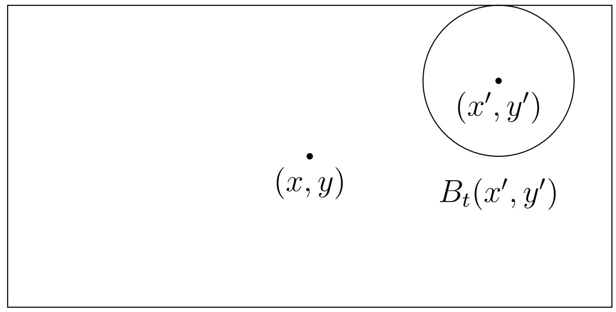
as desired.

Part (c). By part (b), the product topology $\mathcal{T}$ on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is generated by the collection $\mathcal{B}$ consisting of products of balls, i.e., subsets of the form $B_r(x) \times B_s(y)$ for $x \in X$, $y \in Y$, $r, s > 0$. The metric topology $\mathcal{T}'$ is generated by the collection $\mathcal{B}'$ consisting of balls $B_t(x, y)$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and $t > 0$. By part (a) it suffices to show that

(i) Given $(x', y') \in B_r(x) \times B_s(y)$ there is some $t > 0$ such that $B_t(x', y') \subset B_r(x) \times B_s(y)$;

(ii) Given $(x', y') \in B_t(x, y)$, there are $r, s > 0$ such that $B_r(x') \times B_s(y') \subset B_t(x, y)$.

Our strategy to prove (i) is to find the largest ball $B_t(x', y')$ with center (x', y') that is contained in the product $B_r(x) \times B_s(y)$. The situation is illustrated by the following picture.



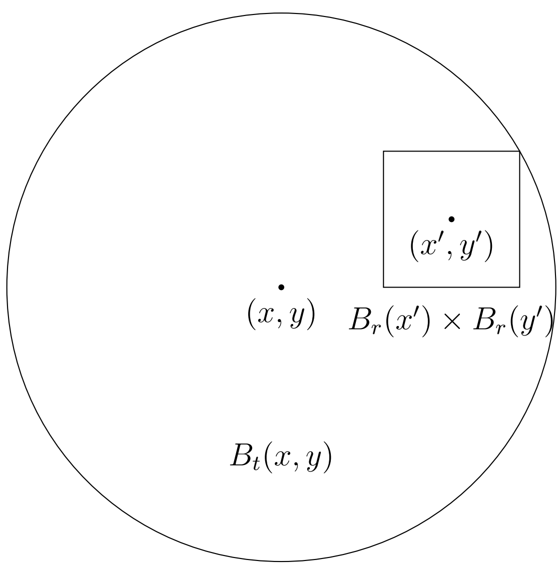
$B_r(x) \times B_s(y)$

We note that $r - \operatorname{dist}(x, x')$ is the distance of the point $x' \in B_r(x)$ to the boundary of the ball; similarly, $s - \operatorname{dist}(y, y')$ is the distance of $y' \in B_s(y)$ to the boundary of that ball. So choosing $t := \min\{r - \operatorname{dist}(x, x'), s - \operatorname{dist}(y, y')\}$ should guarantee $B_t(x', y') \subset B_r(x) \times B_s(y)$. To check this, let $(x'', y'') \in B_t(x', y')$. Then by the triangle inequality

$$\operatorname{dist}(x'', x) \leq \operatorname{dist}(x'', x') + \operatorname{dist}(x', x) < t + \operatorname{dist}(x', x) \leq (r - \operatorname{dist}(x, x')) + \operatorname{dist}(x', x) = r.$$

Similarly, $\operatorname{dist}(y'', y) < s$ and hence $(x'', y'') \in B_r(x) \times B_s(y)$, which proves $B_t(x', y') \subset B_r(x) \times B_s(y)$.

To prove (ii), our strategy is to find the largest product of balls $B_r(x') \times B_r(y')$ of the same radius which is inside the ball $B_t(x, y)$. Here is a picture illustrating the situation:



The picture suggests that we should chose r such that the distance from the center point (x', y') to the corner points of the "square" $\bar{B}_r(x') \times \bar{B}_r(y')$ is equal to the distance of $(x'y')$

to the boundary of the big ball $\bar{B}_t(x,y)$ (here $\bar{B}_r$ denotes the closed ball of radius r). The Euclidean distance from (x',y') to a corner point is $\sqrt{r^2+r^2} = r\sqrt{2}$, the distance of (x',y') to the boundary of $B_t(x,y)$ is $t - \mathrm{dist}((x',y'),(x,y))$. This suggests to define

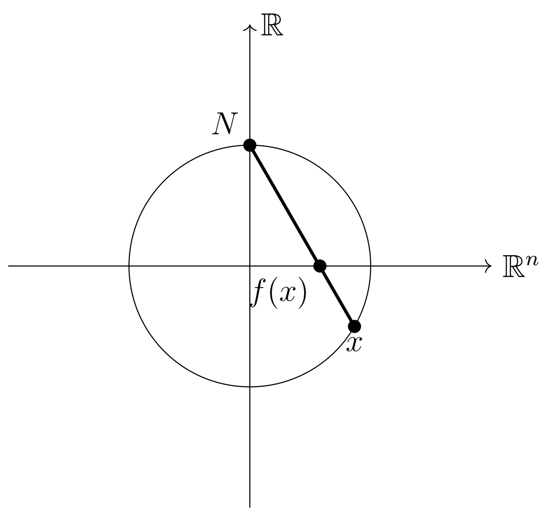
$$r := \frac{1}{\sqrt{2}}(t - \mathrm{dist}((x',y'),(x,y))).$$

Let us check that with this choice of r the product $B_r(x') \times B_r(y')$ is contained in $B_t(x,y)$. So let $(x'',y'') \in B_r(x') \times B_r(y')$. Then

$$\begin{aligned}
\mathrm{dist}((x'',y''),(x,y)) \leq& \,\mathrm{dist}((x'',y''),(x',y')) + \mathrm{dist}((x',y'),(x,y))\\
=&\sqrt{||x''-x'||^2 + ||y''-y'||^2} + \mathrm{dist}((x',y'),(x,y))\\
\leq&\sqrt{2r^2} + \mathrm{dist}((x',y'),(x,y))\\
=&\sqrt{2}r + \mathrm{dist}((x',y'),(x,y))\\
=&(t - \mathrm{dist}((x',y'),(x,y))) + \mathrm{dist}((x',y'),(x,y))\\
=&t
\end{aligned}$$

□

3. (10 points) Let $N \in S^n$ be the "north pole" of S^n, i.e., $N = (0,\dots,0,1) \in S^n$. The *stereographic projection* is the map $f\colon S^n \setminus \{N\}$ which sends a point $x \in S^n \setminus \{N\}$ to the intersection point of the straight line L_x in $\mathbb{R}^{n+1}$ with endpoint N and x with $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Here is a picture of the situation for $n = 1$.



The map $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a bijection. Explicitly, the map f and its inverse are given by the explicit formulas

$$f(x_1,\dots,x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1,\dots,x_n) \qquad f^{-1}(y_1,\dots,y_n) = \frac{1}{||y||^2+1}(2y_1,\dots,2y_n,||y||^2-1)$$

for $(x_1, \dots, x_{n+1}) \in S^n$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$. Provide a careful argument for the continuity of f and f^{-1} (you can use freely that recognize that certain maps $\mathbb{R} \supset U \to \mathbb{R}$ are continuous, but each time you use one of our "continuity criteria" for maps involving sub-spaces, products and quotients, you should be explicit about it).

Proof. The only thing left to show here is the continuity of the map f and its inverse f^{-1}.

The formula $f(x_1, \dots, x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1, \dots, x_n) = (\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}})$ gives a well-defined map from the open subset $U \subset \mathbb{R}^{n+1}$ given by $U = \{(x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_{n+1} \neq 1\}$ to $\mathbb{R}^n$. It is a continuous map since all its components are continuous. Hence the restriction of this map to $S^n \setminus \{N\} \subset U$ is continuous as well.

The codomain of the inverse map $f\colon \mathbb{R}^n \to S^n \setminus \{N\}$ is a subspace of $\mathbb{R}^{n+1}$, and hence to check continuity of f^{-1}, it suffices to check continuity of $i \circ f^{-1}$, where $i\colon S^n \setminus \{N\} \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion map. This is continuous, since its component functions are $\frac{2y_i}{||y||^2+1}$ resp. $\frac{||y||^2-1}{||y||^2+1}$, since they are quotients of polynomial functions on $\mathbb{R}^{n+1}$ with nowhere vanishing denominators. □

4. (10 points) Do the first step towards proving that the quotient space D^n/S^{n-1} is homeomorphic to the sphere S^n by constructing a continuous bijection from one of these spaces to the other (a result we'll cover in class next week will make it easy to conclude that this is in fact a homeomorphism). Hint: produce a bijective map f relating these spaces by writing down an explicit formula, paying attention to have this map go the "natural direction" to make proving its continuity simple.

Proof. Generalizing the case $n = 1$ which we did in class, we define a map

$$f\colon D^n \longrightarrow \mathbb{R}^n \oplus \mathbb{R} = \mathbb{R}^{n+1} \qquad \text{by} \qquad f(v) := (\frac{\sin \pi ||v||}{||v||} v, \cos \pi ||v||)$$

We recall from Calculus that $t \to \frac{\sin \pi t}{t}$ is a continuous function on $\mathbb{R} \setminus \{0\}$ which extends to a continuous function on all of $\mathbb{R}$ (by defining its value at $t = 0$ to be π). It follows that f is a continuous function since all its components are continuous.

It's easy to check that $f(v)$ belongs to $S^n \subset \mathbb{R}^{n+1}$: $f(0) = (0, 1) \in S^n$, and for $v \neq 0$ we have

$$||f(v)||^2 = \sin^2 \pi ||v||^2 + \cos^2 \pi ||v|| = 1$$

We conclude that the map $D^n \to S^n$, $v \mapsto f(v)$ is continuous w.r.t. the subspace topology on S^n since its composition with the inclusion map $i\colon S^n \to \mathbb{R}^{n+1}$ is the map f. Abusing notation, we write again $f\colon D^n \to S^n$.

We note that f maps any unit vector $v \in D^n$ to the point $(0, -1) \in \mathbb{R}^n \oplus \mathbb{R}$. Hence f induces a well-defined map

$$\bar{f}\colon D^n/S^{n-1} \longrightarrow S^n \qquad [v] \mapsto f(v)$$

This map is continuous since its composition with the projection map $p\colon D^n \to D^n/S^{n-1}$ is the map f which is continuous.

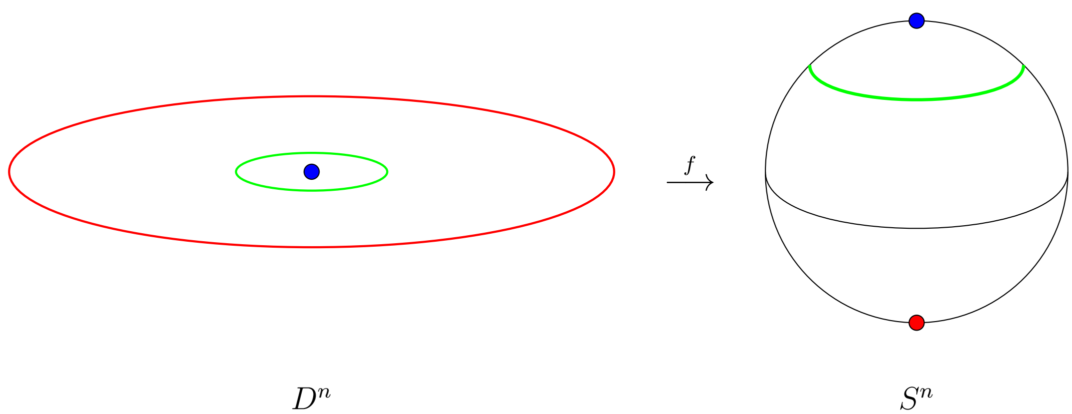
To show that $\bar{f}$ is bijective, we note that f maps the origin $0 \in D^n$ to the point $(0,1) \in S^n$ (points indicated by the blue dots in the picture below illustrating the situation for $n = 2$), and all of the boundary sphere $S^{n-1} \subset D^n$ to the point $(0,-1)$ (indicated by the color red). All points of S^{n-1} are identified to *one point* of the quotient space D^n/S^{n-1} which is mapped bijectively to the "south pole" $(0,-1) \in S^n$. Any point v of the sphere

$$S^{n-1}_r := \{v \in \mathbb{R}^n \mid ||v|| = r\} \qquad \text{for } 0 < r < 1\text{, indicated by the green circle in } D^n,$$

can be written uniquely in the form $v = re$, where e is a unit vector in $\mathbb{R}^n$. Then

$$f(v) = f(re) = (\sin \pi r e, \cos \pi r);$$

i.e., as $e \in S^{n-1}$ varies, the image under f is the $(n-1)$-sphere of radius $\sin \pi r$ with centerpoint $(0, \cos \pi r) \in \mathbb{R}^n \oplus \mathbb{R}$ in the plane parallel to $\mathbb{R}^n$ through that point (the green circle in S^n in the picture below). In particular, for fixed r the map $\bar{f}$ provides a bijection between these two spheres. Since the map $(0,1) \to (-1,1)$ given by $r \mapsto \cos \pi r$ is a bijection, the map f is a bijection between the open punctured ball $\{v \in \mathbb{R}^n \mid 0 < ||v|| < 1\}$ and $S^n \setminus \{(0,1),(0,-1)\}$ (the sphere without the north pole and the south pole).



Hence $\bar{f}\colon D^n/S^{n-1} \to S^n$ is a continuous bijection. $\square$

5. (10 points) Consider the following topological spaces

- The subspace $T_1 := \{v \in \mathbb{R}^3 \mid \operatorname{dist}(v, S) = r\} \subset \mathbb{R}^3$ equipped with the subspace topology, where $S = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $0 < r < 1$.
- The product space $T_2 := S^1 \times S^1$ equipped with the product topology.
- The quotient space $T_3 := ([-1,1] \times [-1,1])/\sim$ equipped with the quotient topology, where the equivalence relation is generated by $(s,-1) \sim (s,1)$ and $(-1,t) \sim (1,t)$.

Construct two bijective continuous maps between these spaces such that each of these three spaces features in at least one of these (once we have the convenient continuity criterion for the inverse of a continuous bijection, this easily implies that these three spaces are all homeomorphic). Hint: as in the previous problem, pick your maps to go in a direction that makes it easy to verify continuity using the Continuity Criterions for maps to/from subspaces, product spaces resp. quotient spaces.

Proof. We recall from class that checking continuity of a map $f\colon X \to Y$ is easy if the domain X is a quotient space and the range Y is a subspace or a product space. This suggests to construct continuous bijections

$$f\colon T_3 \to T_1 \qquad \text{and} \qquad g\colon T_3 \to T_2.$$

We first discuss the simpler map g given by the formula

$$g([s,t]) := ((\cos\pi s, \sin\pi s), (\cos\pi t, \sin\pi t)) \in S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2. \tag{1.1}$$

To argue that this is a well-defined bijection, we note that the map $[-1,1] \to S^1$, $s \mapsto (\cos\pi s, \sin\pi s)$ is surjective, and its only failure to be injective is due to the fact that it maps $s = -1$ and $s = +1$ to the same point in S^1. The map above when precomposed with the projection map $[-1,1] \times [-1,1] \to T_3$ is the product of two copies of this map. Hence it is surjective, and its only failure of injectivity comes from points (s,t) with $s = \pm 1$ or $t = \pm 1$. This shows that g is in fact a bijection. To prove continuity of g it suffices to prove continuity of the composition

$$[-1,1] \times [-1,1] \xrightarrow{pr} [-1,1] \times [-1,1]/\{\pm 1\} \xrightarrow{g} S^1 \times S^1 \xrightarrow{i} \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4\ ,$$

where pr is the projection map, the i is the inclusion map. This map is continuous, since all its component functions, given explicitely in equation (1.1) are continuous.

Now we define the map f, first describing it geometrically. From this point of view, it will be obvious that f is bijective. Then we will derive an explicit formula for f, which will make it easy to argue that f is continuous.

We note that there is a surjective map $p\colon T_1 \to S$ which sends a point $v \in \mathbb{R}^3$ to the unique point $(x,y,0) \in S$ which is closest to v. Similarly, there is a surjective map

$$q\colon T_3 = ([-1,1] \times [-1,1])/\sim \longrightarrow [-1,1]/\{\pm 1\} \qquad \text{given by} \qquad [s,t] \mapsto [s].$$

We want to construct the map f such that it fits into a commutative diagram

$$\begin{tikzcd}
T_3 = ([-1,1] \times [-1,1])/\sim \arrow[r, "f"] \arrow[d, "q"] & T_1\ , \arrow[d, "p"] \\
{[-1,1]/\{\pm 1\}} \arrow[r, "f_0"] & S
\end{tikzcd}$$

where f_0 is the homeomorphism we discussed in class that sends $[s]$ to $(\cos \pi s, \sin \pi s, 0) \in S \subset \mathbb{R}^3$. We note that the fiber $q^{-1}([s])$ for any fixed $[s] \in [-1,1]/\{\pm 1\}$ is a circle. Similarly, for any point $(x, y, 0)$ of the circle $S \subset \mathbb{R}^3$, the fiber $p^{-1}(x, y, 0) \subset T_1$ is a circle of radius r in the plane spanned by the unit vectors $(x, y, 0)$ and $(0, 0, 1)$ with center $(x, y, 0)$. We note that the map

$$\begin{aligned}[-1,1]/\{\pm 1\} \longrightarrow & \mathbb{R}^3 \\ [t] \mapsto & (x, y, 0) + r \cos \pi t (x, y, 0) + r \sin \pi t (0, 0, 1) \\ = & ((1 + r \cos \pi t)x, (1 + r \cos \pi t)y, r \sin \pi t)\end{aligned}$$

is a bijection onto the circle $p^{-1}(x, y, 0)$. In particular, defining f on each fiber $q^{-1}([s])$ to be this map, we obtain a bijection between T_3 and T_1.

To argue that f is continuous, we write down $f\colon T_3 \to T_1$ explicitly: f_0 sends a point $[s] \in [-1,1]/\{\pm 1\}$ to $(x, y, 0) = (\cos \pi s, \sin \pi s, 0)$ and hence

$$\begin{aligned}f([s,t]) =& ((1 + r \cos \pi t)x, (1 + r \cos \pi t)y, r \sin \pi t) \\ =& ((1 + r \cos \pi t) \cos \pi s, (1 + r \cos \pi t) \sin \pi s, r \sin \pi t).\end{aligned} \tag{1.2}$$

Then f is continuous if and only if the composition

$$[-1,1] \times [-1,1] \xrightarrow{pr} [-1,1] \times [-1,1]/\{\pm 1\} \xrightarrow{f} T_1 \xrightarrow{i} \mathbb{R}^3$$

is continuous. This is continuous since its component functions are, which can be read off from equation (1.2). □

2 Homework Assignment # 2

1. (10 points) Show that a closed subspace C of a compact topological space X is compact.

Proof. Assume that $\{U_a\}$, $a \in A$ is an open covering of C. Since the U_a's are open w.r.t. the subspace topology of C, there are open subsets V_a of X such that $U_a = V_a \cap K$. Then the V_a's together with the open subset $X \setminus C$ form an open covering of X. The compactness of X implies that there is a finite subset $A' \subset A$ such that the subsets V_a for $a \in A'$, together with $X \setminus C$ still cover X. It follows that U_a, $a \in A'$ is a finite cover of C, showing that C is compact. □

2. (10 points) Let X be a topological space which is the union of two subspaces X_1 and X_2. Let $f\colon X \to Y$ be a (not necessarily continuous) map whose restriction to X_1 and X_2 is continuous.

(a) Show f is continuous if X_1 and X_2 are open subsets of X.

(b) Show f is continuous if X_1 and X_2 are closed subsets of X.

(c) Give an example showing that in general f is *not continuous*.

Remark. This result is needed to verify that various constructions (e.g., concatenations of paths) in fact lead to *continuous* maps. In a typical situation, we have continuous maps $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ which agree on $X_1 \cap X_2$ and hence there is a well-defined map

$$f\colon X \longrightarrow Y \qquad \text{given by} \qquad f(x) = \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \end{cases}$$

The above result then helps to show that this map is continuous.

Proof. To prove part (a) let $U \subset Y$ be open. Then $U_i := f_i^{-1}(U)$ is an open subset of X_i, i.e., it is of the form $U_i = V_i \cap X_i$, where $V_i \subset X$ is open. If follows that U_i is an open subset of X, and hence $f^{-1}(U) = f_1^{-1}(U) \cup f_2^{-1}(U) = U_1 \cup U_2$ is an open subset of X, proving the continuity of f.

To prove part (b) we note that $f\colon X \to Y$ is continuous if and only if $f^{-1}(U)$ is a closed subset of X for every closed subset $U \subset Y$. Then repeating the previous sentences with *open* replaced by *closed* provides a proof of part (b).

For part (c) consider the map $f\colon X = \mathbb{R} \to Y = \mathbb{R}$ given by $f(t) = 0$ if $t \in (-\infty, 0)$, and $f(t) = 1$ if $t \in [0, \infty)$. The restrictions of f to $(-\infty, 0)$ resp. $[0, \infty)$ are constant and hence continuous, but f is not.

Based on a request from one of you, here is also an example with $X_1 \cap X_2 \neq \emptyset$. Let X be the quotient space $X = [-1, +1]/\{\pm 1\}$, and let $f\colon X \to \mathbb{R}$ be the map defined by $f(t) = t$ for $-1 < t < 1$, and $f(\pm 1) := -1$. Then f is not continuous since the composition $[-1, +1] \to [-1, +1]/\{\pm 1\} \xrightarrow{f} \mathbb{R}$ isn't continuous, but the restriction of f to $X_1 = (-1, +1) \subset X$ and $X_2 = [-1, 0)$ evidently is continuous. □

3. (10 points) Use the Heine-Borel Theorem to decide which of the topological groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are compact. Provide proofs for your statements. Hint: A strategy often useful for proving that a subset C of $\mathbb{R}^n$ is closed is to show that C is of the form $f^{-1}(C')$ for some closed subset $C' \subset \mathbb{R}^k$ (often C' consists of just one point) and some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^k$.

Proof. The special linear group $SL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ is not compact for $n \geq 2$, since it is not bounded. To see this, consider the sequence of diagonal matrices

$$A_i = \begin{pmatrix} i & & & \\ & i^{-1} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in SL_n(\mathbb{R}).$$

Considered as a vector in $\mathbb{R}^{n^2}$, the norm squared of A_i is given by

$$||A_i||^2 = i^2 + i^{-2} + 1^2 + \cdots + 1^2 \geq i^2$$

which shows that $SL_n(\mathbb{R})$ is an unbounded subset of $\mathbb{R}^{n^2}$ for $n \geq 2$. For $n = 1$, the group $SL_1(\mathbb{R}) = \{1\} \subset \mathbb{R}$, and hence is compact.

The general linear group $GL_n(\mathbb{R})$ is unbounded for $n \geq 2$, since it contains the unbounded subgroup $SL_n(\mathbb{R})$. The group $GL_1(\mathbb{R}) = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ is obviously unbounded. Hence $GL_n(\mathbb{R})$ is non-compact for all $n \geq 1$.

We claim that $O(n)$ and $SO(n)$ are compact. We begin by showing that they are bounded subspaces of $\mathbb{R}^{n^2}$. So let $A \in O(n)$ be a matrix with column vectors $v_1, \ldots, v_n \in \mathbb{R}^n$. The assumption that A belongs to $O(n)$ means that the v_i's form an orthogonal basis of $\mathbb{R}^n$. Hence

$$||A||^2 = ||v_1||^2 + \cdots + ||v_n||^2 = n,$$

which shows that $O(n)$ and hence also $SO(n) \subset O(n)$ are bounded subsets of $\mathbb{R}^{n^2}$.

To show that $O(n)$ is a closed subset of $\mathbb{R}^{n^2}$ we note that a matrix A with column vectors $v_i \in \mathbb{R}^n$ belongs to $O(n)$ if and only if the v_i's are orthogonal, i.e., if

$$\langle v_i, v_j \rangle = \delta_{i,j} \qquad \text{for all } 1 \leq i, j \leq n,$$

where $\langle v, w \rangle$ denotes the scalar product of vectors v, w, and the delta symbol $\delta_{i,j}$ is defined by declaring $\delta_{i,i} = 1$, and $\delta_{i,j} = 0$ for $i \neq j$. More elegantly, this can be rephrased by saying $A \in O(n)$ if and only if $A^t A = I$, where A^t is the transpose of A, and I is the identity matrix. In particular, if we define

$$f\colon M_{n\times n} \longrightarrow M_{n\times n} \qquad \text{by} \qquad A \mapsto A^t A,$$

then $O(n) = f^{-1}(I)$. Since f is continuous (since each component is), and the one-element subset $\{I\} \subset M_{n\times n} = \mathbb{R}^{n^2}$ is closed, it follows that $O(n)$ is a closed subset of $\mathbb{R}^{n^2}$.

To show that $SO(n)$ is a closed subset, we note that $SO(n) = O(n) \cap SL_n(\mathbb{R})$, where $SL_n(\mathbb{R})$ is the special linear group consisting of all $n \times n$ matrices with determinant 1. In other words, $SL_n(\mathbb{R}) = \det^{-1}(1)$, where $\det\colon M_{n\times n} \to \mathbb{R}$ is the map that sends a matrix to its determinant. Since the determinant is a polynomial function of the entries of the matrix, this is a continuous function, and hence $SL_n(\mathbb{R}) = \det^{-1}(1)$ is a closed subset of $\mathbb{R}^{n^2}$. This implies that the intersection $SO(n) = O(n) \cap SL_n(\mathbb{R})$ is closed as well. $\square$

4. (10 points) Let M be a manifold of dimension m and let N be a manifold of dimension n. Show that the product $M \times N$ is a manifold of dimension $m + n$. Don't forget to check the technical conditions (Hausdorff and second countable) for $M \times N$.

Proof. First we check that $M \times N$ is locally homeomorphic to $\mathbb{R}^{m+n}$. So let $(x, y) \in M \times N$. Since M is a manifold of dimension m, there exists an open neighborhood U of x and a homeomorphism $\phi\colon U \xrightarrow{\equiv} U'$ between U and an open subset $U' \subset \mathbb{R}^m$. Similarly, there is a homeomorphism $\psi\colon V \xrightarrow{\equiv} V'$ between an open neighborhood of y and an open subset $V \subset \mathbb{R}^n$. It follows that the map

$$U \times V \longrightarrow U' \times V' \qquad \text{given by} \qquad (u, v) \mapsto (\phi(u), \psi(v))$$

is a homeomorphism (this map is continuous, since its components are continous; its inverse is given by

$$U' \times V' \ni (v, w) \mapsto (\phi^{-1}(v), \psi^{-1}(w)) \in U \times V,$$

which is continuous by the same argument). We note that $U' \times V'$ is an open subset of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+1}$, since the product topology on $\mathbb{R}^m \times \mathbb{R}^n$ agrees with the standard metric topology on $\mathbb{R}^{m+n}$. This shows that $M \times N$ is locally homeomorphic to $\mathbb{R}^{m+n}$.

Next we show that $M \times N$ is Hausdorff. Suppose that (x, y), (x', y') are two distinct points in $M \times N$. Then $x \neq x'$ or $y \neq y'$. If $x \neq x'$, then there are disjoint open neighborhoods $U_x, U_{x'} \subset M$ of x resp. x', since M is Hausdorff. It follows that $U_x \times N$, $U_{x'} \times N$ are two disjoint open neighborhoods of (x, y) resp. (x', y'). Reversing the roles of M and N in the argument above similarly proves the statement for $y \neq y'$.

Finally we prove that the assumption that M and N are second countable implies that the product $M \times N$ is second countable (this is a general statement for topological spaces M, N). Let $\{U_\alpha\}_{\alpha \in A}$ be a countable basis for topology of M, and let $\{V_\beta\}_{\alpha \in B}$ be a countable basis for topology of N. Then the product topology on $M \times N$ is generated by the open subsets $U_\alpha \times V_\beta$ for $\alpha \in A$, $\beta \in B$. This is a countable basis, since the countability of A and B implies that countability of $A \times B$. □

5. (10 points) Show that the real projective space $\mathbb{RP}^n$ is a manifold of dimension n. Don't forget to check that $\mathbb{RP}^n$ is second countable (we have proved in class that the projective space is Hausdorff). Hint: to prove that $\mathbb{RP}^n$ is locally homeomorphic to $\mathbb{R}^n$ suitably modify the method we used for the sphere S^n. For showing that $\mathbb{RP}^n$ is second countable, recall from class that if X is second countable, and $p\colon X \twoheadrightarrow Y$ is an open surjection, then Y is second countable.

Proof. Let $U_i \subset \mathbb{RP}^n = S^n/\sim$ be the subset

$$U_i = \{[x_0, \ldots, x_n] \mid (x_0, \ldots, x_n) \in S^n,\ x_i \neq 0\}.$$

This is an open subset of $\mathbb{RP}^n$ since the preimage $p^{-1}(U_i)$ under the projection map $p\colon S^n \twoheadrightarrow \mathbb{RP}^n$ is an open set. Explicitly,

$$p^{-1}(U_i) = \{(x_0, \ldots, x_n) \in S^n \mid x_i \neq 0\},$$

which is an open subset of S^n since it is equal to $p_i^{-1}(\mathbb{R} \setminus \{0\})$, where $p_i\colon S^n \to \mathbb{R}$ is the projection map to the i-th factor (more precisely, the restriction of the projection map $\mathbb{R}^{n+1} \to \mathbb{R}$ to $S^n \subset \mathbb{R}^{n+1}$). Since $\mathbb{R} \setminus \{0\}$ is an open subset of $\mathbb{R}$ and p_i is continuous, the preimage $p_i^{-1}(\mathbb{R} \setminus \{0\})$ is an open subset of S^n.

Let $B_1^n := \{(v_1, \dots, v_n) \in \mathbb{R}^n \mid v_1^2 + \cdots + v_n^2 < 1\}$ be the open n-ball of radius 1 and let

$$\phi_i\colon U_i \longrightarrow B_1^n$$

be the map defined by $\phi_i([x_0, \dots, x_n]) := (x_0, \dots, \widehat{x}_i, \dots, x_n)$ for $x_i > 0$, where the "hat" over x_i indicates that we skip this term. We note that every equivalence class $[x_0, \dots, x_n]$ consists of two antipodal points, exactly one of which will have a positive component x_i; hence the prescription above yields a well-defined map ϕ_i.

We want to show that ϕ_i is a homeomorphism. It is easy to verify that the map

$$B_1^n \longrightarrow U_i \qquad v = (v_1, \dots, v_n) \mapsto [v_1, \dots, v_i, \sqrt{1 - ||v||^2}, v_{i+1}, \dots, v_n]$$

is in fact the inverse to ϕ_i. Here $||v||^2 = v_1^2 + \cdots + v_n^2$ is norm squared of $v \in B_1^n$. Both maps, ϕ_i and its inverse, are continuous since all their components are continuous. This shows that ϕ_i is in fact a homeomorphism, and hence the projective space $\mathbb{RP}^n$ is locally homeomorphic to $\mathbb{R}^n$.

It remains to show that $\mathbb{RP}^n = S^n/\sim$ is second countable. We have proved earlier that S^n is second countable (as subspace of the second countable space $\mathbb{R}^{n+1}$). Hence it suffices to show that the projection map $p\colon S^n \twoheadrightarrow \mathbb{RP}^n$ is open, since by a lemma from class this implies that $\mathbb{RP}^n$ is second countable.

So let $U \subset S^n$ be open. Let $A\colon S^n \to S^n$ be the antipodal map $x \mapsto -x$, which is continuous, since it is the restriction of the map $A\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by the same formula which is continuous since all its components are. To check whether $p(U) \subset \mathbb{RP}^n$ is open, we need to verify that $p^{-1}(p(U))$ is an open subset of S^n.

$$p^{-1}(p(U)) = U \cup A^{-1}(U),$$

which is an open subset of S^n as the union of open subsets. Hence the lemma above implies that $\mathbb{RP}^n$ is second countable. $\square$

3 Homework Assignment # 3

1. (10 points) Show that the complex projective space $\mathbb{CP}^1$ is homeomorphic to the 2-sphere S^2. Hint: recall that $\mathbb{CP}^1$ is a quotient of $\mathbb{C}^2 \setminus \{0\}$ and hence a point of $\mathbb{CP}^1$ is an equivalence class $[z_0, z_1]$ of elements $(z_0, z_1) \in \mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\})$. Construct a bijection f between $\mathbb{CP}^1$ with the point $[0,1]$ removed and $\mathbb{C}$. Compose the map f with the map $g\colon \mathbb{C} = \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\}$ which is the inverse of the stereographic projection map (see the formula from the previous homework set). Simplify the explicit formula for $g \circ f\colon \mathbb{CP}^1 \setminus \{[0,1]\} \longrightarrow S^2 \setminus \{(0,0,1)\}$ to show that it extends to a continuous bijection between $\mathbb{CP}^1$ and S^2.

Proof. The map $f\colon (\mathbb{CP}^1 \setminus \{[0,1]\}) \longrightarrow \mathbb{C}$ given by $f([z_0, z_1]) = \frac{z_1}{z_0}$ is well-defined since

- $z_0 \neq 0$ for any $[z_0, z_1] \neq [0, 1]$, and
- for any $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $f([\lambda z_0, \lambda z_1]) = \frac{\lambda z_1}{\lambda z_0} = f([z_0, z_1])$.

Moreover, f is a bijection, since the map $\mathbb{C} \to \mathbb{CP}^1 \setminus \{[0, 1]\}$, $z \mapsto [1, z]$ is the inverse of f as is easily checked.

The map $g\colon \mathbb{R}^2 \to S^2 \setminus \{(0, 0, 1)\}$ inverse to the stereographic projection is given explicitly by

$$g(y_1, y_2) = \frac{1}{||y||^2 + 1}(2y_1, 2y_2, ||y||^2 - 1).$$

Identifying $y = (y_1, y_2) \in \mathbb{R}^2$ with the complex number $z = y_1 + iy_2 \in \mathbb{C}$, $||y||^2 = ||z||^2 = z\bar{z}$, and hence

$$g(z) = \frac{1}{||z||^2 + 1}(2z, ||z||^2 - 1) = \left(\frac{2z}{||z||^2 + 1}, \frac{||z||^2 - 1}{||z||^2 + 1}\right) \in S^2 \subset \mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}.$$

Hence for $z_0 \neq 0$,

$$\begin{aligned} g(f([z_0, z_1])) = g(z_1/z_0) &= \left(\frac{2\frac{z_1}{z_0}}{\frac{||z_1||^2}{||z_0||^2} + 1}, \frac{\frac{||z_1||^2}{||z_0||^2} - 1}{\frac{||z_1||^2}{||z_0||^2} + 1}\right) \\ &= \left(\frac{2\frac{z_1}{z_0}||z_0||^2}{||z_1||^2 + ||z_0||^2}, \frac{||z_1||^2 - ||z_0||^2}{||z_1||^2 + ||z_0||^2}\right) \\ &= \left(\frac{2z_1\bar{z}_0}{||z_1||^2 + ||z_0||^2}, \frac{||z_1||^2 - ||z_0||^2}{||z_1||^2 + ||z_0||^2}\right) \end{aligned}$$

The composition $\mathbb{CP}^1 \setminus \{[0, 1]\} \xrightarrow{f} \mathbb{C} \xrightarrow{g} S^2 \setminus \{(0, 0, 1)\}$ is a bijection, since f and g are bijections. For any $[z_0, z_1] \in \mathbb{CP}^1$, let $F([z_0, z_1])$ be defined by the above formula. Then for $z_0 \neq 0$, $F([z_0, z_1]) = g(f(z_0, z_1))$, while $F([0, z_1]) = (0, 0, 1) \in S^2$. In other words, F is a map

$$F\colon \mathbb{CP}^1 \longrightarrow S^2,$$

which extends the map $g \circ f$ to all of $\mathbb{CP}^1$, sending $[0, 1] \in \mathbb{CP}^1$ to $(0, 0, 1) \in S^2$. In particular, F is a bijection, since $g \circ f$ is.

To show that F is continuous, consider the composition

$$\mathbb{C}^2 \setminus \{0\} \xrightarrow{p} \mathbb{CP}^1 \xrightarrow{F} S^2 \xrightarrow{i} \mathbb{R}^3 .$$

By the Continuity Criterium for maps out of quotient spaces, F is continuous if and only if $F \circ p$ is continuous. By the Continuity Criterium for maps into subspaces, $F \circ p$ is continuous if and only if $F \circ p \circ i$ is continuous. By the Continuity Criterium for maps to product spaces, $F \circ p \circ i$ is continuous if and only if all of its three components (the real resp. imaginary component of $\frac{2z_1\bar{z}_0}{||z_1||^2+||z_0||^2}$ resp. $\frac{||z_1||^2-||z_0||^2}{||z_1||^2+||z_0||^2}$) are continuous. These are rational functions of components of $(z_0, z_1) \in \mathbb{C}^2 = \mathbb{R}^4$ whose denominators are non-zero on the domain of the map, and hence are continuous. This shows that F is continuous.

The continuous bijection $F\colon \mathbb{CP}^1 \to S^2$ is a homeomorphism, since its codomain S^2 is Hausdorff (as subspace of $\mathbb{R}^3$), and its domain $\mathbb{CP}^1$ is compact (as quotient of S^3 which is compact by Heine-Borel). We mentioned in class that $\mathbb{CP}^n$ can be described as quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by identifying $(z_0, \dots, z_n)$ with $(\lambda z_0, \dots, \lambda z_n)$ for $\lambda \in \mathbb{C} \setminus \{0\}$, or equivalently as quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the same relation as above, but assuming that $(z_0 \dots, z_n) \in S^{2n+1}$ and $\lambda \in S^1$. We proved the analogous statement for real projective spaces in class. □

2. (10 points) Which of the topological groups $GL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are connected? Hint: To show that one of these topological groups is connected, it might be easier to show that it is path-connected. Note that to prove this, it suffices to find a path connecting any element with the identity element (why?). Use without proof the fact that every element in $SO(n)$ (the group of linear maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which are isometries with determinant one) for a suitable choice of basis of $\mathbb{R}^n$ is represented by a matrix of block diagonal form whose diagonal blocks are the 1×1 matrix with entry $+1$ and/or 2×2 rotational matrices

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Here "block diagonal" means that all other entries are zero.

Proof. The determinant function $\det\colon M_{n\times n} \to \mathbb{R}$ is a continuous function. Since the determinant for matrices in $GL_n(\mathbb{R})$ or $O(n)$ is non-zero, its restriction to $G = GL_n(\mathbb{R}), O(n)$ provides a continuous map

$$f\colon G \longrightarrow \mathbb{R} \setminus \{0\}.$$

It follows that $G = f^{-1}((-\infty, 0)) \cup f^{-1}((0, \infty))$ is a decomposition of G into a disjoint union of open subsets. Both of these are not empty, since the determinant of the identity matrix is 1, and the determinant of the diagonal matrix with diagonal entries $(-1, 1, \dots, 1)$ is -1. This shows that the spaces $GL_n(\mathbb{R})$ and $O(n)$ are both not connected.

To prove that $SO(n)$ is connected it suffices to show that $SO(n)$ is path connected, i.e., any two elements $A, B \in SO(n)$ can be connected by a path. We will show this by constructing for every $A \in SO(n)$ a path $\gamma_A\colon [0,1] \to SO(n)$ with $\gamma_A(0) = I$ (the identity matrix) and $\gamma_A(1) = A$. This implies that any two points $A, B \in SO(n)$ can be connected by a path γ_{AB}, obtained by first taking a path from A to I, by running the path γ_A backwards, and then taking the path γ_B from I to B. In formulas, the path γ_{AB} is given by

$$\gamma_{AB}(t) = \begin{cases} \gamma_A(1-2t) & \text{for } t \in [0, \frac{1}{2}] \\ \gamma_B(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

This is in fact a path (i.e., a continuous map) by problem 4 of homework assignment # 2.

Let us first construct γ_A for $n = 2$, using the fact that every $A \in SO(2)$ is a rotation, given by a matrix of the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We define

$$\gamma_A \colon [0,1] \longrightarrow SO(2) \qquad \text{by} \qquad \gamma_A(t) := \begin{pmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{pmatrix}$$

which is continuous since it components are continuous functions of t, and has the desired property $\gamma_A(0) = I$ and $\gamma_A(1) = A$.

For a general dimension n, we use the fact that for any $A \in SO(n)$ there is a choice of basis for $\mathbb{R}^n$ such that the matrix representing the isometry A is of block diagonal form as described in the hint. Replacing for each block R_i the rotation angle θ_i (which may be different for the different blocks) by $t\theta_i$ we obtain a path $\gamma_A \colon [0,1] \to SO(n)$ from I to A, generalizing our construction in the $n = 2$ case. □

3. (10 points) The definition of a manifold involves the technical conditions of being Hausdorff and second countable. Show that these properties are "inherited" by subspaces in the following sense. Let X be a topological space and A a subspace.

(a) Show that if X is Hausdorff, then so is A.

(b) Show that if X is second countable, then so is A.

Proof. **Part (a).** Suppose that X is Hausdorff and let $x, y \in A$ with $x \neq y$. Then there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$. Then $U \cap A$, $V \cap A$ are disjoint open subsets of A (by definition of the subspace topology) with $x \in U \cap A$ and $y \in V \cap A$. This shows that A is Hausdorff.

Part (b). Assume that X is second countable and that $\mathcal{B}$ is a countable collection of subspaces of X which are a basis for the topology of X. We claim that the countable collection of subspaces $B \cap A \subset A$ for $B \in \mathcal{B}$ is a basis for the subspace topology. Let U be an open subset of A. By the definition of the subspace topology, this means there is some open subset V of X such that $U = V \cap A$. Since $\mathcal{B}$ is a basis for the topology on X, the open subset V can be written as $V = \bigcup_{i \in I} B_i$, a (not necessarily finite) union of subsets B_i belonging to the collection $\mathcal{B}$. Then

$$U = V \cap A = \left(\bigcup_{i \in I} B_i \right) \cap A = \bigcap_{i \in I} B_i \cap A,$$

which shows that the subsets of the form $B_i \cap A$ form a countable basis for the topology of A. □

4. (10 points) Let Σ, Σ' be compact 2-manifolds. Show that the Euler characteristic of the connected sum $\Sigma\#\Sigma'$ is given by the following formula:

$$\chi(\Sigma\#\Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2.$$

Proof. Let P resp. P' be a pattern of polygons on Σ resp. Σ'. Without loss of generality we can assume that one of the faces of P is a triangle T, and that one of the faces of P' is a triangle T'. Then we can form the connected sum $\Sigma\#\Sigma'$ by removing the interiors of T and T' from Σ resp. Σ' and gluing the resulting manifolds with boundary by identifying each of the edges of T with an edge each of T'. Each of the polygons of P and P' with the exception of T and T' then gives a polygon on the connected sum $\Sigma\#\Sigma'$, thus leading to a pattern of polygons P'' on $\Sigma\#\Sigma'$. Denoting by $V(P)$ resp. $E(P)$ resp. $F(P)$ the number of vertices resp. edges resp. faces of the pattern P, and similarly for P', P'', we can now calculate the number of vertices/edges/faces of P'':

- $V(P'') = V(P)+V(P')-3$, since the three vertices of the triangle $T \subset \Sigma$ are identified with the three vertices of the triangle $T' \subset \Sigma$ when glue the boundaries of the triangles T and T'.
- $E(P'') = E(P) + E(P') - 3$, since the three edges of the triangle $T \subset \Sigma$ are identified with the three edges of the triangle $T' \subset \Sigma$ when glue the boundaries of the triangles T and T'.
- $F(P'') = F(P) + F(P') - 2$, since the two faces given by the triangles T, T' are no longer faces of the polygonal pattern P''.

It follows that

$$\begin{aligned}
\chi(\Sigma\#\Sigma') =& V(P'') - E(P'') + F(P'') \\
=& (V(P) + V(P') - 3) \; - \; (E(P) + E(P') - 3) \; + \; (F(P) + F(P') - 2) \\
=& (V(P) - E(P) + F(P)) \; + \; (V(P') - E(P') + F(P')) \; - \; 2 \\
=& \chi(\Sigma) + \chi(\Sigma') - 2.
\end{aligned}$$

□

5. (10 points) By the classification theorem for compact connected 2-manifolds, the connected sum $T\#T\#K\#K\#K$ of two copies of the torus T and three copies of the Klein bottle K is homeomorphic to exactly one of the manifolds Σ_g (the surface of genus $g \geq 0$) or X_k (the connected sum of k copies of the real projective plane $\mathbb{RP}^2$). Which one is it? (provide detailed arguments!).

Proof. Let $X := T\#T\#K\#K\#K$. As discussed in class, the Klein bottle K contains a Möbius band. Forming the connected sum $M\#K$ for $M = T\#T\#K\#K$ using a disk in K which is disjoint to the Möbius band in K, we see that also the connected sum $M\#K = T\#T\#K\#K\#K = X$ contains a Möbius band, and hence X is non-orientable.

By the classification Theorem for compact connected 2-manifolds, X is then homeomorphic to X_k, the connected sum of k copies of the projective plane $\mathbb{RP}^2$ for some $k \geq 1$. To determine k, we note that the homeomorphism $X \approx X_k$ implies

$$\chi(X) = \chi(X_k) = 2 - k.$$

To calculate the Euler characteristic $\chi(X)$, we use the formula $\chi(\Sigma\#\Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$, and $\chi(T) = \chi(K) = 0$ (from class). It follows that

$$\chi(\Sigma\#T) = \chi(\Sigma) + 0 - 2 = \chi(\Sigma) - 2 \qquad \text{and} \qquad \chi(\Sigma\#K) = \chi(\Sigma) - 2.$$

In other words, a connected sum with T or K causes the Euler characteristic to drop by 2. Hence starting with T (with $\chi(T) = 0$), the connected sum operation with 1 copy of T and three copies of K causes the Euler characteristic to drop to -8; i.e., $\chi(X) = -8$. From the equation $\chi(X) = 2 - k$ above we conclude $k = 10$, in other words, X is homeomorphic to X_{10}. □

4 Homework Assignment # 4

1. (10 points) Let $\alpha, \beta, \gamma\colon I \to X$ be paths in a topological space X. Assume that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$ which guarantees that the concatenated paths $\alpha*(\beta*\gamma)$ and $(\alpha*\beta)*\gamma$ can be formed. Show that these two paths are homotopic (relative endpoints). Verifying this shows that if α, β, γ are loops based at $x_0 \in X$ representing elements $a = [\alpha]$, $b = [\beta]$, $c = [\gamma]$ in $\pi_1(X; x_0)$, then $a(bc) = (ab)c$. In other words, this proves associativity of multiplication in $\pi_1(X; x_0)$, one of the last things to verify in order to prove that $\pi_1(X; x_0)$ is indeed a group.

Hint: Show that both paths can be written as the composition $\Psi \circ \phi$ of a suitable map $\phi\colon I \to [0, 3]$ and the map

$$[0,3] \xrightarrow{\Psi} X \qquad \text{defined by} \quad \Psi(s) := \begin{cases} \alpha(s) & 0 \leq s \leq 1 \\ \beta(s-1) & 1 \leq s \leq 2 \\ \gamma(s-2) & 2 \leq s \leq 3 \end{cases}$$

Avoid writing down explicit homotopies; instead use the handy fact that any two paths with the same endpoints in a convex subset of $\mathbb{R}^n$ are homotopic relative endpoints.

Proof. Let $\alpha', \beta', \gamma' \colon I \to [0,3]$ be defined by $\alpha'(s) = s$, $\beta'(s) = s+1$ and $\gamma'(s) = s+2$. We note that

$$\Psi \circ \alpha' = \alpha \qquad \Psi \circ \beta' = \beta \qquad \Psi \circ \gamma' = \gamma.$$

It follows that

$$\alpha * (\beta * \gamma) = \Psi \circ (\alpha' * (\beta' * \gamma')) \qquad \text{and} \qquad (\alpha * \beta) * \gamma = \Psi \circ ((\alpha' * \beta') * \gamma').$$

Hence it suffices to find a homotopy ϕ_t (relative endpoints) between the paths $\phi_0 := \alpha' * (\beta' * \gamma')$ and $\phi_1 := (\alpha' * \beta') * \gamma'$ with the same endpoints, since then $\Psi \circ \phi_t$ will then be the desired homotopy between $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$. The paths $\phi_0, \phi_1 \colon [0,1] \to [0,3]$ are paths in the convex subset $[0,3]$ of the real line, and hence the linear homotopy ϕ_t between these two paths does the job. Explicitly, $\phi_t(s)$ is given by the formula

$$\phi_t(s) = (1-t)\phi_0(s) + t\phi(s).$$

$\square$

2. (10 points) Let X be a topological space and let β be a path from x_0 to x_1. Show that the map

$$\Phi_\beta \colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \qquad [\gamma] \mapsto [\bar{\beta} * \gamma * \beta]$$

is an isomorphism of groups. In particular, the isomorphism class of the fundamental group $\pi(X, x_0)$ of a path connected space does not depend on the choice of the base point $x_0 \in X$. Hint: for any path γ in X, there are homotopies

$$\gamma * \bar{\gamma} \simeq c_{\gamma(0)} \qquad \bar{\gamma} * \gamma \simeq c_{\gamma(1)} \qquad c_{\gamma(0)} * \gamma \simeq \gamma, \qquad \gamma * c_{\gamma(1)} \simeq \gamma$$

where c_x for $x \in X$ denotes the constant path at x. Make use of these (we proved one of these in class; no need to prove the others).

Proof. First we show that $\Phi_\beta \colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is a homomorphism. So let γ, δ be loops in X based at x_0. Then

$$\begin{aligned} \Phi_\beta([\gamma][\delta]) = \Phi_\beta([\gamma * \delta]) &= [\bar{\beta} * \gamma * \delta * \beta] = [\bar{\beta} * \gamma * \beta * \bar{\beta} * \delta * \beta] \\ &= [\bar{\beta} * \gamma * \beta][\bar{\beta} * \delta * \beta] = \Phi_\beta([\gamma])\Phi_\beta([\delta]) \end{aligned}$$

Here we avoid using parantheses for the concatenation operation; this is ok by the fact that the paths resulting from different ways of putting in parantheses leads to homotopic paths. The first and second equality is just the definition of multiplication in $\pi_1(X, x_0)$ resp. the definition of Φ_β. The third equation holds due to the homotopy $\beta * \bar{\beta} \simeq c_{x_0}$ (an application of the second homotopy of the hint), and the fact that we can insert at any point an appropriate constant path into an iterated concatenation without changing the homotopy class of the path (follows from either the third or the fourth homotopy). The last two equations again hold by construction of multiplication in the fundamental group resp. the map Φ_β.

It remains to show that Φ_β is an isomorphism. We claim that its inverse is given by $\Phi_{\bar\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$.

$$\Phi_{\bar\beta}\Phi_\beta(\gamma) = \Phi_{\bar\beta}([\beta * \gamma * \bar\beta]) = [\bar\beta * \beta * \gamma * \bar\beta * \beta] = [c_{x_0} * \gamma * c_{x_0}] = [\gamma]$$

Here the first two equations is just by construction of Φ_β resp. $\Phi_{\bar\beta}$, the third equation holds thanks to the first two homotopies mentioned in the hint, and the last equation holds due to the last two homotopies. □

3. (10 points) A *pointed topological space* is a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. Let (X, x_0), (Y, y_0) be pointed topological spaces and let $f\colon (X, x_0) \to (Y, y_0)$ be a basepoint preserving map, i.e., a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$.

(a) Show that the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $f_*([\gamma]) = [f \circ \gamma]$ is a well-defined.

(b) Show that f_* is a group homomorphism.

The map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is called the *homomorphism of fundamental groups induced by f*.

Proof. **Part (a).** Let γ, γ' be two based loops in (X, x_0) which are homotopic relative endpoints, and let $H\colon I \times I \to X$ be such a homotopy from γ to γ', i.e., $H(s, 0) = \gamma(s)$, $H(s, 1) = \gamma'(s)$, $H(0, t) = x_0 = H(1, t)$ for all $s, t \in I$. Then the composition

$$I \times I \xrightarrow{H} X \xrightarrow{f} Y$$

has the properties $f \circ H(s, 0) = f \circ \gamma(s)$, $f \circ H(s, 1) = f \circ \gamma'(s)$, $f \circ H(0, t) = f(x_0) = y_0$ and $f \circ H(1, t) = f(x_0) = y_0$. In other words, $f \circ H$ is a homotopy relative endpoints from $f \circ \gamma$ to $f \circ \gamma'$, and hence $[f \circ \gamma] = [f \circ \gamma'] \in \pi_1(Y, y_0)$ showing that the map f_* is well-defined.

Part (b). Let $[\gamma], [\gamma'] \in \pi_1(X, x_0)$. We recall that the multiplication in the group $\pi_1(X, x_0)$ is induced by concatenation of paths, i.e., $[\gamma] \cdot [\gamma'] := [\gamma * \gamma']$. Then

$$f_*([\gamma] \cdot [\gamma']) = f_*([\gamma * \gamma']) = [f \circ (\gamma * \gamma')]$$
$$f_*([\gamma]) \cdot f_*([\gamma']) = [f \circ \gamma] \cdot [f \circ \gamma'] = [(f \circ \gamma) * (f \circ \gamma')]$$

For $0 \le s \le 1/2$, $(f \circ (\gamma * \gamma'))(s) = f((\gamma * \gamma')(s)) = f(\gamma(s))$ and $((f \circ \gamma) * (f \circ \gamma'))(s) = (f \circ \gamma)(s) = f(\gamma(s))$. Similarly, the paths $f \circ (\gamma * \gamma')$ and $(f \circ \gamma) * (f \circ \gamma')$ also agree for $1/2 \le s \le 1$. Hence f_* is a homomorphism. □

4. (10 points) Let (X, x_0), (Y, y_0) be pointed topological spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the Cartesian product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of (X, x_0) and (Y, y_0).

Hint: use the base point preserving projection maps $p^X : X \times Y \to X$, $p^Y : X \times Y \to Y$, and the induced homomorphisms (see the previous problem)

$$p^X_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \qquad p^Y_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0).$$

Proof. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projection maps, and let

$$(p_X)_* : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \qquad (p_Y)_* : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(Y, y_0)$$

be the induced homomorphisms on fundamental groups. We claim that the group homomorphism

$$\Psi : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

given by

$$[\gamma] \mapsto ((p_X)_*([\gamma]), (p_Y)_*([\gamma]) = ([p_X \circ \gamma], [p_Y \circ \gamma])$$

is an isomorphism.

To show that Ψ is surjective, let $([\gamma_1], [\gamma_2])$ be an element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$, i.e., γ_1 is a based loop in (X, x_0), and γ_2 is a based loop in (Y, y_0). Let $\gamma : I \to X \times Y$ be the continuous map whose components are the maps γ_1, γ_2; i.e., $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. Then $\gamma(0) = \gamma(1) = (x_0, y_0)$; in other words, γ is a based loop in $(X \times Y, (x_0, y_0))$. By construction, Ψ sends $[\gamma] \in \pi_1(X \times Y, (x_0, y_0))$ to $([\gamma_1], [\gamma_2])$, which proves that Ψ is surjective.

To show that Ψ is injective, it suffices to show that the kernel of Ψ is trivial. So assume that $[\gamma] \in \pi_1(X \times Y, (x_0, y_0))$ is in the kernel of Ψ. This means that both components paths $\gamma_1 := p_X \circ \gamma$ and $\gamma_2 := p_Y \circ \gamma$ are homotopic to the constant path c_{x_0} resp. c_{y_0} via homotopies $H_1 : I \times I \to X$ resp. $H_2 : I \times I \to Y$; more explicitly,

$$H_i(s, 0) = \gamma_i(s) \quad \text{for } i = 1, 2 \qquad H_i(s, 1) = \begin{cases} x_0 & i = 1 \\ y_0 & i = 2 \end{cases}$$

Let $H : I \times I \longrightarrow X \times Y$ be the map with component maps H_1 and H_2. Then H is the desired homotopy between γ and the constant path at the basepoint $(x_0, y_0) \in X \times Y$, showing that $[\gamma]$ is the trivial element in $\pi_1(X \times Y, (x_0, y_0))$. $\square$

5. (10 points) The goal of this problem is to show that the winding number map

$$W\colon \pi_1(S^1,1)\to\mathbb{Z}$$

is a group isomorphism. We recall that for a based loop γ in $(S^1,1)$, the winding number $W(\gamma)\in\mathbb{Z}$ is defined by $W(\gamma):=\widetilde{\gamma}(1)\in\mathbb{Z}$, where $\widetilde{\gamma}\colon I\to\mathbb{R}$ is the unique lift of $\gamma\colon I\to S^1$ (i.e., $p\circ\widetilde{\gamma}=\gamma$) with starting point $\widetilde{\gamma}(0)=0\in\mathbb{R}$. We assume that W is well-defined, i.e., that the winding number $W(\gamma)$ of a based loop γ depends only on the homotopy class of γ relative endpoints (which will be proved in class on Tuesday, 9-19). Let

$$\Phi\colon \mathbb{Z}\longrightarrow\pi_1(S^1,1)\qquad\text{be defined by}\qquad\Phi(n):=[\gamma_n],$$

where $\gamma_n\colon I\to(S^1,1)$ is the based loop $\gamma_n(s)=e^{2\pi ins}$.

(a) Show that the composition $\mathbb{Z}\xrightarrow{\Phi}\pi_1(S^1,1)\xrightarrow{W}\mathbb{Z}$ is the identity.

(b) Show that the composition $\pi_1(S^1,1)\xrightarrow{W}\mathbb{Z}\xrightarrow{\Phi}\pi_1(S^1,1)$ is the identity. Hint: If $[\gamma]\in\pi_1(S^1,1)$ and $\Phi(W([\gamma]))=[\gamma']$, you need to show that $\gamma\sim\gamma'$. Let $\widetilde{\gamma},\widetilde{\gamma}'\colon I\to\mathbb{R}$ be the unique lifts of γ (resp. γ') with $\widetilde{\gamma}(0)=0=\widetilde{\gamma}'(0)$. Try to show that $\widetilde{\gamma}(1)=\widetilde{\gamma}'(1)$. Using the fact that paths with the same endpoints in a convex subspace are homotopic relative endpoints, why does this imply $\gamma\sim\gamma'$?

(c) Show that W is a group homomorphism.

Proof. **Part (a).** For $n\in Z$, $\Phi(n)=[\gamma_n]$, and hence $W(\Phi(n))=W(\gamma_n)$ is the winding number of γ_n, given by $W(\gamma_n)=\widetilde{\gamma}_n(1)$, where $\widetilde{\gamma}_n\colon I\to\mathbb{R}$ is the unique lift of $\gamma_n(s)=e^{2\pi ins}$ with $\widehat{\gamma}_n(0)=0$. So we need to show $\widetilde{\gamma}_n(1)=n$.

We observe that the map $\widetilde{\gamma}_n\colon I\to\mathbb{R}$ given by $\widetilde{\gamma}_n(s)=ns$ satisfies the two properties that uniquely characterize $\widetilde{\gamma}_n$:

- this map $\widetilde{\gamma}_n$ is a lift of γ_n, i.e., $p\circ\widehat{\gamma}_n=\gamma_n$, since $p(\widehat{\gamma}_n(s))=p(ns)=e^{2\pi ins}=\gamma_n(s)$
- $\widehat{\gamma}_n(0)=0$.

It follows that $W(\gamma_n)=\widehat{\gamma}_n(1)=n$, which proves part (a).

Part (b). Let $[\gamma]\in\pi_1(S^1,1)$, and let $n:=W(\gamma)=\widehat{\gamma}(1)\in\mathbb{Z}$. Then $\Phi(W(\gamma))=[\gamma_n]$, and we need to show $\gamma\sim\gamma_n$. The based loops γ and γ_n in $(S^1,1)$ have lifts $\widehat{\gamma}$, $\widehat{\gamma}_n$ which are paths in $\mathbb{R}$ with starting point 0. We know that $\widehat{\gamma}_n(s)=ns$ (from part (a)), and $\widehat{\gamma}(1)=n$. In particular, the paths $\widehat{\gamma}_n,\widehat{\gamma}\colon I\to\mathbb{R}$ have the *same* endpoints. Since these are paths in $\mathbb{R}$ (which in particular is a convex subspace of itself), it follows that $\widehat{\gamma}_n\sim\widehat{\gamma}$. This in turn implies $\gamma_n=p\circ\widehat{\gamma}_n\sim p\circ\widehat{\gamma}=\widehat{\gamma}$, which proves part (b).

Part (c). To show that W is a group homomorphism, let γ, γ' be based loops in $(S^1, 1)$, and let $\widetilde{\gamma}, \widetilde{\gamma}'\colon I \to \mathbb{R}$ be their lifts for the covering map $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi it}$ with starting point $0 \in \mathbb{R}$. We need to determine the lift of the concatenated loop $\gamma * \gamma'$. It is tempting to say that that is $\widetilde{\gamma} * \widetilde{\gamma}'$, but this concatenation not be defined since the endpoint of the first path, i.e., $\widetilde{\gamma}(1) = W(\gamma)$, in general is not equal to the starting point of the second path $\widetilde{\gamma}'(0) = 0$. We can fix this problem by replacing $\widetilde{\gamma}'$ by the path $\widehat{\gamma}'\colon I \to \mathbb{R}$ defined by

$$\widehat{\gamma}'(s) := W(\gamma) + \widetilde{\gamma}'(s).$$

The path $\widehat{\gamma}'$ also is a lift of γ', but its starting point is $\widehat{\gamma}'(0) = W(\gamma)$, and hence the concatenation $\widetilde{\gamma} * \widehat{\gamma}'$ is a continuous path which is a lift of $\gamma * \gamma'$ with starting point 0 and endpoint

$$(\widetilde{\gamma} * \widehat{\gamma}')(1) = \widehat{\gamma}'(1) = W(\gamma) + \widetilde{\gamma}'(1) = W(\gamma) + W(\gamma').$$

This shows that the winding number of $\gamma * \gamma'$ is $W(\gamma) + W(\gamma')$. □

5 Homework Assignment # 5

1. (10 points) A subspace $A \subset X$ of a topological space X is called a *retract of* X if there is a continuous map $r\colon X \to A$ whose restriction to A is the identity.

(a) Show that S^1 is not a retract of D^2. Hint: Show that the assumption that there is a continuous map $r\colon D^2 \to S^1$ which restricts to the identity on S^1 leads to a contradiction by contemplating group homomorphisms $r_*\colon \pi_1(D^2, x_0) \to \pi_1(S^1, x_0)$ and $i_*\colon \pi_1(S^1, x_0) \to \pi_1(D^2, x_0)$ induced by the retraction f resp. the inclusion map $i\colon S^1 \hookrightarrow D^2$.

(b) Brouwer's Fixed Point Theorem states that every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e., a point x with $f(x) = x$. Prove this for $n = 2$. Hint: show that if f has no fixed point, then a retraction map $r\colon D^2 \to S^1$ can be constructed out of f.

Proof. We prove part (a) by contradiction. Suppose that $r\colon D^2 \to S^1$ is a retraction of D^2 onto S^1. This means that r makes the following diagram commutative:

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & D^2 \arrow[d, "r"] \\ & S^1 \end{tikzcd}$$

Choosing a basepoint $x_0 \in S^1$ and applying the fundamental group functor gives the commutative diagram

$$\begin{tikzcd} \pi_1(S^1, x_0) \arrow[r, "i_*"] \arrow[dr, "\mathrm{id}_* = \mathrm{id}"'] & \pi_1(D^2, x_0) \arrow[d, "r_*"] \\ & \pi_1(S^1, x_0) \end{tikzcd}$$

This gives the desired contradiction since according to the diagram the identity map on $\pi_1(S^1, x_0) \cong \mathbb{Z}$ factors through the trivial group $\pi_1(D^2, x_0)$.

To prove part (b) we assume that $f\colon D^2 \to D^2$ does not have a fixed point, i.e., that for all $x \in D^2$, $f(x) \neq x$. Then the line through the points $f(x)$ and x intersects the sphere S^1 in two points. Let $r(x) \in S^1$ be the intersection point closer to x. It is clear from the construction that $r\colon D^2 \to S^1$ has the property $r(x) = x$ for $x \in S^1$, and so it remains to show the continuity of r to derive the desired contradiction from part (a).

To show that r is continuous, we note that $r(x)$ can be written in the form

$$r(x) = x + \alpha(x)(x - f(x)),$$

where $\alpha(x) \in \mathbb{R}$ is the unique non-negative solution of the quadratic equation

$$||x + \alpha(x)(x - f(x))||^2 = 1 \tag{5.1}$$

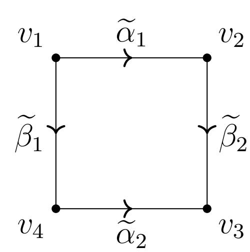
(the solutions of this equation correspond to the two intersections of the line through x and $f(x)$ with S^1; hence it is clear geometrically, that there are two solution for every x, and that exactly one solution is non-negative). Writing out the quadratic equation (5.1) explicitly as

$$||x - f(x)||^2\alpha^2(x) + 2\langle x, x - f(x)\rangle\alpha(x) + ||x||^2 - 1 = 0$$

shows that its coefficients are continuous functions of x, and hence the quadratic formula shows that its non-negative solution $\alpha(x)$ is continuous function of x. We conclude that $r(x)$ is a continuous function of x since its components are linear combinations of products of continuous functions. $\square$

2. (10 points) The goal of this problem is to construct elements in the fundamental group of the torus T, the Klein bottle K and the projective plane $\mathbb{RP}^2$ and to show that they satisfy certain relations.

(a) Recall that T is homeomorphic to $\Sigma(aba^{-1}b^{-1})$, the quotient of the polygon P_4 with four edges (also known as "square") according using the edge identification determined by the word $aba^{-1}b^{-1}$. Let $p\colon P_4 \to \Sigma(aba^{-1}b^{-1})$ be the projection map. Let $\widetilde{\alpha}_i, \widetilde{\beta}_i\colon I \to P_4$ be the linear edge paths as shown in the the picture below (the edge $\widetilde{\alpha}_1$ is identified with $\widetilde{\alpha}_2$ and $\widetilde{\beta}_1$ is identified with $\widetilde{\beta}_2$ to obtain $\Sigma(aba^{-1}b^{-1}) \approx P_4/\sim$)



Let $\alpha = p \circ \widetilde{\alpha}_i$, $\beta = p \circ \widetilde{\beta}_i$ be the based loops in $(\Sigma(aba^{-1}b^{-1}), v)$, with $v := p(v_i) \in \Sigma(aba^{-1}b^{-1})$, and let $a := [\alpha]$, $b := [\beta]$ be the elements of the fundamental group $\pi_1(\Sigma(aba^{-1}b^{-1}), v)$ represented by these based loops. Show that these elements satisfy the relation $aba^{-1}b^{-1} = 1 \in \pi_1(\Sigma(aba^{-1}b^{-1}), v)$.

(b) Similarly, construct elements a, b in the fundamental group of the Klein bottle $K \approx \Sigma(aba^{-1}b)$ and show that they satisfy the relation $aba^{-1}b = 1$.

(c) Similarly, construct an element a in the fundamental group of the real projective plane $\mathbb{RP}^2 \approx \Sigma(aa)$ and show that it satisfies the relation $a^2 = 1$.

(d) Recall that $\mathbb{RP}^2$ is also homeomorphic to $\Sigma(abab)$. Can we use the same techniques as above to construct elements a, b in the fundamental group of $\Sigma(abab)$ which satisfy the relation $abab = 1$? If yes, construct these elements and prove the relation; if no, explain the difference to the previous cases.

Proof. **Part (a).** As the picture shows, the paths $\widetilde{\alpha}_1$, $\widetilde{\beta}_2$, $\overline{\widetilde{\alpha}}_2$ and $\overline{\widetilde{\beta}}_1$ can be concatenated to obtain a based loop

$$\widetilde{\alpha}_1 * \widetilde{\beta}_2 * \overline{\widetilde{\alpha}}_2 * \overline{\widetilde{\beta}}_1 \qquad \text{in } (P_4, v_1).$$

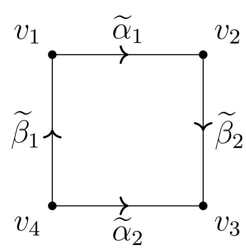
Since P_4 is a convex subset of $\mathbb{R}^2$, this loop is homotopic to the constant loop c_{v_1}. Composing with the projection map $p\colon P_4 \twoheadrightarrow \Sigma(aba^{-1}b^{-1})$ we obtain

$$c_v = c_{p(v_1)} = p \circ c_{v_1} \sim p \circ (\widetilde{\alpha}_1 * \widetilde{\beta}_2 * \overline{\widetilde{\alpha}}_2 * \overline{\widetilde{\beta}}_1) \sim \alpha * \beta * \bar{\alpha} * \bar{\beta}$$

for the based loops $\alpha = p \circ \widetilde{\alpha}$, $\beta = p \circ \widetilde{\beta}$ in $(\Sigma(aba^{-1}b^{-1}), v)$. Passing to homotopy classes, it follows that

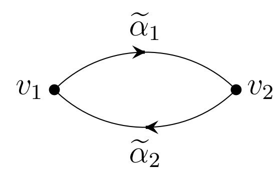
$$1 = [c_v] = [\alpha][\beta][\bar{\alpha}][\bar{\beta}] = [\alpha][\beta][\alpha]^{-1}[\beta]^{-1} = aba^{-1}b^{-1} \in \pi_1(\Sigma(aba^{-1}b^{-1}), v).$$

Part (b). The construction of the elements $a, b \in \pi_1(\Sigma(aba^{-1}b), v)$ is completely analogous to the construction in part (a), based on the following picture.



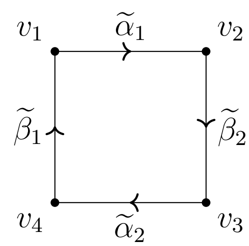
The concavity of P_4 implies that the based loop $\widetilde{\alpha}_1 * \widetilde{\beta}_2 * \overline{\widetilde{\alpha}}_2 * \widetilde{\beta}_1$ is homotopic to the constant loop c_{v_1}. Composing with the projection map $p\colon P_4 \twoheadrightarrow \Sigma(aba^{-1}b)$, it follows that $\alpha * \beta * \bar{\beta} * \beta$ is homotopic to c_v, which in turn proves the relation $aba^{-1}b = 1 \in \pi_1(\Sigma(aba^{-1}b), v)$.

Part (c). Again, the construction is analogous to the construction in part (a), but based on the following picture of the bigon P_2 on the edge paths $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, which project to the same based loop α in $(\Sigma(aa), v = p(v_1))$.



Again, concavity of P_2 implies that the based loop $\widetilde{\alpha}_1 * \widetilde{\alpha}_2$ is homotopic to the constant loop c_{v_1}. Composing with the projection map $p\colon P_2 \twoheadrightarrow \Sigma(aa)$ then yields the based loop $\alpha * \alpha$ which is homotopic to the constant loop c_v at $v = p(v_1)$, which implies the relation $a^2 = 1$ for $a = [\alpha] \in \pi_1(\Sigma(aa), v)$.

Part (d). The construction is now based on the picture



which shows that the concatenation $\widetilde{\alpha}_1 * \widetilde{\beta}_2 * \widetilde{\alpha}_2 * \widetilde{\beta}_1$ is a loop in P_4 based at v_1. Thanks to the convexity of P_4, it is again homotopic to the constant loop c_{v_1}. The difference to the previous cases shows up when we compose with the projection map p. Unlike in the previous cases, the vertices v_i *do not* all map to the same vertex v under p; rather, $p(v_1) = p(v_3) \neq p(v_2) = p(v_4)$. In particular, the paths $\widetilde{\alpha}_i$, $\widetilde{\beta}_i$ yield *paths* $\alpha = p \circ \widetilde{\alpha}_1 = p \circ \widetilde{\alpha}_2$, $\beta = p \circ \widetilde{\beta}_1 = p \circ \widetilde{\beta}_2$ in $\Sigma(abab)$, not loops, and consequently, they don't represents elements a, b of the fundamental group of $\Sigma(abab)$. $\square$

3. (10 points) Two topological spaces X, Y are *homotopy equivalent* if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f\colon X \to X$ is homotopic to id_X and $f \circ g\colon Y \to Y$ is homotopic to id_Y. Show that the following five topological spaces are all homotopy equivalent:

(1) the circle S^1,

(2) the open cylinder $S^1 \times \mathbb{R}$,

(3) the annulus $A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$,

(4) the solid torus $S^1 \times D^2$,

(5) the Möbius strip

Hint: A subspace $A \subset X$ is a *retract of* X if there is map $r\colon X \to A$ which restricts to the identity on A. It is a *deformation retract* of X if in addition the composition $X \xrightarrow{r} A \overset{i}{\hookrightarrow} X$ with the inclusion map i is homotopic to the identity on X. Note that if A is a deformation retract of X, then $r \circ i = \mathrm{id}_A$ and $i \circ r \sim \mathrm{id}_X$. In particular, A is homotopy equivalent to X. Show that each of the spaces (2)-(5) contains a subspace S homeomorphic to the circle S^1 which is a deformation retract of the bigger space it is contained in.

Proof. As suggested by the hint, for each of the spaces X in cases (2)-(5) we construct a continuous injection $i\colon S^1 \to X$. This gives a continuous bijection from S^1 to the subspace $S := i(S^1) \subset X$. Since S^1 is compact and the spaces X under consideration are all Hausdorff, the subspace S is homeomorphic to S^1. To show that S is a deformation retract of X we need to construct a continuous map

$$H\colon X \times I \longrightarrow X$$

with the properties

(a) $H(x, 1) = x$ for all $x \in X$;

(b) $H(x, 0) \in S$ for all $x \in X$;

(c) $H(x, t) = x$ for $x \in S$.

This map is the homotopy between the identity of X and the retraction $r\colon X \to S \subset X$ given by $r(x) := H(x, 0)$, showing that S is a deformation retract.

Here are the maps $i\colon S^1 \to X$ and $H\colon X \times I \to X$ for each of the cases (2)-(5). It is straightforward to show that the maps i, H are continuous, since we describe them by explicit formulas, to prove the injectivity of i, and to show that H has properties (a), (b) and (c).

For the cylinder $S^1 \times \mathbb{R}$ we define

$$\begin{aligned} i\colon S^1 &\longrightarrow X = S^1 \times \mathbb{R} & \qquad H\colon S^1 \times \mathbb{R} \times I &\longrightarrow S^1 \times \mathbb{R} \\ z &\mapsto (z, 0) & (z, s, t) &\mapsto (z, st) \end{aligned}$$

For the annulus $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 2\} = \{z \in \mathbb{C} \mid 1 \le |z|^2 \le 2\}$ we define

$$\begin{aligned} i\colon S^1 &\longrightarrow A & \qquad H\colon A \times I &\longrightarrow A \\ z &\mapsto z & (z, t) &\mapsto (1 - t)\frac{z}{|z|} + tz \end{aligned}$$

For the solid torus $S^1 \times D^2$ we define

$$i\colon S^1 \longrightarrow S^1 \times D^2 \qquad\qquad H\colon S^1 \times D^2 \times I \longrightarrow S^1 \times D^2$$
$$z \mapsto (z, 0) \qquad\qquad (z, w, t) \mapsto (z, tw)$$

For the Möbius strip $M = (I \times [-1,1])/\sim$ with the equivalence relation $(0,s) \sim (1,-s)$ it will be convenient to replace S^1 by the homeomorphic space $[0,1]/\{0,1\}$ obtained by identifying the two endpoints of the interval $[0,1]$. We define

$$i\colon \mathbb{R}/\mathbb{Z} \longrightarrow M \qquad\qquad H\colon M \times I \longrightarrow M$$
$$[r] \mapsto [r, 0] \qquad\qquad ([r,s], t) \mapsto [r, ts]$$

$\square$

4. (10 points) Let $f\colon (S^1, 1) \to (S^1, 1)$ be the basepoint preserving map defined by $f(z) = z^n$ for some $n \in \mathbb{Z}$ and let

$$f_*\colon \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$$

be the induced homomorphism on the fundamental group. Calculate explicitly the group homomorphism f_*. By this, we mean the following: the winding number gives an explicit isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$. Via this isomorphism, the automorphism f_* of the group $\pi_1(S^1, 1)$ corresponds to an automorphism of the group $\mathbb{Z}$. Any automorphism of $\mathbb{Z}$ is of the form $\mathbb{Z} \to \mathbb{Z}$, $m \mapsto km$, i.e., is given by multiplication by some integer $k \in \mathbb{Z}$. In other words, "calculate explicitly" means "determine the integer $k \in \mathbb{Z}$ such that the following diagram commutes":

$$\begin{tikzcd}
\pi_1(S^1,1) \arrow[r, "f_*"] \arrow[d, "W"', "\cong"] & \pi_1(S^1,1) \arrow[d, "\cong"', "W"] \\
\mathbb{Z} \arrow[r, "k"] & \mathbb{Z}
\end{tikzcd}$$

Proof. We recall that the based loop $\gamma_n\colon I \to S^1$ given by $\gamma_n(s) = e^{2\pi i n s}$ has winding number $W(\gamma_n) = n$. In particular $W(\gamma_1) = 1$. We note

$$f_*[\gamma_1] = [f \circ \gamma_1] = [f(e^{2\pi i s})] = [(e^{2\pi i s})^n] = [e^{2\pi i n s}] = [\gamma_n] \in \pi_1(S^1, 1),$$

and hence $W(f_*[\gamma_1]) = W(\gamma_n) = n$. It follows that

$$k = kW([\gamma_1]) = W(f_*[\gamma_1]) = n,$$

where the second equality is a consequence of the commutativity of the diagram. $\square$

5. (10 points) A *d-fold covering map* is a covering map $p\colon \widetilde{X} \to X$ such that for each point $x \in X$, the fiber $p^{-1}(x)$ consists of d points.

(a) Let X be compact 2-manifold and let $p\colon \widetilde{X} \to X$ be a d-fold covering map. Show that $\chi(\widetilde{X}) = d \cdot \chi(X)$. Hint: Choose a pattern of polygons Γ on X such that each polygon is contained in some evenly covered subset $U \subset X$. Argue that Γ determines a compatible pattern of polygons $\widetilde{\Gamma}$ on $\widetilde{X}$.

(b) Let $\widetilde{X} \to X$ is a d-fold covering of orientable compact connected 2-manifolds. Give a formula expressing the genus $\widetilde{g}$ of $\widetilde{X}$ in terms of the genus g of X.

Proof. **Part (a).** Let Γ be a pattern of polygons on X such that each polygon (and hence also each edge) is contained an evenly covered subset $U \subset X$. Then $p^{-1}(U)$ is the disjoint union of d subsets $U_1, \dots, U_d$ such that $p_{U_k}\colon U_k \to U$ is a homeomorphism. Let $\widetilde{\Gamma}$ be the graph on $\widetilde{X}$ whose vertices/edges/faces belonging to $p^{-1}(U)$ are the images of the vertices/edges/faces of Γ that belong to U under the homeomorphisms $U \approx U_k$ for $k = 1, \dots, d$. Denoting by $\#V_\Gamma$, $\#E_\Gamma$, $\#F_\Gamma$ the number of vertices/edges/faces of Γ, and similarly for $\widetilde{\Gamma}$, we see that

$$\#V_{\widetilde{\Gamma}} = d\#V_\Gamma \qquad \#E_{\widetilde{\Gamma}} = d\#E_\Gamma \qquad \#F_{\widetilde{\Gamma}} = d\#F_\Gamma$$

and hence

$$\chi(\widetilde{X}) = \#V_{\widetilde{\Gamma}} - \#E_{\widetilde{\Gamma}} - \#F_{\widetilde{\Gamma}} = d\,(\#V_\Gamma - \#E_\Gamma - \#F_\Gamma) = d\chi(X).$$

Part (b). We recall that Euler characteristic of the surface Σ_g of genus g is given by $\chi(\Sigma_g) = 2 - 2g$ and hence $g = \frac{1}{2}(2 - \chi(\Sigma_g) = 1 - \frac{1}{2}\chi(\Sigma_g)$. In particular,

$$\widetilde{g} = 1 - \frac{1}{2}\chi(\widetilde{X}) = 1 - \frac{d}{2}\chi(X) = 1 - \frac{d}{2}(2 - 2g) = 1 - d + dg.$$

□

6 Homework Assignment # 6

1. (10 points) Let (X, x_0), (Y, y_0) be pointed spaces. We recall that writing $f\colon (X, x_0) \to (Y, y_0)$ means that f is a map from X to Y which is basepoint-preserving in the sense that $f(x_0) = y_0$. Maps $f_0, f_1\colon (X, x_0) \to (Y, y_0)$ *basepoint-preserving homotopic*, notation $f_0 \sim_{\mathrm{bp}} f_1$, if there is a homotopy $H\colon X \times I \to Y$ from f_0 to f_1 which is *basepoint-preserving* in the sense that $H(x_0, t) = y_0$ for all $t \in I$. A map $f\colon (X, x_0) \to (Y, y_0)$ is a *basepoint-preserving homotopy equivalence* if there is a map $g\colon (Y, y_0) \to (X, x_0)$ such that $g \circ f \sim_{\mathrm{bp}} \mathrm{id}_X$ and $f \circ g \sim_{\mathrm{bp}} \mathrm{id}_Y$.

(a) Show that if $f, g\colon (X, x_0) \to (Y, y_0)$ are basepoint-preserving homotopic, then the induced homomorphisms $f_*, g_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ are equal.

(b) Show that if $f\colon (X,x_0)\to(Y,y_0)$ is a basepoint-preserving homotopy equivalence, then the induced map $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,y_0)$ is an isomorphism.

Proof. **Part (a).** Let $H\colon X\times I\to Y$ be a basepoint-preserving homotopy from f to g, i.e., $H(x,0)=f(x)$, $H(x,1)=g(x)$ and $H(x_0,t)=y_0$. Let $\gamma\colon (I,\partial I)\to(X,x_0)$ be a based loop in (X,x_0), and let H' be the composition

$$H'\colon I\times I \xrightarrow{(\gamma\circ p_1)\times p_2} X\times I \xrightarrow{H} Y$$

Here $p_i\colon I\times I\to I$ denotes the projection onto the i^{th} factor, and the first map is the map to the product $X\times I$ whose first component is $\gamma\circ p_1$ and whose second component is p_2 (recall that a map to a product of topological spaces is given by its component maps, or, categorically speaking, the cartesian product of topological spaces is their *categorical product* in the category of topological spaces. Then

$$H'(s,0)=H(\gamma(s),0)=f(\gamma(s)) \qquad H'(s,1)=H(\gamma(s),1)=g(\gamma(s))$$

and $H'(0,t)=y_0=H'(1,t)$; in other words, H' is a homotopy from $f\circ\gamma$ to $g\circ\gamma$ relative endpoints. In particular,

$$f_*([\gamma])=[f\circ\gamma]=[g\circ\gamma]=g_*([\gamma])\in\pi_1(Y,y_0).$$

Part (b). Let $g\colon (Y,y_0)\to(X,x_0)$ be the homotopy inverse of $f\colon (X,x_0)\to(Y,y_0)$, i.e., $g\circ f\sim_{\text{bp}}\text{id}_X$ and $f\circ g\sim_{\text{bp}}\text{id}_Y$. Then

$$g_*\circ f_*=(g\circ f)_*=(\text{id}_X)_*=\text{id}_{\pi_1(X,x_0)},$$

where the first and third equation follow from the functor property of the fundamental group, and the second equation holds by part (a). Similarly, $f_*\circ g_*=\text{id}_{\pi_1(Y,y_0)}$, and hence $f_*\colon\pi_1(X,x_0)\to\pi_1(Y,y_0)$ is a group isomorphism with inverse g_*. $\square$

2. (10 points) In this problem you are ask to show that an object X in a category $\mathcal{C}$ is the categorical product of two other objects. Recall that this means that you need to construct morphisms $p_1\colon X\to X_1$ and $p_2\colon X\to X_2$ and show that the following diagram in $\mathcal{C}$ has the property of being a *product diagram* discussed in the lectures:

$$X_1 \xleftarrow{p_1} X \xrightarrow{p_2} X_2$$

(a) Show that the cartesian product $G_1\times G_2$ of two groups G_1, G_2, equipped with the usual multiplication given by $(g_1,g_2)\cdot(h_1,h_2):=(g_1h_1,g_2h_2)$ is the categorical product of G_1 and G_2 in the category Grp of groups and group homomorphisms.

(b) Let (X_1, x_1), (X_2, x_2) be pointed topological spaces. Show that the pointed space $(X_1 \times X_2, (x_1, x_2))$ is the categorical product of (X_1, x_1) and (X_2, x_2) in the category Top_* of pointed topological spaces and basepoint-preserving maps.

Proof. **Part (a).** For $i = 1, 2$, let $p_i \colon G_1 \times G_2 \to G_i$ be the projection map (this is a group homomorphism). To prove that the cartesian product $G_1 \times G_2$ is the categorical product of G_1, G_2 in Grp, it suffices to show that the diagram

$$G_1 \xleftarrow{p_1} G_1 \times G_2 \xrightarrow{p_2} G_2$$

is a *product diagram*, i.e., that it satisfies the universal property

$$\begin{tikzcd} & G_1 \\ H \arrow[ur, "f_1"] \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_2"'] & G_1 \times G_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & G_2 \end{tikzcd}$$

for any group homomorphisms $f_i \colon H \to G_i$. Given f_1, f_2, we define

$$f \colon H \longrightarrow G_1 \times G_2 \qquad \text{by} \qquad f(h) := (f_1(h), f_2(h)).$$

It is clear that this map f makes the diagram commutative, and that f is determined by f_1, f_2.

Part (b). The proof of this part proceeds completely analogous to that of part (a): it suffices to show that the diagram

$$(X_1, x_1) \xleftarrow{p_1} (X_1 \times X_2, (x_1, x_2)) \xrightarrow{p_2} X_2$$

is a product diagram in the category Top_* of pointed topological spaces and basepoint-preserving maps (here p_1, p_2 denote the projection map to the first resp. second factor). In other words, we need to check the universal property expressed by the diagram

$$\begin{tikzcd} & (X_1, x_1) \\ (Y, y_0) \arrow[ur, "f_1"] \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_2"'] & (X_1 \times X_2, (x_1, x_2)) \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & (X_2, x_2) \end{tikzcd}$$

where f_1, f_2 are arbitrary basepoint-preserving maps. It is clear that the map $f = (f_1, f_2)$ with components f_1, f_2 that makes the above diagram commutative, and that it is the only map with this property. $\square$

3. (10 points)

(a) Show that the free product $G_1 * G_2$ of groups G_1, G_2 is the coproduct of G_1 and G_2 in the category Grp of groups and group homomorphisms. Hint: proving this amounts to constructing homomorphisms $i_1\colon G_1 \to G_1 * G_2$ and $i_2\colon G_2 \to G_1 * G_2$ and verifying that the diagram

$$G_1 \xrightarrow{i_1} G_1 * G_2 \xleftarrow{i_2} G_2 \tag{6.1}$$

is a coproduct diagram.

(b) Let $G_1 \xleftarrow{j_1} H \xrightarrow{j_2} G_2$ be a diagram of groups and homomorphisms. Show that the free product with amalgamation $G_1 *_H G_2$ is a pushout of the diagram above in the category of groups. Hint: Showing that $G_1 *_H G_2$ is a pushout of the diagram above means that there are homomorphisms $k_1\colon G_1 \to G_1 *_H G_2$ and $k_2\colon G_2 \to G_1 *_H G_2$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "j_1"] \arrow[d, "j_2"] & G_1 \arrow[d, "k_1"] \\ G_2 \arrow[r, "k_2"'] & G_1 *_H G_2 \end{tikzcd} \tag{6.2}$$

is commutative, and has the property of being a *pushout diagram.*

Proof. **Part(a).** We recall that the free product $G_1 * G_2$ is given by equivalence classes of words with letters belonging to $G_1 \amalg G_2$. Let $i_1\colon G_1 \to G_1 * G_2$ be the map given by sending $g \in G_1$ to the equivalence class $[g] \in G_1 * G_2$ of the one-letter word g. This is a homomorphism since

$$i_1(gg') = [(gg')] = [gg'] = [g][g'] = i_1(g)i_1(g').$$

Here (gg') is a one-letter (with letter $gg' \in G_1$), and gg' is a two-letter word; while these are different *words*, they are equivalent, and hence $[(gg')] = [gg'] \in G_1 * G_2$. Similarly, $i_2\colon G_2 \to G_1 * G_2$, given by $g \mapsto [g]$, is a homomorphism. To prove part (a) it then suffice to show that diagram (6.1) is a coproduct diagram, i.e., satisfies the universal property given by the diagram

$$\begin{tikzcd} G_1 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ G_1 * G_2 \arrow[r, dashed, "\exists! f"] & H \\ G_2 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

for arbitrary homomorphisms f_1, f_2 to a group H.

Any element of $G_1 * G_2$ is represented by a word $g_1 \dots g_k$, for $g_i \in G_1 \amalg G_2$. If $f\colon G_1 * G_2 \to H$ is a homomorphism making the above diagram commutative, then

$$f([g_1 \dots g_k]) = f([g_1] \dots [g_k]) = f([g_1]) \dots f([g_k]) = f_{\epsilon(g_1)}(g_1) \cdots f_{\epsilon(g_k)}(g_k) \in H,$$

where $\epsilon(g) \in \{1,2\}$ for $g \in G_1 \amalg G_2$ is given by $\epsilon(g) = i$ if $g \in G_i$. In particular, f is determined by f_1, f_2, and hence there is at most one homomorphism f that makes the diagram commutative.

To show that there is a homomorphism f that makes the above diagram commutative, we *define* f by

$$f([g_1 \dots g_k]) := f_{\epsilon(g_1)}(g_1) \cdots f_{\epsilon(g_k)}(g_k).$$

It remains to show that f is well-defined and a homomorphism. If the elements g_i is an identity element in G_1 or G_2, then $[g_1 \dots g_k] = [g_1 \dots \widehat{g}_i \dots g_k]$, where $\widehat{g}_i$ indicates that the letter g_i has been removed from the word $g_1 \dots g_k$. Then $f([g_1 \dots \widehat{g}_i \dots g_k]) \in H$ does not have the factor $f_{\epsilon(g_i)}(g_i)$, but since this is the unit element in H, removing this factor does not change the value of the product.

Similarly, if g_i, g_{i+1} belong to the same factor, say G_1, then

$$[g_1 \dots g_i g_{i+1} \dots g_k] = [g_1 \dots (g_i g_{i+1}) \dots g_k] \in G_1 * G_2.$$

Both of these elements have the same image under f, since $f_1(g_i g_{i+1}) = f_1(g_i) f_1(g_{i+1})$, since f_1 is a homomorphism. Analogous arguments apply for $g_i, g_{i+1} \in G_2$. This shows that f is well-defined.

To verify that f is homomorphism, let $g_1, \dots, g_k, g_{k+1}, \dots g_\ell \in G_1 \amalg G_2$. Then

$$\begin{aligned} f([g_1 \dots g_k g_{k+1} \dots g_\ell]) =& f_{\epsilon(g_1)}(g_1) \cdots f_{\epsilon(g_k)}(g_k) \cdot f_{\epsilon(g_{k+1})}(g_{k+1}) \cdots f_{\epsilon_{g_\ell}}(g_\ell) \\ =& f([g_1 \dots g_k]) \cdot f([g_{k+1} \dots g_\ell]) \end{aligned}$$

This proves part (a).

Part(b). We recall that $G_1 *_H G_2$, the free product of G_1 and G_2 with amalgamation over H is the quotient of the free product $G_1 * G_2$ by the normal subgroup N generated by the elements $[j_1(h) j_2(h)^{-1}] \in G_1 * G_2$. Following the hint, we construct a homomorphism k_1 as the composition

$$G_1 \xrightarrow{i_1} G_1 * G_2 \xrightarrow{p} G_1 * G_2/N = G_1 *_H G_2 \ ,$$

and similarly for $k_2\colon G_2 \to G_1 * G_2$. Then it suffices to show that diagram (6.2) is a pushout diagram. To verify this universal property, we look at the following big diagram. Removing the object $G_1 * G_2$ from this diagram, and all morphisms with that (co)domain, the remaining diagram is the diagram expressing the universal property of the pushout.

$$\begin{tikzcd}
 & & G_1 \arrow[dl, "i_1"'] \arrow[dr, "k_1"] \arrow[drr, bend left, "f_1"] & & \\
H \arrow[urr, bend left, "j_1"] \arrow[drr, bend right, "j_2"'] & G_1 * G_2 \arrow[rr, "p"] \arrow[rrr, dotted, bend right, "\widetilde{f}"'] & & G_1 *_H g_2 \arrow[r, dashed, "\exists ! f"] & K \\
 & & G_2 \arrow[ul, "i_1"] \arrow[ur, "k_1"'] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

It should be emphasized that this is *not* a commutative diagram, since the left square of the diagram is not commutative. All other parts of the diagram are required to be commutative. We will be using this bigger diagram to construct the homomorphism f. By part (a), there is a unique homomorphism $\widetilde{f}$ that makes the diagram commutative. To construct f, it remains to show that $\widetilde{f}\colon G_1 * G_2 \to K$ factors through the quotient map p, i.e., that $\widetilde{f}(n) = 1$ for elements n belonging to the normal subgroup $N \subset G_1 * G_2$. Since N is the normal subgroup generated by the elements $[j_1(h)j_2(h)^{-1}] \in G_1 * G_2$ it suffices to show

$$\widetilde{f}([j_1(h)j_2(h)^{-1}] = 1 \qquad \text{or, equivalently} \qquad \widetilde{f}([j_1(h)]) = \widetilde{f}([j_2(h)]).$$

Now,

$$\widetilde{f}([j_1(h)]) = \widetilde{f}(i_1(j_1(h))) = f_1(j_1(h)) = f_2(j_2(h)) = \widetilde{f}(i_2(j_2(h))) = \widetilde{f}([j_2(h)]),$$

where the first and fifth equality is by definition of the maps i_1, i_2, the second and fourth equality is by construction of $\widetilde{f}$, and the third equality is the commutativity of the outer square.

This shows that $\widetilde{f}$ factors through $G_1 *_H G_2$, giving us a homomorphism $f\colon G_1 *_H G_2 \to K$ that makes the required parts of the diagram commutative. It is clear that there is at most one such homomorphism, since f is determined by f_1 and f_2. $\square$

4. (10 points) Let M, N be path-connected manifolds of dimension $n \geq 3$. The goal of this problem is to compute the fundamental group of their connected sum $M \# N$ in terms of the fundamental groups of M and N. We provide an alternative description of the connected sum $M \# N$, which is easier for the problem at hand, works for smooth manifolds, and uses pushout diagrams (it is not hard to show that this version of $M \# N$ is homeomorphic to the version presented in class).

For the construction of the connected sum we pick points $x_0 \in M$, $y_0 \in N$ and maps $\phi\colon B^n_2 \to M$, $\psi\colon B^n_2 \to N$ which are are homeomorphisms onto their image with $\phi(0) = x_0$,

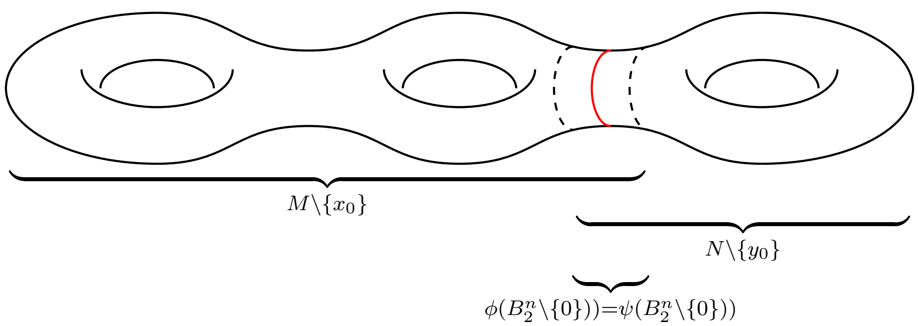
$\psi(0) = y_0$; here $B_2^n = \{v \in \mathbb{R}^n \mid ||v|| < 2\} \subset \mathbb{R}^n$ is the open ball of radius 2. Let α be the homeomorphism

$$\alpha\colon S^{n-1} \times (-1,1) \xrightarrow{\approx} B_2^n \setminus \{0\} \qquad \text{given by} \qquad (v,t) \mapsto (1-t)v,$$

and let $g\colon S^{n-1} \times (-1,1) \xrightarrow{\approx} S^{n-1} \times (-1,1)$ be the homeomorphism given by $g(v,t) = g(v,-t)$. Let $M\#N$ be the space determined by the pushout diagram

$$\begin{tikzcd} S^{n-1} \times (-1,1) \arrow[r, "\psi\circ\alpha"] \arrow[d, "\phi\circ\alpha\circ g"'] & N \setminus \{y_0\} \arrow[d] \\ M \setminus \{x_0\} \arrow[r] & M\#N. \end{tikzcd} \tag{6.3}$$

In other words, $M\#N = (M \setminus \{x_0\}) \cup_{S^{n-1}\times(-1,1)} (N \setminus \{y_0\})$ is obtained from the disjoint union $(M \setminus \{x_0\}) \amalg (N \setminus \{y_0\})$ by identifying the point $\phi \circ \alpha \circ g(v,t) \in M \setminus \{x_0\}$ with the point $\psi \circ \alpha(v,t) \in N \setminus \{y_0\}$ for $(v,t) \in S^{n-1} \times (-1,1)$. Here is a picture of $M\#N$, where the red circle is the image of $S^{n-1} \times \{0\} \subset S^{n-1} \times (-1,1)$ under either map in the commutative diagram above.



(a) Determine the fundamental group of $M \setminus \{x_0\}$ in terms of the fundamental group of M. Hint: use the Seifert van Kampen Theorem.

(b) Determine the fundamental group of $M\#N$ in terms of the fundamental groups of M and N.

Proof. **Part (a).** M is the union of the open subsets $M_1 := M\setminus\{x_0\}$ and $M_2 := \phi(B_2^n) \subset M$. Next we determine the fundamental groups of M_2 and $M_1 \cap M_2$.

- The map $\phi\colon B_2^n \to M_2 \subset M$ is a homeomorphism, and hence the induced map $\phi_*\colon \pi_1(B_2^n, v) \to \pi_1(M_2, \phi(v))$ is an isomorphism for any base point $v \in B_2^n$. Since B_2^n is convex, $\pi_1(B_2^n, v)$ is trivial, and hence $\pi_1(M_2, \phi(v))$ is trivial.

- The intersection $M_1 \cap M_2 = \phi(B_2^n) \setminus \{x_0\} = \phi(B_2^n \setminus \{0\})$ is homeomorphic to $B_2^n \setminus \{0\}$ via ϕ. The subspace $S^{n-1} \subset B_2^n$ is a deformation retract of $B_2^n \setminus \{0\}$, with retraction map $r\colon B_2^n \to S^{n-1}$ given by $r(v) := v/||v||$, and the homotopy between $i \circ r$ and the identity on B_2^n provided by the linear homotopy $H\colon B_2^n \times I \to B_2^n$ given by $H(v,t) := (1-t)v/||v|| + tv$. Hence the inclusion map $i\mathcal{S}^{n-1} \hookrightarrow B_2^n$ induces an isomorphism $i*\colon \pi_1(S^{n-1}, v) \to \pi_1(B_2^n \setminus \{0\}$ for any basepoint $v \in S^{n-1}$. The assumption $n \geq 3$ implies that $\pi_1(S^{n-1}, v)$, and hence $\pi_1(M_1 \cap M_2, \phi(v))$ are trivial.

In addition, since S^{n-1} is path-connected, the homotopy equivalent space $M_1 \cap M_2$ is also path-connected, allowing us to apply the Seifert van Kampen Theorem to the decomposition $M = M_1 \cup M_2$, and hence

$$\pi_1(M) \cong \pi_1(M_1) *_{\pi_1(M_1 \cap M_2)} \pi_1(M_2) \cong \pi_1(M \setminus \{x_0\}) *_{\{} 1\}\{1\} \cong \pi_1(M).$$

Part (b). The maps in the pushout diagram (6.3) are open embeddings; in particular, each of these maps is a homeomorphism onto its image. Identifying $M \setminus \{x_0\}$ (resp. $N \setminus \{y_0\}$ resp. $S^{n-1} \times (-1,+1)$) with its image in $M\#N$, $M \setminus \{x_0\}$ and $N \setminus \{y_0\}$ are open subsets of $M\#N$ with intersection $S^{n-1} \times (-1,+1)$. Since $S^{n-1} \times (-1,+1)$ is path-connected, according to the Seifert van Kampen Theorem,

$$\begin{aligned}\pi_1(M\#N) &\cong \pi_1(M \setminus \{x_0\}) *_{\pi_1(S^{n-1} \times (-1,+1))} \pi_1(N \setminus \{y_0\}) \\ &\cong \pi_1(M) *_{\{1\}} \pi_1(N) \cong \pi_1(M) * \pi_1(N)\end{aligned}$$

□

5. (10 points) Let X be the subspace of $\mathbb{R}^3$ given by the union of the 2-sphere S^2 and the segment S of the x-axis given by $S = \{(t,0,0) \in \mathbb{R}^3 \mid -1 \leq t \leq 1\}$. Calculate the fundamental group of X. Hint: use the Seifert van Kampen Theorem.

Proof. In order to apply the Seifert van Kampen Theorem we write $X = U \cup V$, where U, V are the following open subsets of X:

- $U = (S^2 \setminus \{(0,0,1)\}) \cup S$, and
- $V = S^2 \cup (S \setminus \{(0,0,0)\})$

To identify the fundamental groups of U, V and $U \cap V$ we show that these spaces are homotopy equivalent to simpler spaces whose fundamental group we are familiar with. We begin with the observation that X is homeomorphic to the quotient space of the disjoint union of S^2 and the interval $I = [-1,1]$ where the point $(-1,0,0) \in S^2$ is identified with $-1 \in I$ and $(1,0,0) \in S^2$ is identified with $1 \in I$ (the obvious map from the disjoint union $S^2 \amalg I$ to X factors through the quotient space and provides a continuous bijection between this quotient space and X; this is a homeomorphism since the quotient space is compact as the image of the compact space $S^2 \amalg I$ and $X \subset \mathbb{R}^3$ is Hausdorff). From now on we will identify X with this quotient space.

To understand the fundamental group of U, we note that via the stereographic projection $S^2 \setminus \{(0,0,1)\}$ is homeomorphic to $\mathbb{R}^2$, with the two points $(\pm 1,0,0)$ corresponding to $(\pm 1,0) \in \mathbb{R}^2$). It follows that U is homeomorphic to $(\mathbb{R}^2 \amalg I)/\sim$ where the endpoints $\pm 1 \in I$ are identified with $(\pm 1,0) \in \mathbb{R}^2$. We claim that $(\mathbb{R}^2 \amalg I)/\sim$ has the subspace $(J \amalg I)/\sim$ as deformation retract where $J = \{(s,0) \in \mathbb{R}^2 \mid -1 \leq 0 \leq 1\}$. The retract map

$$r\colon (\mathbb{R}^2 \amalg I)/\sim \longrightarrow (J \amalg I)/\sim$$

is given by the identity on I, while its restriction $r_{|\mathbb{R}^2}\colon \mathbb{R}^2 \to J$ to $\mathbb{R}^2$ is given by

$$r(x,y) = \begin{cases} (-1,0) \in J & \text{for } x \leq -1 \\ (x,0) \in J & \text{for } -1 \leq x \leq +1 \\ (+1,0) \in J & \text{for } x \geq +1 \end{cases}$$

The homotopy H_t between the identity map on $(\mathbb{R}^2 \amalg I)/\sim$ and $i \circ r$ is similarly given by the identity map on I and the linear homotopy $H_t(x,y) = (1-t)(x,y) + tr(x,y)$ for $(x,y) \in \mathbb{R}^2$. It follows that the retraction map r induces an isomorphism of fundamental groups

$$r_*\colon \pi_1(U,x_0) \xrightarrow{\cong} \pi_1((J \cup I)/\sim, x_0) \cong \mathbb{Z},$$

where we use the basepoint $x_0 = [(1,0)]$. Since $(J \cup I)/\sim$ is homeomorphic to the circle, its fundamental group is isomorphic to $\mathbb{Z}$.

The subspace V of $X = (S^2 \amalg I)/\sim$ is given by $(S^2 \amalg [-1,0) \amalg (0,+1])/\sim$ where the endpoints ± 1 of the intervals $[-1,0)$ resp. $(0,+1]$ are identified with $(-1,0,0) \in S^2$ resp. $(1,0,0) \in S^2$. Since these half-open intervals deformation retract to their endpoints via a linear homotopy, the space V deformation retracts to S^2 and hence $\pi_1(V,x_0)$ is trivial.

The subspace $U \cap V \subset U$ is homeomorphic via the stereographic projection to $(\mathbb{R}^2 \amalg [-1,0) \amalg (0,+1])/\sim$, where the endpoints ± 1 of these half-open intervals are identified with $(\pm 1,0)$. Deformation retracting these intervals to their endpoints shows that $\mathbb{R}^2$ is a deformation retract of $U \cap V$ and hence the fundamental group of $U \cap V$ is trivial. Moreover, since $\mathbb{R}^2$ is path connected, this implies that $U \cap V$ is path connected.

If follows by the Seifert van Kampen Theorem that

$$\pi_1(X,x_0) \cong \pi_1(U,x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V,x_0) \cong \{1\} *_{\{1\}} \mathbb{Z} \cong \mathbb{Z}.$$

$\square$

7 Homework Assignment # 7

1. (10 points) We recall that if $G \times X \to X$ is the action of a group G on a set X, then the subgroup $G_x := \{g \in G \mid gx = x\} \subseteq G$ is the *isotropy subgroup* of the point $x \in X$. The action is called *free* if the isotropy subgroup G_x is the trivial group for all $x \in X$. If X is a topological space, the action is called *continuous* if for every $g \in G$ the map $X \to X$ given by $x \mapsto gx$ is continuous.

(a) Show that if G is a finite group which acts freely and continuously on a Hausdorff space X, then the projection map $p\colon X \to G\backslash X$ to the orbit space $G\backslash X$ is a covering map. Hint: Use the assumptions that the action is free and X is Hausdorff to show that for every $x \in X$ there is an open neighborhood U such that the subsets $gU \subset X$ for $g \in G$ are mutually disjoint.

(b) Show that if X is a manifold of dimension n, then also the orbit space $G\backslash X$ is a manifold of dimension n (it is true, but in order to make this problem a little shorter, don't worry about proving that $G\backslash X$ is Hausdorff and second countable).

(c) Show that the map $\mathbb{Z}/2 \times S^n \to S^n$ given by $(m, v) \mapsto (-1)^m v$ is a continuous free action. We note that the orbit space $\mathbb{Z}/2\backslash S^n$ is the real projective space $\mathbb{RP}^n$, and hence part (b) of this problem provides a different way to show that $\mathbb{RP}^n$ is a manifold of dimension n.

(d) Show that the map

$$\mathbb{Z}/k \times S^{2n-1} \longrightarrow S^{2n-1} \qquad \text{given by} \qquad (m, v) \mapsto e^{2\pi i m/k} v$$

is a continuous free action of the cyclic group $\mathbb{Z}/k$ on the sphere $S^{2n-1} \subset \mathbb{C}^n$. By part (b) the orbit space $\mathbb{Z}/k\backslash S^{2n-1}$ is then a manifold of dimension $2n-1$, which is known as a *lens space*. Note that for $k = 2$, this is the real projective space $\mathbb{RP}^{2n-1}$.

Proof. **Part (a).** For any point $x \in X$ the points gx for $g \in G$ are all distinct, since the action is free (if $gx = hx$ for $g \neq h$, then $h^{-1}gx = x$ and hence $h^{-1}g \in G_x$, contradicting the freeness assumption). This implies that there are open neighborhoods U_{gx} of $gx \in X$ which are mutually disjoint, since X is Hausdorff. We note that $g^{-1}U_{gx}$ is another open neighborhood of x (it is open, since it is the image of the open set U_{gx} under the homeomorphism given by the action of the group element g^{-1}). Hence the finite intersection

$$U := \bigcap_{g \in G} g^{-1} U_{gx}$$

is again an open neighborhood of x.

We claim that the subsets gU for $g \in G$ are mutually disjoint. To prove this, assume that $y \in gU \cap hU$ for $g \neq h$. Then in particular y belongs to $g(g^{-1}U_{gx}) = U_{gx}$ and to $h(h^{-1}U_{hx}) = U_{hx}$ contradicting $U_{gx} \cap U_{hx} = \emptyset$.

Given the point $[x] \in G\backslash X$ the subset $V = p(U) \subset G\backslash X$ is an open neighborhood of $[x]$; it is open since $p^{-1}(V) = \bigcup_{g \in G} gU$ is open. This open set is evenly covered, since for any $g \in G$ the restriction

$$p_{|gU} \colon gU \longrightarrow V$$

is a homeomorphism, since it is a continuous bijection and an open map. To argue that p is open, suppose that $U \subset X$ is an open subset. Then $p(U)$ is an open subset of $G\backslash X$ since $p^{-1}(p(U)) = \bigcup_{g \in G} gU$ is an open subset of X.

Part (b). To show that $G\backslash X$ is locally homeomorphic to $\mathbb{R}^n$, let $y \in G\backslash X$ and let $V \subset G\backslash X$ be an evenly covered open neighborhood of y. Then $p^{-1}(V)$ is the disjoint of open subsets $V_\alpha \subset X$ such that $p_{|V_\alpha} \colon V_\alpha \to V$ is a homeomorphism. Let x be the unique point in V_α with $p(x) = y$. Since X is a manifold of dimension n a possibly smaller neighborhood of x is homeomorphic to an open subset of $\mathbb{R}^n$. Via the restriction of $p_{|V_\alpha}$ to this smaller neighborhood we conclude that a neighborhood of y is homeomorphic to an open subset of $\mathbb{R}^n$.

Part (c). To show that the $\mathbb{Z}/2$-action on S^n is free we note that for any point $v \in S^n$ the action of the non-trivial element $1 \in \mathbb{Z}/2$ maps v to its antipodal point $-v$. Since $-v \neq v$, the isotropy subgroup is trivial group, i.e., the action is free.

Part (d). To show that the $\mathbb{Z}/k$ action on S^{2n-1} is free, assume $[m] \in \mathbb{Z}/k$ belongs to the isotropy subgroup of some $v = (v_1, \dots, v_n) \in S^{2n-1} \subset \mathbb{C}^n$, that is, $e^{2\pi i m/k} v = v$. Since v is a unit vector, it has a non-zero component $v_i \in \mathbb{C}$. Then $e^{2\pi i m/k} v_i = v_i$ which implies $e^{2\pi i m/k} = 1$, which in turn implies that $m \in \mathbb{Z}$ must be a multiple of k. In particular, $[m]$ is the identity element in $\mathbb{Z}/k$, proving that the action is free. $\square$

2. (10 points) Let $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Let Y be a path-connected and let $f \colon (Y, y_0) \to (X, x_0$ be a map such that the image $f_*\pi_1(Y, y_0)$ is contained in the image $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$. We proved in class that then there exists a unique (not necessarily continuous) map $\widetilde{f}$ making the diagram

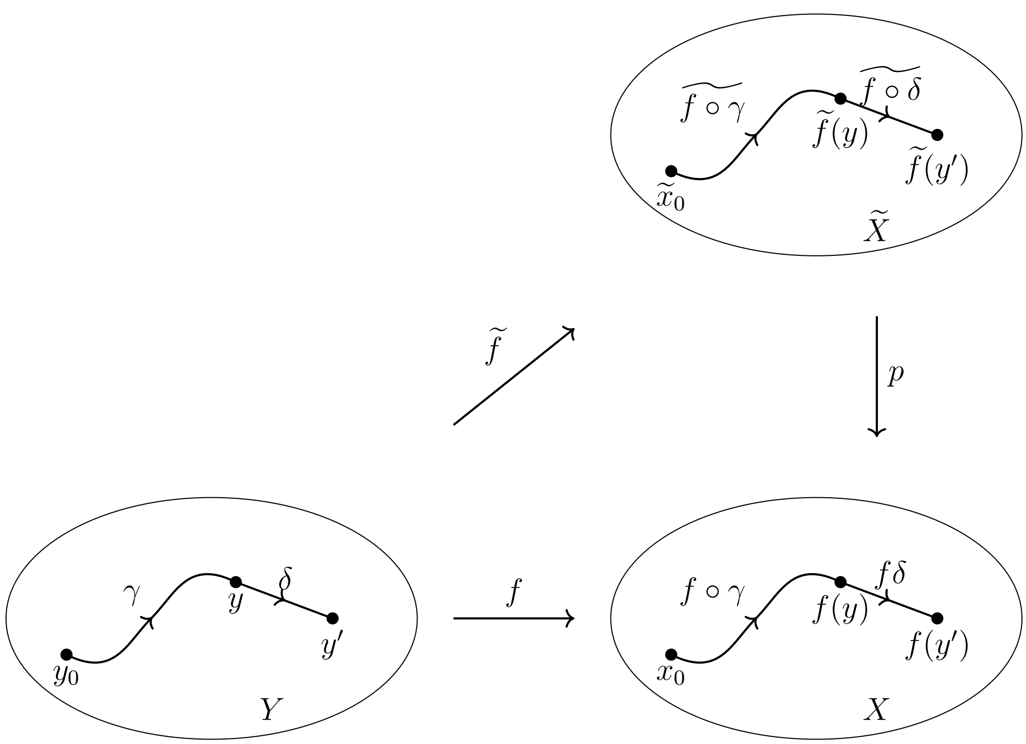
$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

commutative. We constructed $\widetilde{f}(y)$ by picking a path $\gamma \colon I \to Y$ from y_0 to y, composed with the map $f \colon Y \to X$ to obtain the path $f \circ \gamma \colon I \to X$, and defined $\widetilde{f}(y) := \widetilde{f\gamma}(1)$, where $\widetilde{f \circ \gamma} \colon I \to \widetilde{X}$ is the unique lift of $f \circ \gamma$ with starting point $\widetilde{x}_0$.

Show that $\widetilde{f}$ is continuous under the additional assumption that Y is locally path-connected. Hint: It suffices to show that $\widetilde{f}$ is continuous in some open neighborhood V of every point $y \in Y$. Show that the assumption that Y is locally path-connected can be used to choose for every point $y \in Y$ a path-connected neighborhood V such that $f(V)$ is contained in a evenly covered open subset $U \subset X$. To analyze $\widetilde{f}(y')$ for $y' \in V$, use the concatenation $\gamma * \delta$ of a path γ from y_0 to y and $\delta \colon I \to V$ from y to y'.

Proof. To prove continuity of $\widetilde{f}$ at a point $y \in Y$, it suffices to prove continuity of f restricted to some open neighborhood $V \subset Y$ of y. Let U be an evenly covered open neighborhood of $f(y) \in X$. Then $f^{-1}(U)$ is an open neighborhood of y, and thanks to the assumption that Y is locally path-connected, there is path-connected open neighborhood V of y contained in $f^{-1}(U)$.

To construct $\widetilde{f}(y')$ for $y' \in V$, we can use any path $\gamma' \colon I \to Y$ from y_0 to y', for example $\gamma' = \gamma * \delta$, the concatenation of the path $\gamma \colon I \to Y$ from y_0 to y and some path $\delta \colon I \to V$ from y to y'. Let $\widetilde{f \circ \gamma} \colon I \to \widetilde{X}$ be the lift of $f \circ \gamma$ from $\widetilde{x}_0$ to $\widetilde{f}(y)$ (by definition of $\widetilde{f}(y)$) and let $\widetilde{f \circ \delta}$ be the unique lift of $f \circ \delta$ with starting point $\widetilde{f}(y)$. These paths are shown in the picture below.



Then the path $\widetilde{f \circ \gamma} * \widetilde{f \circ \delta}$ is a lift of the path $f \circ \gamma * f \circ \delta = f \circ (\gamma * \delta)$, and hence by definition $\widetilde{f}(y') = \widetilde{f \circ \gamma} * \widetilde{f \circ \delta}(1) = \widetilde{f \circ \delta}(1)$. The key observation is that since δ is a path in

V, $f \circ \delta$ is a path in the evenly covered subset U, and hence its lift $\widetilde{f \circ \delta}$ with starting point $f(y) \in \widetilde{U}$ is simply given by $\widetilde{f \circ \delta} = p_{|\widetilde{U}}^{-1} f \circ \delta$, the image of $f \circ \delta$ under the inverse of the homeomorphism $p_{|\widetilde{U}} \colon \widetilde{U} \to U$. In particular, $\widetilde{f}(y') = p_{|\widetilde{U}}^{-1} f(y')$. This holds for every point $y' \in V$, and hence $\widetilde{f}_{|V} = p^{-1} \circ f_{|V}$. In particular, $\widetilde{f}_{|V}$ is continuous. $\square$

3. (10 points) Let $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering of a path-connected and locally path-connected space X.

(a) It follows from the General Lifting Criterion that for $g \in G := \pi_1(X, x_0)$ there is a unique map $\phi_g \colon (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, g\widetilde{x}_0)$ making the diagram

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{x}_0) \arrow[rr, "\phi_g"] \arrow[dr, "p"'] & & (\widetilde{X}, g\widetilde{x}_0) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

commutative. Here $g\widetilde{x}_0 := \widetilde{\gamma}(1)$ is the endpoint of a lift $\widetilde{\gamma} \colon I \to \widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ of any based loop γ in (X, x_0) which represents $g \in \pi_1(X, x_0)$ (we have shown that $\widetilde{\gamma}(1)$ depends only on $[\gamma] \in \pi_1(X, x_0)$, not on the particular loop γ). Show that the map

$$G \times \widetilde{X} \longrightarrow \widetilde{X} \qquad\qquad (g, \widetilde{x}) \mapsto \phi_g(\widetilde{x})$$

is an action map.

Hint: it might be helpful to have the following explicit description of $\phi_{[\alpha]}(\widetilde{x})$ for an element $[\alpha] \in \pi_1(X, x_0)$ and $\widetilde{x} \in \widetilde{X}$. Let $\widetilde{\alpha} \colon I \to \widetilde{X}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}$, and let $\alpha := p \circ \widetilde{\alpha}$ be the corresponding path in X. Then

$$\phi_{[\gamma]}(\widetilde{x}) = \widetilde{(\gamma * \alpha)}(1) \qquad \text{where } \widetilde{(\gamma * \alpha)} \text{ is the unique lift of } \gamma * \alpha \text{ with } \widetilde{(\gamma * \alpha)}(0) = \widetilde{x}_0$$

In particular, to evaluate $\Phi_{[\gamma]}(\widetilde{x}_0)$ we can choose $\alpha = c_{\widetilde{x}_0}$ and hence $\Phi_{[\gamma]}(\widetilde{x}_0) = \widetilde{\gamma}(1)$, where $\widetilde{\gamma} \colon I \to \widetilde{X}$ is a lift of γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$.

(b) Show that the action is free, i.e., for every $\widetilde{x} \in \widetilde{X}$, the only element of $g \in G$ with $g\widetilde{x} = \widetilde{x}$ is the identity element. Hint: According to Proposition on p. 1 of the notes from the lecture on Oct. 10 (specializing the statement to a universal covering space), the map $G \to p^{-1}(x_0)$ given by $[\gamma] \mapsto \widetilde{\gamma}(1)$ is a bijection.

(c) Show that the action is transitive on the fiber $p^{-1}(x)$ for all $x \in X$, i.e., for $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$ there is some $g \in G$ such that $g\widetilde{x} = \widetilde{x}'$.

Proof. **Part (a).** To show that it is an action map, let $g = [\gamma] \in G$ and $h = [\delta] \in G$. Let $\widetilde{\gamma}\colon I \to \widetilde{X}$ resp. be the lifts of γ resp. δ with starting point $\widetilde{x}_0$. Then by construction, $\widetilde{\gamma}(1) = g\widetilde{x}_0 = \phi_g(\widetilde{x}_0)$ and $\widetilde{\delta}(1) = h\widetilde{x}_0 = \phi_h(\widetilde{x}_0)$. To determine $\phi_{gh} = \phi_{[\gamma*\delta]}$ we need to determine the unique lift $\widetilde{\gamma * \delta}$ of $\gamma * \delta$ with starting point $\widetilde{x}_0$. Noting that the starting point of the path $\phi_g\widetilde{\delta}$ is $\phi_g(\widetilde{\delta}(0)) = \phi_g(\widetilde{x}_0) = \widetilde{\gamma}(1)$, and hence we can form the path $\widetilde{\gamma} * (\phi_g\delta)$ which is the desired lift $\widetilde{\gamma * \delta}$ with starting point $\widetilde{x}_0$. Then

$$\phi_{gh}(\widetilde{x}_0) = (gh)\widetilde{x}_0 = (\widetilde{\gamma * \delta})(1) = (\widetilde{\gamma} * (\phi_g\delta))(1) = (\phi_g\delta)(1) = \phi_g(\delta(1)) = \phi_g(\phi_h(\widetilde{x}_0)).$$

Hence ϕ_{gh} and $\phi_g\phi_h$ are two deck transformations that map $\widetilde{x}_0$ to the same point. By the uniqueness part of the General Lifting Criterion, then $\phi_{gh} = \phi_g \circ \phi_h$. This proves that this is an action, since $g(h\widetilde{x}) = \phi_g(\phi_h\widetilde{x}) = \phi_{gh}(\widetilde{x}) = (gh)\widetilde{x}$.

Part (b). Let $g \in G$ and let $\phi_g\colon \widetilde{X} \to \widetilde{X}$ be the associated deck transformation. If $\widetilde{x} \in \widetilde{X}$ is some point fixed by the G-action, i.e., $\widetilde{x} = g\widetilde{x} = \phi_g(\widetilde{x})$, then the deck transformation ϕ_g is the identity map, again by the uniqueness part of the General Lifting Criterion.

Part (c). We first show that G acts transitively on the fiber $p^{-1}(x_0$. If $\widetilde{x} \in p^{-1}(x_0)$ let $\widetilde{\gamma}\colon I \to \widetilde{X}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}$, let $\gamma = p \circ \widetilde{\gamma}$, and $g := [\gamma] \in \pi_1(X, x_0)$. Then by construction of the action $g\widetilde{x}_0 = \widetilde{x}$. Given some other $\widetilde{x}' \in p^{-1}(x_0)$, then there is some $g' \in G$ with $g'\widetilde{x}_0 = \widetilde{x}'$. It follows that $(g'g^{-1})\widetilde{x} = g'(g^{-1}\widetilde{x}) = g'\widetilde{x}_0 = \widetilde{x}'$, which proves that G acts transitively on the fiber $p^{-1}(x_0)$ over the base point.

To show that G acts transitively on the fiber $p^{-1}(x')$ over some other point $x' \in X$, let $\delta\colon I \to X$ be a path from x_0 to x'. This path determines a bijection

$$\Psi_\delta\colon p^{-1}(x_0) \xrightarrow{\cong} p^{-1}(x')$$

by sending $\widetilde{x} \in p^{-1}(x_0)$ to $\delta_{\widetilde{x}_0}(1)$ to the endpoint of the unique lift $\delta_{\widetilde{x}_0}\colon I \to \widetilde{X}$ of the path δ with starting point $\delta_{\widetilde{x}}(0) = \widetilde{x}$. The inverse is given similarly by considering lifts of $\bar{\delta}$. This map is compatible with deck transformations $\phi\colon \widetilde{X} \to \widetilde{X}$ in the sense that $\phi(\Psi_\delta(\widetilde{x})) = \Psi_\delta(\phi(\widetilde{x}))$. This follows from the fact that if $\widetilde{\delta}$ is a path from $\widetilde{x} \in p^{-1}(x_0)$ to $\widetilde{x}' \in p^{-1}(x')$, then $\phi \circ \delta$ is a lift of δ from $\phi(\widetilde{x})$ to $\phi(\widetilde{x}')$. In particular, the map Ψ_δ is *equivariant*, i.e., $\Psi_\delta(g\widetilde{x}) = g\Psi_\delta(\widetilde{x})$. Hence transitivity of the action on $p^{-1}(x_0)$ implies transitivity of the action on $p^{-1}(x')$. □

4. (10 points) Let (X, x_0) be a pointed space which is path-connected, locally path-connected, and semilocally simply connected. The goal of this assignment is the classification of isomorphism classes of objects of the category $\mathrm{Cov}_*(X, x_0)$ of based path connected covering spaces $p\colon (E, e_0) \to (X, x_0)$. More precisely, the goal is to show that there is a bijection Ψ between

$$\{\text{based coverings } p\colon (E, e_0) \to (X, x_0) \text{ with } E \text{ path-connected}\}/\text{isomorphism}$$

and

$$\{\text{subgroups of } \pi_1(X, x_0)\}.$$

It is given by sending a covering p to the subgroup $p_*\pi_1(E, e_0) \subset \pi_1(X, x_0)$.

(a) Show that Ψ is injective. Hint: use the general lifting criterion to show that any two path-connected based coverings $p\colon (E, e_0) \to (X, x_0)$ and $p'\colon (E', e'_0) \to (X, x_0)$ are isomorphic.

(b) Let $p\colon \widetilde{X} \to X$ be the universal covering of X, on which the fundamental group $G = \pi_1(X, x_0)$ acts freely by covering maps; this action is transitive on all fibers $p^{-1}(x)$ for $x \in X$. Let H be a subgroup of G and let $H\backslash\widetilde{X}$ be the orbit space of action of the subgroup H and let $p^H\colon (H\backslash\widetilde{X}, [\widetilde{x}_0]) \to (X, x_0)$, $[\widetilde{x}] \mapsto p(\widetilde{x})$ be the projection map. Here $[\widetilde{x}] = H\widetilde{x}$ denotes the orbit through the point $\widetilde{x}$. Show that p^H is a covering and that $p^H_*\pi_1(H\backslash\widetilde{X}, [\widetilde{x}_0]) \subset G$ is the subgroup H.

Proof. **Part (a).** Let $p\colon (E, e_0) \to (X, x_0)$ and $p'\colon (E', e'_0) \to (X, x_0)$ be two coverings of X with path-connected total spaces E, E' such that $p_*\pi_1(E, e_0) = p'_*\pi_1(E', e'_0)$. The assumption that X is locally path-connected implies that E and E' are locally path-connected. Hence we can use the General Lifting Criterion to find unique pointed maps making the following diagram commutative

$$\begin{tikzcd}
(E, e_0) \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & (E', e'_0) \arrow[ll, shift left, "f'"] \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

Then the composition $f' \circ f\colon E \to E$ is a covering transformation of E which fixes the point e_0. By the uniqueness part of the General Lifting Criterion, this is the identity of E. Similarly, $f \circ f'$ is the identity of E', and hence the based coverings p and p' are isomorphic.

Part (b). Let $U \subset X$ be evenly covered subset for the universal covering $p\colon \widetilde{X} \to X$. Let $\widetilde{U} \subset p^{-1}(U)$ be an open subset such that $p_{|\widetilde{U}}\colon \widetilde{U} \to U$ is a homeomorphism.

We claim that the subsets $g\widetilde{U}$ for $g \in G$ are mutually disjoint. So assume $g\widetilde{x} = g'\widetilde{x}'$ for $g, g' \in G$ and $\widetilde{x}, \widetilde{x}' \in \widetilde{U}$. Then $p(\widetilde{x}) = p(\widetilde{x}')$ since the G-action preserves the fibers. Hence $\widetilde{x} = \widetilde{x}'$, since $p_{\widetilde{U}}$ is a homeomorphism. Finally, $g\widetilde{x} = g'\widetilde{x}$ implies $g = g'$, since the G-action is free. Summarizing, $p^{-1}(U)$ is the union of the mutually disjoint subsets $g\widetilde{U}$ for $g \in G$ and the restriction $p_{|g\widetilde{U}}\colon g\widetilde{U} \to U$ is a homeomorphism.

We note that in the orbit space $H\widetilde{X}$ a point $\widetilde{\in}\widetilde{X}$ is identified with $h\widetilde{x}$ for $h \in H$. It follows that $(p^H)^{-1}(U)$ is the disjoint union of $Hg\widetilde{U}$ for $Hg \in H\backslash G$. Hence p^H restricted to $Hg\widetilde{U}$ maps $Hg\widetilde{U}$ homeomorphically to U, showing that U is evenly covered and that $p^H\colon H\backslash\widetilde{X} \to X$ is a covering.

To show that $p^H_*\pi_1(H\backslash\widetilde{X},[\widetilde{x}_0])$ is the subgroup $H\subset\pi_1(X,x_0)$, we recall that a based loop γ in (X,x_0) represents an element in the image of the fundamental group of the covering $p^H\colon H\backslash\widetilde{X}\to X$ if and only if its lift $\widetilde{\gamma}^H\colon I\to H\backslash\widetilde{X}$ with starting point $[x_0]$ is a *loop*. Let $\widetilde{\gamma}\colon I\to\widetilde{X}$ be the lift of γ with starting point $\widetilde{x}_0$, and let $q\colon\widetilde{X}\to H\backslash\widetilde{X}$ be the projection map. Then $q\circ\widetilde{\gamma}$ an explicit description of the lift $\widetilde{\gamma}^H$. In particular, $\widetilde{\gamma}^H$ is a loop in p^H if and only if $\widetilde{\gamma}(1)\in H$ if and only if $[\gamma]\in H$.

□

8 Homework Assignment # 8

1. (10 points) A standard atlas for the sphere S^n is given by the hemisphere atlas, given by the open subsets $U_i^\epsilon := \{(x_0,\dots,x_n)\in\mathbb{R}^{n+1}\mid \epsilon x_i>0\}\subset S^n$ for $i=0,\dots,n$ and $\epsilon\in\{\pm1\}$, and the homeomorphism

$$\phi_i^\epsilon\colon U_i^\epsilon\longrightarrow B_1^n \qquad \text{given by} \qquad \phi_i^\epsilon(x_0,\dots,x_n)=(x_0,\dots,\widehat{x}_i,\dots,x_n)$$

(a) Show that $\{(U_i^\epsilon,\phi_i^\epsilon)\}$ is a smooth atlas for S^n. You can use your calculus knowledge about smooth functions on open subsets of $\mathbb{R}$. Beware that the function $\sqrt{x}$ is defined, but not smooth at $x=0$.

(b) Show that with respect to the smooth structure on S^n given by the smooth atlas from part (a), the inclusion map $i\colon S^n\hookrightarrow\mathbb{R}^{n+1}$ is smooth.

Proof. **Part (a).** A short calculation shows that the inverse $(\phi_i^\epsilon)^{-1}\colon B^n\to U_i^\epsilon$ is given by

$$(\phi_i^\epsilon)^{-1}(y)=(y_1,\dots,y_{i-1},\epsilon\sqrt{1-||y||^2},y_{i+1},\dots,y_n)\qquad\text{for } y=(y_1,\dots,y_n)\in B_1^n$$

To prove that the atlas is smooth, we need to verify that the transition map $\phi_j^\delta\circ(\phi_i^\epsilon)^{-1}$ is a smooth map on $\phi_i^\epsilon(U_j^\delta\cap U_i^\epsilon)\subset B_1^n$. This amounts to checking that the components of this map are smooth. Now, composing with ϕ_j^δ just forgets the j^{th} component, and hence it suffices to show that the components of

$$(\phi_i^\epsilon)^{-1}\colon\phi_i^\epsilon(U_j^\delta\cap U_i^\epsilon)\longrightarrow S^n\subset\mathbb{R}^{n+1}$$

are smooth. This is obvious for all components except the component $\epsilon\sqrt{1-||y||^2}$, but since y is an element of the *open* ball B_1^n, $1-||y||^2\neq0$, and hence the map $B_1^n\to\mathbb{R}$, $y\mapsto\epsilon\sqrt{1-||y||^2}$ is smooth.

Part (b). To check that $i\colon S^n\hookrightarrow\mathbb{R}^{n+1}$ is smooth, we need to check that the composition

$$B_1^n\xrightarrow{(\phi_i^\epsilon)^{-1}}U_i^\epsilon\xhookrightarrow{i}\mathbb{R}^{n+1}$$

is smooth. This amounts to showing that all components of this map are smooth, but this is what we already proved in part (a). □

2. (10 points) We recall that the stereographic projection provides a homeomorphism between the open subsets $U_\pm := S^n \setminus \{(\mp 1, 0, \dots, 0)\}$ of S^n and $\mathbb{R}^n$. More explicitly, the stereographic projection is the map

$$\psi_\pm \colon U_\pm \longrightarrow \mathbb{R}^n \qquad \text{is defined by} \qquad \psi_\pm(x_0, \dots, x_n) := \frac{1}{1 \pm x_0}(x_1, \dots, x_n),$$

and its inverse $\psi_\pm^{-1} \colon \mathbb{R}^n \to U_\pm$ is given by the formula

$$\psi_\pm^{-1}(y_1, \dots, y_n) = \frac{1}{||y||^2 + 1}(\pm(1 - ||y||^2), 2y_1, \dots, 2y_n).$$

In particular, the two charts (U_+, ψ_+), (U_-, ψ_+) form an atlas for S^n.

(a) Show that $\{(U_+, \psi_+), (U_-, \psi_-)\}$ is a *smooth* atlas for S^n.

(b) Show that the atlas above determines the same smooth structure on S^n as the smooth atlas of the previous problem.

Proof. **Part (a).** We need to show that the transition function between the two charts, given by the composition

$$\psi_+(U_+ \cap U_-) \xrightarrow{\psi_+^{-1}} U_+ \cap U_- \xrightarrow{\psi_-} \psi_-(U_+ \cap U_-)$$

is a diffeomorphism. The intersection $U_+ \cap U_-$ consists of all points $(x_0, \dots, x_n) \in S^n$ with $x_0 \neq \pm 1$. The formula for $\psi_\pm$ shows that $\psi_+(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ as well as $\psi_-(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$. Explicitly, for $y = (y_1, \dots, y_n) \in R^n \setminus \{0\}$ we have

$$\begin{aligned}
\psi_-(\psi_+^{-1}(y)) &= \psi_-(\frac{1}{||y||^2 + 1}\left((||y||^2 - 1), 2y_1, \dots, 2y_n\right)) \\
&= \frac{1}{1 + \frac{(||y||^2 - 1)}{||y||^2 + 1}} \left(\frac{2y_1}{||y||^2 + 1}, \dots, \frac{2y_n}{||y||^2 + 1} \right) \\
&= \frac{1}{||y||^2 + 1 + (||y||^2 - 1)} (2y_1, \dots, 2y_n) \\
&= \frac{1}{||y||^2}(y_1, \dots, y_n)
\end{aligned}$$

This map is smooth, since all its components are smooth functions (the function $1/||y||^2$ is smooth for $y \in \mathbb{R}^n \setminus \{0\}$. We note that the inverse of this map is the maps itself, and hence the map is a diffeomorphism.

Part (b). The map $\phi_i^\epsilon\colon U_i^\epsilon \to B_1^n$ (the open ball of radius 1 in $\mathbb{R}^n$) is given by forgetting the i-th coordinate. Hence the composition $\phi_i^\epsilon \circ \psi_\delta^{-1}$ for $\delta \in \{\pm 1\}$ is a smooth function, since all components of ψ_δ^{-1} are smooth.

A short calculation shows that the other composition $\psi_\delta \circ (\phi_i^\epsilon)^{-1}$ is given by

$$\psi_\delta \circ (\phi_i^\epsilon)^{-1}(y) = \begin{cases} \frac{1}{1-\delta\epsilon\sqrt{1-||y||^2}}(y_1, \ldots, y_n) & i = 0 \\ \frac{1}{1-\delta y_1}(y_2, \ldots, y_{i-1}, \epsilon\sqrt{1-||y||^2}, y_i, \ldots, y_n) & i \neq 0 \end{cases}$$

We note that there are two potential problems with the smoothness of these maps:

- The square root function is not differentiable at 0; so there is a potential issue when $1 - ||y||^2$, the argument of the square root, becomes zero. This doesn't happen since $y \in B_1^n$, the open n-ball of radius 1.
- The function is undefined if the denominator becomes 0. We could discuss for which y this happens in either case ($i = 0$ or $i \neq 0$) and show that those y are not in the domain of the transition map. It is easier to say that *by construction* the transition map is well-defined on its domain, and hence we don't need to worry about those points.

□

3. (10 points) Let $\mathbb{RP}^n$ be the real projective space of dimension n, which can be defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by identifying x with λx for a non-zero $\lambda \in \mathbb{R}$.

(a) Show that the atlas $\{(U_i, \phi_i)\}_{i=0,\ldots,n}$ with $U_i := \{[x_0, \ldots, x_n] \mid x_i \neq 0\}$ and the homeomorphism $\phi_i\colon U_i \to \mathbb{R}^n$ given by

$$\phi_i([x_0, \ldots, x_n]) = \frac{1}{x_i}(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

is a smooth atlas.

(b) Show that the function $h\colon \mathbb{RP}^n \to \mathbb{R}$ defined by

$$h([x]) = \frac{1}{||x||^2} \sum_{\ell=0}^{n} \ell x_\ell^2 \qquad \text{for } x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$$

is smooth.

Proof. **Part (a).** We need to show that the transition maps

$$\mathbb{R}^n \supset \phi_i(U_i \cap U_j) \xrightarrow{\phi_i^{-1}} U_i \cap U_j \xrightarrow{\phi_j} \phi_j(U_i \cap U_j) \subset \mathbb{R}^n$$

are smooth for all $i, j = 0, \dots, n$. For $w = (w_1, \dots, w_n) \in \phi_i(U_i \cap U_j)$, $i \neq j$ we compute

$$\begin{aligned}\phi_j(\phi_i^{-1}(w) &= \phi_j([w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n] \\ &= \begin{cases} \frac{1}{w_{j+1}}(w_1, \dots, \widehat{w}_{j+1}, \dots, w_i, 1, \dots, w_n) & 0 \leq j < i \\ \frac{1}{w_j}(w_1, \dots, w_i, 1, w_{i+1}, \dots, \widehat{w}_j, \dots, w_n) & i < j \leq n \end{cases}\end{aligned}$$

As in the previous homework problem this shows that the transition map is smooth since its components are all smooth functions (since the transition function is well-defined those ws for which the denominator is 0 can't be in the domain of the transition function).

Part (b). It is clear that h is well-defined since for $\lambda \in \mathbb{R} \setminus \{0\}$

$$h([\lambda x]) = \frac{1}{||\lambda x||^2} \sum_{\ell=0}^{n} \ell (\lambda x_\ell)^2 = \frac{1}{||x||^2} \sum_{\ell=0}^{n} \ell x_\ell^2 = h([x]).$$

To show that h is smooth it suffices to show that the composition

$$f := h \circ \phi_k^{-1} \colon B_1^n \to \mathbb{R}$$

is smooth for $0 \leq k \leq n$. We compute:

$$\begin{aligned} f(w_1, \dots, w_n) &= h([w_1, \dots, w_k, 1, w_{k+1}, \dots, w_n]) \\ &= \frac{1}{||w||^2 + 1} \left(\sum_{\ell=0}^{k-1} \ell w_{\ell+1}^2 + k + \sum_{\ell=k+1}^{n} \ell w_\ell^2 \right) \end{aligned} \tag{8.1}$$

The numerator and denominator of this fraction are both quadratic functions and hence smooth. The denominator function $||w||^2 + 1$ is non-where vanishing and hence f is a smooth function. □

4. (10 points) Show that the Cartesian product of $M \times N$ of smooth manifolds of dimension m resp. n is a smooth manifold of dimension $m + n$.

Proof. Let $(U_i, \phi_i)_{i \in I}$ be a smooth atlas for M, and $(V_j, \psi_j)_{j \in J}$ for N. We claim that the collection of charts of $M \times N$ given by

$$M \times N \underset{\text{open}}{\supset} U_i \times V_j \xrightarrow{\phi_i \times \psi_j} \phi_i(U_i) \times \psi_j(V_j) \underset{\text{open}}{\subset} \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$$

for $(i, j) \in I \times J$ is a smooth atlas for $M \times N$. In particular, $M \times N$ is a smooth manifold of dimension $m + n$.

It is clear that $\phi_i \times \psi_j$ is a homeomorphism with inverse given by $\phi_i^{-1} \times \psi_j^{-1}$. Also it is clear that the open subsets $U_i \times V_j$ cover $M \times N$. Hence this collection is an atlas, and it only remains to show that this is a *smooth* atlas.

The transition map between the charts $\phi_i \times \psi_j$ and $\phi_{i'} \times \psi_{j'}$ is given by (after restricting the domains in the obvious way) by

$$(\phi_{i'} \times \psi_{j'}) \circ (\phi_i \times \psi_j)^{-1} = (\phi_{i'} \times \psi_{j'}) \circ (\phi_i^{-1} \times \psi_j^{-1}) = (\phi_{i'} \circ \phi_i^{-1}) \times (\psi_{j'} \circ \psi_j^{-1})$$

The map $\phi_{i'} \circ \phi_i^{-1}$ is a transition map for a smooth atlas for M and hence a smooth map between open subsets of $\mathbb{R}^m$. Similarly the map $\psi_{j'} \circ \psi_j^{-1}$ is a smooth map between open subsets of $\mathbb{R}^n$. It follows that the Cartesian products of these maps is a smooth map (between open subsets of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$) and hence the atlas constructed above is in fact smooth. □

5. (10 points) We recall that for an open subset $U \subset \mathbb{R}^m$ and $p \in U$ the map

$$D^U \colon T_p^{\mathrm{geo}} U \longrightarrow \mathbb{R}^m \qquad \text{given by} \qquad [\gamma] \mapsto \gamma'(0)$$

is a bijection. Show that for a smooth map $\mathbb{R}^m \underset{\text{open}}{\supset} U \xrightarrow{F} V \underset{\text{open}}{\subset} \mathbb{R}^n$ and $p \in U$, the diagram

$$\begin{tikzcd}
T_x^{\mathrm{geo}} U \arrow[r, "DF_x^{\mathrm{geo}}"] \arrow[d, "D^U"', "\cong"] & T_{F(p)}^{\mathrm{geo}} V \arrow[d, "D^V"', "\cong"] \\
\mathbb{R}^m \arrow[r, "dF_x"] & \mathbb{R}^n
\end{tikzcd}$$

is commutative. We note that this expresses the compatibility of the Jacobian dF_x (the traditional calculus definition of the derivative of the map F) and DF_x^{geo} (the new definition of the derivative, which generalizes to manifolds).

Proof. Let $[\gamma] \in T_p^{\mathrm{geo}} U$. Then

$$D^V(F_*([\gamma])) = D^V([F \circ \gamma]) = (F \circ \gamma)'(0) = dF_{\gamma(0)}(\gamma'(0)) = dF_x(D^U([\gamma])).$$

Here the third equality is consequence of the chain rule (the ordinary calculus version). □

9 Homework Assignment # 9

1. (10 points) Let M, N be smooth manifolds, and let $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ be the projection maps. Show that for any $(x, y) \in M \times N$ the map

$$\alpha \colon T_{(x,y)}(M \times N) \longrightarrow T_x M \oplus T_y N$$

defined by
$$\alpha(v) = (D\pi_1(v), D\pi_2(v))$$
is an isomorphism. Here we suppress the subscripts of the differentials that indicate the point of the domain, i.e., we write $D\pi_1$ instead of $(D\pi_1)_{x,y}$. Hint: To prove this, it is unnecessary to “unpack” the definition of the tangent space of manifolds by using either the geometric or algebraic definition. Rather, only the functorial properties of the tangent space, i.e., the chain rule, is needed, applied to suitable projection/inclusion maps. Remark: Using this isomorphism, we will routinely identify T_xM and T_yN with subspaces of $T_{(x,y)}(M \times N)$.

Proof. For $\ell = 1, 2$ let $i_\ell \colon M_\ell \to M_1 \times M_2$ be the inclusion map given by $i_1(x_1) = (x_1, p_2)$, $i_2(x_2) = (p_1, x_2)$. Then for $\ell, j = 1, 2$ the composition
$$M_\ell \xrightarrow{i_\ell} M_1 \times M_2 \xrightarrow{\pi_j} M_j$$
is the identity map for $\ell = j$, and the constant map $x_\ell \mapsto p_j$ otherwise. Differentiating it, it follows that the composition
$$T_{p_\ell}M_\ell \xrightarrow{Di_\ell} T_{(p_1,p_2)}(M_1 \times M_2) \xrightarrow{D\pi_j} T_{p_j}M_j$$
is the identity map for $\ell = j$ and the trivial map otherwise. It follows that the composition
$$T_{p_\ell}M_\ell \xrightarrow{Di_\ell} T_{(p_1,p_2)}(M_1 \times M_2) \xrightarrow{\alpha} T_{p_1}M_1 \oplus T_{p_2}M_2$$
sends a vector $X_\ell \in T_{p_\ell}M_\ell$ to $(X_1, 0)$ for $\ell = 1$ and $(0, X_2)$ for $\ell = 2$. Since every pair (X_1, X_2) can be written as a sum of pairs of this type, the map α is surjective. This implies that α is an isomorphism, since the dimension of domain and range is both $\dim M_1 + \dim M_2$. □

2. (10 points) Let M be a smooth n manifold. For a point $p \in M$ let
$$D^M \colon T_p^{\text{geo}}M \longrightarrow T_p^{\text{alg}}M = \text{Der}(C_p^\infty(M), \mathbb{R})$$
be the map that sends $[\gamma] \in T_p^{\text{geo}}M$ to the derivation D_γ. More explicitly, if f is (the germ of) a function $f \colon M \to \mathbb{R}$ then $D_\gamma f \in \mathbb{R}$ is the directional derivative of f in the direction of γ defined by
$$D_\gamma f := (f \circ \gamma)'(0) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(p)}{t}.$$

(a) Show that the geometric and the algebraic definition of the differential of a smooth map $F \colon M \to N$ are compatible in the sense that for $p \in M$ the following diagram is commutative:
$$\begin{array}{ccc} T_p^{\text{geo}}M & \xrightarrow{DF_p^{\text{geo}}} & T_{F(p)}^{\text{geo}}N \\ {\scriptstyle D^M}\downarrow & & \downarrow{\scriptstyle D^N} \\ T_p^{\text{alg}}M & \xrightarrow{DF_p^{\text{alg}}} & T_{F(p)}^{\text{alg}}N \end{array}$$

(b) Show that the map D^M is a bijection for any manifold M. Hint: use a chart for M and part (a) to reduce to the case of open subsets $U \subset \mathbb{R}^n$; we have proved in class that the map $D^U \colon T_p^{\text{geo}} U \to T_p^{\text{alg}} U$ is a bijection in that case.

Proof. **Part (a).** To calculate $(DF_p^{\text{alg}} \circ D^M)([\gamma]) \in T_{F(p)}^{\text{alg}} N$, we evaluate it on $g \in C_{F(p)}^\infty(N)$. Since DF_p^{alg} is defined by precomposing with F^*, we obtain

$$\begin{aligned}((DF_p^{\text{alg}} \circ D^M)([\gamma]))(g) =&(DF_p^{\text{alg}}(D^M([\gamma])))(g) = (D^M([\gamma]))(F^*g)\\ =&(F^*g \circ \gamma)'(0) = (g \circ F \circ \gamma)'(0)\end{aligned}$$

$$((D^N \circ DF_p^{\text{geo}})([\gamma]))(g) = (D^N([F \circ \gamma])(g) = (g \circ (F \circ \gamma))'(0)$$

This shows that the diagram is commutative.

Part (b). Let (U, ϕ) be a smooth chart for M with $p \in U$, i.e., the chart (U, ϕ) is one of the charts of the maximal smooth atlas of the smooth manifold M. Unwinding this we have smooth maps

$$M \xhookleftarrow{\;i\;} U \xrightarrow[\cong]{\;\phi\;} V \subset \mathbb{R}^n.$$

The map ϕ is a diffeomorphism and hence its differential ϕ_* is a bijection. The inclusion map i of an open subset of a manifold induces a bijection on tangent spaces. So we have the following sequence of bijections of tangent spaces (of the geometric or the algebraic flavor):

$$T_pM \xleftarrow[\cong]{Di_p} T_pU \xrightarrow[\cong]{D\phi_p} T_{\phi(p)}V$$

By part (a) the isomorphisms between the geometric and algebraic flavor of tangent spaces is compatible with induced maps and hence we have a commutative diagram

$$\begin{tikzcd} T_p^{\text{geo}} M \arrow[d, "D^M"'] & T_p^{\text{geo}} U \arrow[l, "Di_p^{\text{geo}}"', "\cong"] \arrow[r, "D\phi_p^{\text{geo}}", "\cong"'] \arrow[d, "D^U"'] & T_{\phi(p)}^{\text{geo}} V \arrow[d, "D^V"', "\cong"] \\ T_p^{\text{alg}} M & T_p^{\text{alg}} U \arrow[l, "Di_p^{\text{alg}}"', "\cong"] \arrow[r, "D\phi_p^{\text{alg}}", "\cong"'] & T_{\phi(p)}^{\text{alg}} V \end{tikzcd}$$

By a lemma from class the right hand side vertical map D^V is a bijection. The commutativity of the diagram and the fact that all horizontal maps are bijections then implies that the left vertical map D^M is a bijection. $\square$

3. (10 points) Let M, N be smooth manifolds, let $F \colon M \to N$ be a smooth map, and $p \in M$.

(a) Show that the map

$$F_p^*\colon C^\infty_{F(p)}(N) \longrightarrow C^\infty_p(M) \qquad \text{given by} \qquad [f] \mapsto [f \circ F]$$

is a well-defined algebra homomorphism.

(b) Show that if $D\colon C^\infty_p(M) \to \mathbb{R}$ is a derivation, then the composition

$$C^\infty_{F(p)}(N) \xrightarrow{F_p^*} C^\infty_p(M) \xrightarrow{D} \mathbb{R}$$

is a derivation. In particular, we can define the (algebraic) differential

$$D^{\mathrm{alg}}F_p\colon T_p^{\mathrm{alg}}M = \mathrm{Der}(C^\infty_p(M),\mathbb{R}) \;\longrightarrow\; T^{\mathrm{alg}}_{F(p)}N = \mathrm{Der}(C^\infty_{F(p)}(M),\mathbb{R}) \tag{9.1}$$

by $DF_p^{\mathrm{alg}}(D) := D \circ F_p^*$.

(c) Show that the differential DF_p^{alg} is a linear map.

(d) If $G\colon N \to Q$ is a smooth map, show that

$$D(G\circ F)^{\mathrm{alg}}_p = DG^{\mathrm{alg}}_{F(p)} \circ DF_p^{\mathrm{alg}}.$$

We note that this statement is the *chain rule* (for the algebraic construction of the tangent space).

Proof. **Part (a).** Let $f, f' \in C^\infty(N)$ represent the same element in the vector space $C^\infty_q(N)$, $q = F(p)$. That means that there is an open neighborhood $V \subset N$ of q such that $f_{|V} = f'_{|V}$. This implies that $(F\circ f)(x) = (F\circ f')(x)$ for $x \in F^{-1}(V)$, which is an open neighborhood of p by continuity of F. In particular, $[F\circ f] = [F \circ f'] \in C^\infty_q(M)$, which shows that $[f] \mapsto [F\circ f]$ is a well-defined map $F_p^*\colon C^\infty_{F(p)}(N) \to C^\infty_p(M)$.

To show that F_p^* is an algebra homomorphism we have to show that it is compatible with scalar multiplication, addition and multiplication, which is verified in the following computations for $f, g \in C^\infty(N)$, $c \in \mathbb{R}$.

$$\begin{aligned}
F_p^*(c[f]) &= F_p^*([cf]) = [cf\circ F] = c[f\circ F] = cF_p^*([f])\\
F_p^*([f]+[g]) &= F_p^*([f+g]) = [(f+g)\circ F] = [(f\circ F)+(g\circ F)]\\
&= [f\circ F] + [g\circ F] = F_p^*([f]) + F_p^*([g])\\
F_p^*([f]\cdot[g]) &= F_p^*([f\cdot g]) = [(f\cdot g)\circ F] = [(f\circ F)\cdot(g\circ F)]\\
&= [f\circ F]\cdot[g\circ F] = F_p^*([f])\cdot F_p^*([g])
\end{aligned}$$

Part (b). To show that $D \circ F^*$ is a derivation, we need to show that it is a linear map and that it satisfies the product property. The linearity is clear, since D and F^* are linear maps. To check the product property, let $f, g \in C^\infty_{F(p)}(N)$. Then

$$\begin{aligned} D \circ F_p^*(fg) =& D((F_p^* f)(F_p^* g)) = D(F_p^* f)(F_p^* g)(p) + (F_p^* f)(p) D(F_p^* g) \\ =& (D \circ F_p^*)(f) g(F(p)) + f(F(p)(D \circ F_p^*)(g), \end{aligned}$$

which shows that $D \circ F_p^*$ is a derivation $C^\infty_{F(p)}(N) \to \mathbb{R}$.

Part (c). For $D, D' \in T_p^{\mathrm{alg}} M = \mathrm{Der}(C_p^\infty(M), \mathbb{R})$ we have

$$DF_p^{\mathrm{alg}}(D + D') = (D + D') \circ F_p^* = D \circ F_p^* + D' \circ F_p^* = DF_p^{\mathrm{alg}}(D) + DF_p^{\mathrm{alg}}(D').$$

A similar calculation shows $DF_p^{\mathrm{alg}}(cD) = cDF_p^{\mathrm{alg}}(D)$ for $c \in \mathbb{R}$, and hence DF_p^{alg} is a linear map.

Part (d). Since $(G \circ F)_*^{\mathrm{alg}}$ is defined in terms of the induced map

$$(G \circ F)_p^* \colon C^\infty_{(G \circ F)(p)}(Q) \longrightarrow C_p^\infty(M),$$

we first express $(G \circ F)_p^*$ in terms of $G^*_{F(p)}$ and F_p^*. We note that for $f \in C^\infty(Q)$

$$(G \circ F)^*(f) = f \circ (G \circ f) = (f \circ G) \circ F = G^* f \circ F = F^*(G^*(f)) = (F^* \circ G^*)(f).$$

If $Q \supset V \xrightarrow{f} \mathbb{R}$ represents an element of $C^\infty_{(G \circ F)(p)}(Q)$, then $[(G \circ F)^* f]_p = [F^*(G^* f)]_p \in C_p^\infty(M)$ and hence

$$(G \circ F)_p^*[f]_{(G \circ F)(p)} = [(G \circ F)^* f]_p = [F^*(G^* f)]_p = F_p^*([G^* f]_{F(p)}) = F_p^*(G^*_{F(p)}[f]_p.$$

Let $D \in T_p^{\mathrm{alg}} M = \mathrm{Der}(C_p^\infty(M), \mathbb{R})$. Then

$$\begin{aligned} (G \circ F)_*^{\mathrm{alg}}(D) =& D \circ (G \circ F)^* = (D \circ F^*) \circ G^* \\ =& (F_*^{\mathrm{alg}}(D)) \circ G^* = G_*^{\mathrm{alg}}(F_*^{\mathrm{alg}}(D)). \end{aligned}$$

$\square$

4. (10 points) Let $M_{n \times k}(\mathbb{R})$ be the vector space of $n \times k$-matrices. For $A \in M_{n \times k}(\mathbb{R})$ let $A^t \in M_{k \times n}(\mathbb{R})$ be the transpose of A, and let $\mathsf{Sym}(\mathbb{R}^k) = \{B \in M_{k \times k}(\mathbb{R}) \mid B^t = B\}$ be the vector space of *symmetric* $k \times k$*-matrices.*

(a) Show that the map $\Phi \colon M_{n \times k}(\mathbb{R}) \to \mathsf{Sym}(\mathbb{R}^k)$, $A \mapsto A^t A$ is smooth, and that its differential

$$D\Phi_A \colon T_A M_{n \times k}(\mathbb{R}) = M_{n \times k}(\mathbb{R}) \longrightarrow T_{\Phi(A)} \mathsf{Sym}(\mathbb{R}^k) = \mathsf{Sym}(\mathbb{R}^k)$$

is given by $D\Phi_A(C) = C^t A + A^t C$. Hint: Use the geometric description of tangent spaces. More explicitly, the tangent space $T_A^{\mathrm{geo}} M_{n \times k}(\mathbb{R})$ can be identified with $M_{n \times k}(\mathbb{R})$ by sending a matrix $C \in M_{n \times k}(\mathbb{R})$ to the path $\gamma(t) := A + tC$.

(b) Show that the identity matrix is a regular value of the map Φ. This implies in particular that the level set Φ^{-1}(identity matrix) is a smooth manifold. We recall that we showed in class that Φ^{-1}(identity matrix) is the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$. Hint: to show that $D\Phi_A\colon T_A M_{n\times n}(\mathbb{R}) \to T_e\mathsf{Sym}(\mathbb{R}^k)$ is surjective for $e =$ identity matrix, compute $D\Phi_A(C)$ for $C = AB$ for $B \in \mathsf{Sym}(\mathbb{R}^k)$.

(c) What is the dimension of $V_k(\mathbb{R}^n)$?

We remark that identifying $M_{n\times k}(\mathbb{R})$ in the usual way with the vector space $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n)$ of linear maps $f\colon \mathbb{R}^k \to \mathbb{R}^n$, a matrix belongs to $V_k(\mathbb{R}^n)$ if and only if the corresponding linear map f is an *isometry*, that is, if f preserves the length of vectors in the sense that $||f(v)|| = ||v||$, or equivalently, if f preserves the scalar product in the sense that

$$\langle f(v), f(w)\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in \mathbb{R}^k.$$

The manifold $V_k(\mathbb{R}^n)$ is called the *Stiefel manifold*. We observe that $V_n(\mathbb{R}^n)$ is the orthogonal group $O(n)$ of isometries $\mathbb{R}^n \to \mathbb{R}^n$.

Proof. **Part (a).** The entries of the matrix A^tA are quadratic polynomials in the entries of A and hence the map Φ is smooth. As suggested in the hint, we will calculate the differential of Φ using the geometric version of tangent spaces described in terms of equivalence classes of paths. So let $C \in M_{n\times n}(\mathbb{R})$ and interpret C as the tangent vector to $A \in M_{n\times n}(\mathbb{R})$ represented by the linear path $\gamma(s) := A + sC$.

To calculate

$$D\Phi_A([\gamma]) = \frac{d\Phi(\gamma(s))}{ds}(0) = \frac{d\Phi(A+sC)}{ds}(0)$$

we evaluate

$$\Phi(A+sC) = (A+sC)^t(A+sC) = A^tA + s(C^tA + A^tC) + s^2C^tC$$

and hence

$$\frac{\partial\Phi(A+sC)}{\partial s}(0) = C^tA + A^tC.$$

This proves that $\Phi_*(C) = C^tA + A^tC$.

Part (b). To show that for $A \in \Phi^{-1}(I)$ ($I \in M_{k\times k}(\mathbb{R})$ is the identity matrix) the differential

$$\Phi_*\colon M_{n\times k}(\mathbb{R}) \to \mathsf{Sym}(\mathbb{R}^k)$$

is surjective, let $B \in \mathsf{Sym}(\mathbb{R}^k)$ and let $C = AB \in M_{n\times k}(\mathbb{R})$. Then

$$\Phi_*(C) = C^tA + A^tC = (AB)^tA + A^t(AB) = B^tA^tA + A^tAB = B^t + B = 2B,$$

which proves surjectivity.

Part (c). According to a theorem proved in class, if $F\colon N \to Q$ is a smooth map, then the preimage $F^{-1}(c)$ of a regular value $c \in Q$ is a manifold of dimension $\dim N - \dim Q$. Part (b) show that this theorem is applicable in our situation and hence $V_k(\mathbb{R}^n)$ is a manifold of dimension

$$\dim M_{n\times k}(\mathbb{R}) - \dim \mathsf{Sym}(\mathbb{R}^k)$$

To calculate the dimensions of these vector spaces, we exhibit bases for them. A basis of $M_{n\times k}(\mathbb{R})$ is provided by the matrices $\{A(i,j)\}$, $1 \le i \le n$, $1 \le j \le k$, whose entries are all zero except the ij-th entry which is 1. There are nk matrices $A(i,j)$ and hence $\dim M_{n\times n}(\mathbb{R}) = nk$.

Similarly, a basis for the symmetric $k \times k$ matrices is given by the matrices $\{B(i,j)\}$, $1 \le i \le j \le k$; here $B(i,j)$ is the matrix whose entries are zero except the ij-th entry and the ji-th entry which are both 1 (obviously, these are the same entries for $i = j$). For a fixed j between 1 and n, the number of matrices $B(i,j)$ with $1 \le k \le j$ is j. It follows that

$$\dim \mathsf{Sym}(\mathbb{R}^k) = 1 + 2 + 3 + \cdots + k = k(k+1)/2$$

Putting these statements together, we conclude

$$\dim V_k(\mathbb{R}^n) = \dim M_{n\times k}(\mathbb{R}) - \dim \mathsf{Sym}(\mathbb{R}^k) = nk - \frac{k(k+1)}{2}$$

$\square$

5. (10 points) Recall that the special linear group $SL_n(\mathbb{R})$ and the orthogonal group $O(n)$ are both submanifolds of the vector space $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices. In particular, the tangent spaces $T_A SL_n(\mathbb{R})$ for $A \in SL_n(\mathbb{R})$ and $T_A O(n)$ for $A \in O(n)$ are subspaces of the tangent space $T_A M_{n\times n}(\mathbb{R})$, which can be identified with $M_{n\times n}(\mathbb{R})$, since $M_{n\times n}(\mathbb{R})$ is a vector space.

(a) Show that $T_e SL_n(\mathbb{R}) = \{C \in M_{n\times n} \mid \mathrm{tr}(C) = 0\}$, where e is the identity matrix, and $\mathrm{tr}(C)$ denotes the trace of the matrix C.

(b) Show that $T_e O(n) = \{C \in M_{n\times n} \mid C^t = -C\}$.

Hint for parts (a) and (b): $SL_n(\mathbb{R})$ and $O(n)$ can be both be described as level sets $F^{-1}(c)$ of a regular value c for a suitable smooth map F (as we did in class for $SL_n(\mathbb{R})$ and you did for $O(n)$ in problem 4 of this homework assignment; note that $O(n)$ is equal to the Stiefel manifold $V_n(\mathbb{R}^n)$).

Remark: A *Lie group* is a group G which also is a smooth manifold and these structures are compatible in the sense that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are smooth maps. The tangent space $T_e G$ at the identity element $e \in G$ is called the *Lie algebra of* G. In other words, this problem asks you to calculate the Lie algebra for the Lie groups $SL_n(\mathbb{R})$ resp. $O(n)$.

Proof. **Part (a).** The special linear group $SL_n(\mathbb{R})$ can be described as the level set $\det^{-1}(1)$ of the determinant function $\det\colon \mathbb{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$. In class we proved that $1 \in \mathbb{R}$ is a regular value of det, and hence hence $SL_n(\mathbb{R})$ is a submanifold of dimension $n-1$ of $M_{n\times n}(\mathbb{R})$. Moreover, the tangent space at any $A \in SL_n(\mathbb{R})$ according to our general theorem in class is given by the kernel of the differential

$$D\det_A\colon T_A M_{n\times n}(\mathbb{R}) = M_{n\times n}(\mathbb{R}) \longrightarrow T_{\det(A)}\mathbb{R} = \mathbb{R}.$$

Also in class we calculated $D\det_A(C)$ for $C \in \mathbb{M}_{n\times n}(\mathbb{R})$ and found

$$D\det_A(C) = \det(c_1, a_2, \ldots, a_n) + \det(a_1, c_2, \ldots, a_n) + \cdots + \det(a_1, a_2, \ldots, c_n),$$

where $a_i \in \mathbb{R}^n$ are the column vectors of A, and similarly for c_i. In particular, if $A = e$, the identity matrix, then $a_i \in \mathbb{R}^n$ is the i-th standard basis vector, all of whose entries are 0, except the i-th which is 1. It follows that the determinants above greatly simplify and we obtain

$$D\det_e(C) = c_{11} + c_{22} + \cdots + c_{nn} = \operatorname{tr}(C)$$

where c_{ij} are the coefficients of C. This implies the statement of part (a).

Part (b). Using the same strategy as in part (a) and the calculation of $D\Phi_A$ from the previous problem for $k = n$ we find

$$\begin{aligned} T_e O(n) =& \ker\left(D\Phi_e\colon T_e M_{n\times n}(\mathbb{R}) \to T_e\mathsf{Sym}(\mathbb{R}^n)\right) \\ =& \{C \in M_{n\times n}(\mathbb{R}) \mid C^t + C\} \\ =& \{C \in M_{n\times n}(\mathbb{R}) \mid C^t = -C\}. \end{aligned}$$

□

10 Homework Assignment # 10

1. (10 points) Let M be a smooth manifold of dimension n. If $f\colon M \to \mathbb{R}$ is a smooth function, then for $p \in M$ its differential

$$df_p\colon T_pM \longrightarrow T_{f(p)}\mathbb{R} = \mathbb{R}$$

is an element of $\operatorname{Hom}(T_pM, \mathbb{R})$. This vector space dual to the tangent space T_pM is called the *cotangent space*, and is denoted T_p^*M.

(a) Let $x^i\colon \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate function, which maps $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to $x_i \in \mathbb{R}$. Show that for any point $q \in \mathbb{R}^n$ a basis of the cotangent space $T_q^*\mathbb{R}^n$ is given by $\{dx_q^i\}_{i=1,\ldots,n}$.

(b) If $M \supset U \xrightarrow{\phi} V \subset \mathbb{R}^n$ is a smooth chart of M, the component functions of ϕ, given by $y^i := x^i \circ \phi$ are called *local coordinates*. Show that for $p \in U$, a basis of the cotangent space T_p^*M is given by $\{dy_p^i\}_{i=1,\dots,n}$.

Hint for part (b): let $(D\phi_p)^*\colon T_q^*\mathbb{R}^n \to T_p^*M$, $q = \phi(p)$ be the linear map dual to the differential $D\phi_p\colon T_pM \to T_q\mathbb{R}^n$ defined by

$$(D\phi_p)^*(\xi)(v) = \xi(D\phi_p(v)) \qquad \text{for } \xi \in T_q^*\mathbb{R}^n \text{ and } v \in T_pM.$$

Show first that $(D\phi_p)^*(dx_q^i) = dy_p^i$.

Proof. **Part (a).** We claim that the standard basis $\{e_i\}_{i=1,\dots,n}$ of $\mathbb{R}^n = T_q\mathbb{R}^n$ is dual to $\{dx_q^i\}_{i=1,\dots,n}$ in the sense that $dx_q^i(e_j) = \delta_{ij}$. In particular, $\{dx_q^i\}_{i=1,\dots,n}$ is a basis of $T_q^*\mathbb{R}^n = \mathrm{Hom}(T_q\mathbb{R}^n, \mathbb{R})$.

To prove the claim, we recall that for $f \in C^\infty(\mathbb{R}^n)$ and $v \in T_q\mathbb{R}^n = \mathbb{R}^n$

$$df_q(v) = \langle (\mathrm{grad} f)_q, v\rangle$$

and hence

$$dx_q^i(e_j) = \langle (\mathrm{grad}\ x^i)_q, e_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}.$$

Part (b). The differential $D\phi_p\colon T_pM \to T_q\mathbb{R}^n$ is an isomorphism since the smooth chart $M \supset U \xrightarrow{\phi} V \subset \mathbb{R}^n$ is a diffeomorphism. It follows that its dual $(D\phi_p)^*\colon T_q^*\mathbb{R}^n \to T_p^*M$ is an isomorphism as well. We claim that $(D\phi_p)^*(dx_q^i) = dy_q^i$. This shows that $\{dy_q^i\}_{i=1,\dots,n}$ is a basis, since it is the image of the basis $\{dx_p^i\}_{i=1,\dots,n}$ under the isomorphisms ϕ^*.

To prove the claim we calculate for $v \in T_pM$:

$$((D\phi_p)^*(dx_q^i))(v) = dx_q^i(D\phi_p(v)) = ((dx^i)_q \circ D\phi_p)(v) = d(x^i \circ \phi)_p(v) = dy^i(v).$$

□

2. (10 points) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Show that the differential df is a smooth section of the cotangent bundle T^*M. Hint: smoothness of a section s is a local property and hence to check smoothness it suffices to check that the composition $\Phi_\alpha \circ s$ is smooth for local trivializations Φ_α of the cotangent bundle T^*M.

Proof. Following the hint, to check smoothness of the section df of the cotangent bundle T^*M, it suffices to check smoothness of the composition of df with the local trivializations of T^*M. We recall the construction of the local trivializations of T^*M. Let

$$M \underset{\text{open}}{\supseteq} U \xrightarrow{\phi} V \underset{\text{open}}{\subseteq} \mathbb{R}^n$$

be a smooth chart for M, and let

$$TM_{|U} \xrightarrow{\Phi} U \times \mathbb{R}^n \qquad (p, v) \mapsto (p, D\phi_p(v))$$

be the associated local trivialization of the tangent bundle, where $D\phi_p \colon T_pM \to T_{\phi(p)}V = \mathbb{R}^n$ is the differential of ϕ at the point $p \in U$. Then the corresponding local trivialization for T^*M is the map $T^*M_{|U} \xrightarrow{\Psi} U \times (\mathbb{R}^n)^* = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ given by

$$(p, \xi) \mapsto (p, ((D\phi_p)^{-1})^*(\xi)) = (p, \xi \circ (D\phi_p)^{-1}) = (p, \xi \circ (D\phi^{-1})_{\phi(p)}).$$

Here $(D\phi_p)^{-1} \colon \mathbb{R}^n \to T_pM$ is the inverse of the linear isomorphism $D\phi_p$, which by the functoriality of the differential (i.e., the chain rule) can be written as $(D\phi^{-1})_{\phi(p)}$.

To show that $df \colon U \to T^*M_{|U}$, $p \mapsto T_pf \in T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R})$ is smooth it suffices to show that the composition

$$\begin{array}{ccccc} U & \xrightarrow{df} & T^*M_U & \xrightarrow{\quad\Psi\quad} & U \times (\mathbb{R}^n)^* \\ \\ p & \longmapsto & (p, df_p) & \longmapsto & (p, df_p \circ (D\phi^{-1})_{\phi(p)}) \end{array}$$

is smooth. We note that $df_p \circ (D\phi^{-1})_{\phi(p)} = D(f \circ \phi^{-1})_{\phi(p)}$, where the last equation is by functoriality of the differential, i.e., the chain rule.

The function $g := f \circ \phi^{-1} \colon V \to \mathbb{R}$ is smooth as the composition of smooth maps. Its differential $dg \colon V \to (\mathbb{R}^n)^*$, $x \mapsto dg_x$ is smooth, since a short calculation shows that $dg_x(e_i) = \frac{\partial f}{\partial x^i}(x)$ where $\{e_i\}_{i=1,\dots,n}$ is the standard basis of $\mathbb{R}^n$. Hence

$$D(f \circ \phi^{-1})_{\phi(p)} = Dg_{\phi(p)},$$

the second component of the map $\Phi \circ df$ is a smooth function of p (as the composition of the smooth maps $\phi \colon U \to V$ and $dg \colon V \to (\mathbb{R}^n)^*$). This shows that df is a smooth section of the cotangent bundle T^*M. $\square$

3. (10 points) We recall that the projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n whose underlying set is the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. In particular, each point $p \in \mathbb{RP}^n$ determines tautologically a 1-dimensional subspace $E_p \subset \mathbb{R}^{n+1}$. Let E be the disjoint union $E = \amalg_{p \in \mathbb{RP}^n} E_p$ of the vector spaces E_p. More explicitly,

$$E = \{([x], v) \mid [x] \in \mathbb{RP}^n, v \in \langle x \rangle\},$$

where $x \in \mathbb{R}^{n+1} \setminus \{0\}$, $\langle x \rangle \subset \mathbb{R}^{n+1}$ is the one-dimensional subspace spanned by x, and $[x] \in \mathbb{RP}^n$ is the corresponding point in the projective space.

(a) Use the Vector Bundle Construction Lemma to show that E is a smooth vector bundle of rank 1 over $\mathbb{RP}^n$ (which is called the *tautological line bundle over* $\mathbb{RP}^n$; *line bundle* is a synonym for *vector bundle of rank* 1). Hint: Construct local trivializations of E restricted to $U_i = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$.

(b) Show that the complement of the zero section in E is diffeomorphic to $\mathbb{R}^{n+1} \setminus \{0\}$.

(c) Show that the line bundle E is not isomorphic to the trivial line bundle. Hint: consider the complement of the zero-section of E and compare it with the complement of the zero-section of the trivial line bundle.

Proof. **Part (a).** Let $\Phi_i \colon E_{|U_i} \to U_i \times \mathbb{R}$ be the map defined by $\Phi_i([x], v) = ([x], v_i)$, where $v = (v_0, \dots, v_n) \in \mathbb{R}^{n+1}$. Evidently, Φ_i commutes with the projection maps to $\mathbb{RP}^n$ and restricts to a vector space isomorphism on the fibers. It is easy to check that the inverse is given explicitly by $\Phi_i^{-1}([x], t) = ([x], t\frac{x}{x_i})$. To use the Vector Bundle Construction Lemma we just need to check the transition maps

$$(U_i \cap U_j) \times \mathbb{R} \xrightarrow{\Phi_i^{-1}} E_{|U_i \cap U_j} \xrightarrow{\Phi_j} (U_i \cap U_j) \times \mathbb{R}$$

are smooth. Explicitly, that composition is given by

$$(\Phi_j \circ \Phi_i^{-1})([x], t) = \Phi_j([x], t\frac{x}{x_i}) = ([x], t\frac{x_j}{x_i}).$$

Hence it suffices to show that the map $f_{ij} \colon U_i \cap U_j \to \mathbb{R}$ given by $[x] \mapsto \frac{x_j}{x_i}$ is smooth. Since the domain is an open subset of the smooth manifold $\mathbb{RP}^n$, we need to use smooth charts for $\mathbb{RP}^n$ to check smoothness of this map. We recall that a smooth atlas for $\mathbb{RP}^n$ is given by $\{(U_i, \phi_i)\}_{i=0,\dots,n}$ where

$$\mathbb{RP}^n \supset U_i \xrightarrow[\approx]{\phi_i} \mathbb{R}^n \qquad \text{is given by} \qquad [x] \mapsto (\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}),$$

where the hat over the term $\frac{x_i}{x_i}$ means to skip it. The inverse

$$\phi_i^{-1} \colon \mathbb{R}^n \longrightarrow U_i \qquad \text{is given explicitly by} \qquad \phi_i^{-1}(y_1, \dots, y_n) = [y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n].$$

It follows that

$$(f_{ij} \circ \phi_i^{-1})(y_1, \dots, y_n) = f_{ij}([y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n] = \begin{cases} y_j & j > i \\ y_{j+1} & j < i \end{cases}$$

This shows that $f_{ij} \circ \phi_i^{-1}$ and hence f_{ij} are smooth maps. Hence the transition maps $\Phi_j \circ \Phi_i^{-1}$ are smooth, allowing us to conclude that E is a smooth vector bundle.

Part (b). The zero-section is the section $s\colon \mathbb{RP}^n \to E$ given by $s([x]) = ([x], 0)$. Let $Z \subset E$ be the image of the zero-section, which slightly abusively is also often called the zero-section. Then the map

$$E \setminus Z = \{([x], v) \mid [x] \in \mathbb{RP}^n,\ v \in \langle x \rangle,\ v \neq 0\} \quad \xrightarrow{F} \mathbb{R}^{n+1} \setminus \{0\}$$

given by $F([x], v) = v$ is a bijection with inverse given by $F^{-1}(v) = ([v], v)$. So we need to show that F and F^{-1} are smooth. We note that $\mathbb{R}^{n+1} \setminus \{0\}$ obtains its smooth structure as open subset of $\mathbb{R}^{n+1}$, while the smooth structure on $E \setminus Z$ is obtained as an open subset of the smooth manifold E. So we can check smoothness of these maps by using smooth charts for E. These are obtained by combining the local trivializations for the vector bundle E constructed in part (a) with the smooth charts (U_i, ϕ_i) for the base space $\mathbb{RP}^n$; the composition

$$\psi_i\colon E_{|U_i} \xrightarrow{\Phi_i} U_i \times \mathbb{R} \xrightarrow{\phi_i \times \mathrm{id}_{\mathbb{R}}} \mathbb{R}^n \times \mathbb{R}$$

is then a smooth chart for E. Explicitly,

$$\begin{aligned} \psi_i([x], v) =& (\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}); v_i) \\ \psi_i^{-1}(y_1, \dots, y_n; t) =& ([y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n], t(y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n)). \end{aligned}$$

We note that the diffeomorphism ψ_i restricts to a diffeomorphism $\psi_i\colon (E \setminus Z)_{|U_i} \cong \mathbb{R}^n \times \mathbb{R}^\times$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

Hence to prove smoothness of F^{-1} and F is suffices to show that the following compositions are smooth for $i = 0, \dots, n$:

$$\mathbb{R}^n \times \mathbb{R}^\times \xrightarrow[\cong]{\psi_i^{-1}} (E \setminus Z)_{|U_i} \xrightarrow{F} \mathbb{R}^{n+1} \setminus \{0\}$$

$$\{v \in \mathbb{R}^{n+1} \mid v_i \neq 0\} \xrightarrow{F^{-1}} (E \setminus Z)_{|U_i} \xrightarrow[\cong]{\psi_i} \mathbb{R}^n \times \mathbb{R}^\times$$

Explicitly,

$$\begin{aligned} F\psi_i^{-1}(y_1, \dots, y_n; t) =& F([y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n], t(y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n) \\ =& (ty_1, \dots, ty_i, t, ty_{i+1}, \dots, ty_n) \end{aligned}$$

and

$$\psi F^{-1}(v_0, \dots, v_n) = \psi_i([v], v) = (\frac{v_0}{v_i}, \dots, \frac{\widehat{v_i}}{v_i}, \dots, \frac{v_n}{v_i}; v_i)$$

These are smooth maps.

Part (c). If there were a vector bundle isomorphism Φ between $E \to \mathbb{RP}^n$ and the trivial line bundle $\mathbb{RP}^n \times \mathbb{R} \to \mathbb{RP}^n$, then Φ would map the zero-section $Z \subset E$ to the zero section $\mathbb{RP}^n \times \{0\} \subset \mathbb{RP}^n \times \mathbb{R}$ (since the restriction of Φ to the fibers is a linear isomorphism). Hence Φ would restrict to a homeomorphism

$$\Phi_{|E \setminus Z} \colon E \setminus Z \xrightarrow{\approx} (\mathbb{RP}^n \times \mathbb{R}) \setminus (\mathbb{RP}^n \times \{0\}) = \mathbb{RP}^n \times (\mathbb{R} \setminus \{0\}).$$

This is the desired contradiction, since by part (b), $E \setminus Z$ is homeomorphic to $\mathbb{R}^{n+1} \setminus \{0\}$, which is connected, while $\mathbb{RP}^n \times (\mathbb{R} \setminus \{0\})$ is not connected. □

4. (10 points) The goal of this problem is to prove the following construction lemma for vector bundles over topological spaces.

Lemma 10.1. *Let M be a topological space, and let $\{E_p\}$ be a collection of vector spaces parametrized by $p \in M$. Let E be the set given by the disjoint union of all these vector spaces, which we write as*

$$E := \coprod_{p \in M} E_p = \{(p, v) \mid p \in M,\ v \in E_p\}$$

and let $\pi \colon E \to M$ be the projection map defined by $\pi(p, v) = p$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M, and let for each $\alpha \in A$, let $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ be maps with the following properties

(i) *The diagram*

$$\begin{tikzcd} E_{|U_\alpha} := \pi^{-1}(U_\alpha) \arrow[rr, "\Phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd} \tag{10.2}$$

is commutative, where π_1 is the projection onto the first factor.

(ii) *For each $p \in U_\alpha$, the restriction of Φ_α to $E_p = \pi^{-1}(p)$ is a vector space isomorphism between E_p and $\{p\} \times \mathbb{R}^k = \mathbb{R}^k$ (which implies that Φ_α is a bijection).*

(iii) *For $\alpha, \beta \in A$, the composition*

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\Phi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\Phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^k \tag{10.3}$$

is continuous.

Then the total space E can be equipped with a topology such that $\pi \colon E \to M$ is a topological vector bundle of rank k with local trivializations Φ_α.

(a) Construct a topology on E by declaring $U \subset E$ to be *open* if $\Phi_\alpha(U \cap E_{|U_\alpha})$ is an open subset of $U_\alpha \times \mathbb{R}^k$ for all $\alpha \in A$. Show that this satisfies the conditions for a topology.

(b) Show that with this topology on E the projection map $\pi\colon E \to M$ is continuous

(c) Show that the map Φ_α is a homeomorphism (for the subspace topology on $E_{|U_\alpha}$). That implies that (U_α, Φ_α) is a bundle atlas for $\pi\colon E \to M$ and which then finishes the proof of the Lemma.

Proof. **Part (a).** We first check the this does indeed define a topology:

- $\Phi_\alpha(E \cap E_{|U_\alpha}) = \Phi_\alpha(E_{|U_\alpha}) = U_\alpha \times \mathbb{R}^k$, since Φ_α is a bijection. This is an open subset of $U_\alpha \times \mathbb{R}^k$, and hence E is open. Similarly, it follows that $\emptyset \subset E$ is open.

- Suppose that $V_1, \ldots, V_n \subset E$ are open subsets. Then $\Phi_\alpha(V_i \cap E_{|U_\alpha})$ is an open subset of $U_\alpha \times \mathbb{R}^k$ and hence

$$\Phi_\alpha(V_1 \cap \cdots \cap V_n \cap E_{|U_\alpha}) = \Phi_\alpha(V_1 \cap E_{|U_\alpha}) \cap \cdots \cap \Phi_\alpha(V_n \cap E_{|U_\alpha})$$

  is an open subset of $U_\alpha \times \mathbb{R}^k$ for all $\alpha \in A$. This implies that $V_1 \cap \cdots \cap V_n$ is an open subset of E.

- Similarly, if V_i, $i \in I$ is an open subset of E, then $\Phi_\alpha(V_i \cap E_{|U_\alpha})$ is an open subset of $U_\alpha \times \mathbb{R}^k$ for each $i \in I$ and $\alpha \in A$ and hence

$$\Phi_\alpha(\left(\bigcup_{i\in I} V_i\right) \cap E_{|U_\alpha}) = \bigcup_{i \in I} \Phi_\alpha(V_i \cap E_{|U_\alpha})$$

  is an open subset of $U_\alpha \times \mathbb{R}^k$.

Part (b). To show that $\pi\colon E \to M$ is continuous, let V be an open subset of M. To show that $\pi^{-1}(V) \subset E$ is open, we consider

$$\Phi_\alpha(\pi^{-1}(V) \cap E_{|U_\alpha}) = \Phi_\alpha(E_{|V \cap U_\alpha}) = (V \cap U_\alpha) \times \mathbb{R}^k.$$

Since $V \cap U_\alpha \subset U_\alpha$ is open for all $\alpha \in A$, $(V \cap U_\alpha) \times \mathbb{R}^k$ is an open subset of $U_\alpha \times \mathbb{R}^k$ and hence $\pi^{-1}(V)$ is open.

Part (c). Next we show that for $\beta \in A$ the bijection $\Phi_\beta\colon E_{|U_\beta} \to U_\beta \times \mathbb{R}^k$ is a homeomorphism (we find it convenient to give the fixed element in A a different name than our generic element $\alpha \in A$ involved in the definition of the topology on E). To verify that that Φ_β is a homeomorphism it suffices to prove that a subset V of $E_{|U_\beta}$ is open (with respect to the subspace topology on $E_{|U_\beta} \subset E$) if and only if $\Phi_\beta(V)$ is an open subset of $U_\beta \times \mathbb{R}^k$. If

V is open in the subspace topology, then $V = V' \cap E_{|U_\beta}$ with V' an open subset of E. In particular, $\Phi_\beta(V' \cap E_{U_\beta}) = \Phi_\beta(V)$ is an open subset of $U_\beta \times \mathbb{R}^k$. Conversely, assume that $\Phi_\beta(V)$ is open in $U_\beta \times \mathbb{R}^k$, then for any $\alpha \in A$

$$\begin{aligned}\Phi_\alpha(V \cap E_{|U_\alpha}) =& \Phi_\alpha(V \cap E_{|U_\beta \cap U_\alpha}) = \Phi_\alpha \circ \Phi_\beta^{-1} \circ \Phi_\beta(V \cap E_{|U_\beta \cap U_\alpha}) \\ =& \Phi_\alpha \circ \Phi_\beta^{-1}\left(\Phi_\beta(V) \cap (U_\alpha \times \mathbb{R}^k)\right)\end{aligned}$$

is an open subset of $U_\alpha \times \mathbb{R}^k$, since $\Phi_\beta(V) \cap (U_\alpha \times \mathbb{R}^k)$ is an open subset of $(U_\beta \cap U_\alpha) \times \mathbb{R}^k$, and the transition map

$$\Phi_\alpha \circ \Phi_\beta^{-1} \colon (U_\beta \cap U_\alpha) \times \mathbb{R}^k \longrightarrow (U_\beta \cap U_\alpha) \times \mathbb{R}^k$$

is a homeomorphism by assumption. □

5. (10 points) The goal of this problem is to prove the vector bundle construction lemma from class. We recall that this has slightly stronger assumptions than the lemma of the previous problem, namely M is required to be a smooth manifold, and the transition maps (10.3) are required to be smooth.

(a) Show that E, equipped with the topology E constructed in the previous problem, is a topological manifold of dimension $n+k$ (don't bother to check the technical conditions of being Hausdorff and second countable). Hint: Let $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ be an atlas for M. Show that the bundle chart Φ_α and the manifold chart ψ_β can be used to construct a chart

$$\chi_{\alpha,\beta} \colon E \underset{\text{open}}{\supset} E_{|U_\alpha \cap V_\beta} \longrightarrow \mathbb{R}^{n+k}.$$

(b) Show that the charts $\{(E_{|U_\alpha \cap V_\beta}), \chi_{\alpha,\beta})\}$ for $(\alpha, \beta) \in A \times B$ form a smooth atlas for E.

(c) Show that $\pi \colon E \to M$ is a smooth vector bundle of rank k with local trivializations provided by Φ_α.

Proof. **Part (a).** Let $\chi_{\alpha,\beta}$ be the composition of the following homeomorphisms

$$E_{|U_\alpha \cap V_\beta} \xrightarrow[\approx]{\Phi_\alpha} (U_\alpha \cap V_\beta) \times \mathbb{R}^k \xrightarrow[\approx]{\psi_\beta \times \mathrm{id}_{\mathbb{R}^k}} \psi_\beta(U_\alpha \cap V_\beta) \times \mathbb{R}^k \subset \mathbb{R}^{n+k} \qquad (10.4)$$

Strictly speaking, the maps involved in this composition are not the homeomorphisms Φ_α and ψ_β, but the homeomorphisms obtained by restricting the domains of these maps in the evident way. Since the open subsets $U_\alpha \cap V_\beta \subset M$ form an open cover of M, their preimages $E_{|U_\alpha \cap V_\beta} = \pi^{-1}(U_\alpha \cap V_\beta)$ form an open covering of E. This shows that E is locally Euclidean.

Part (b). We need to show that for any $\alpha, \alpha' \in A$, $\beta, \beta' \in B$ the transition map $\chi_{\alpha,\beta} \circ \chi_{\alpha',\beta'}^{-1}$ is smooth. Setting $U := U_\alpha \cap U_{\alpha'} \cap V_\beta \cap V_{\beta'}$, the transition map is given by the following composition

$$\psi_{\beta'}(U) \times \mathbb{R}^k \xrightarrow{(\psi_{\beta'}\times \mathrm{id})^{-1}} U \times \mathbb{R}^k \xrightarrow{\Phi_{\alpha'}^{-1}} E_{|U} \xrightarrow{\Phi_\alpha} U \times \mathbb{R}^k \xrightarrow{\psi_\beta \times \mathrm{id}} \psi_\beta(U) \times \mathbb{R}^k$$

We observe that the maps ψ_β, $\psi_{\beta'}$ are diffeomorphisms (since they belong to a smooth atlas for M), and hence $\psi_\beta \times \mathrm{id}$, $\psi_{\beta'} \times \mathrm{id}$ are diffeomorphisms. The composition $\Phi_{\alpha'}^{-1} \circ \Phi_\alpha$ is required to be smooth map (by assumption (iii) of Lemma **??**) and hence all transition maps for the charts $(E_{U_\alpha \cap V_\beta}, \chi_{\alpha,\beta})$ are smooth.

Part (c). After constructing a smooth structure on E, it only remains to show that $\pi\colon E \to M$ is smooth and that the local trivializations $\Phi_\alpha\colon E_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ are diffeomorphisms. The first statement is an immediate consequence of the second one, since by the commutative diagram (10.2) the restriction of π to $E_{|U_\alpha}$ is the composition $\pi_1 \circ \Phi_\alpha$ of smooth maps, and hence $\pi\colon E \to M$ is smooth.

To show that $\Phi_\alpha\colon E_{|U_\alpha} \to U_\alpha \times \mathbb{R}^k$ is a diffeomorphism, it suffices to show that its restriction $E_{U_\alpha \cap V_\beta} \longrightarrow (U_\alpha \cap V_\beta) \times \mathbb{R}^k$ is a diffeomorphism for all $\alpha \in A$. The composition of this map with the diffeomorphism $\psi_\beta \times \mathrm{id}_{\mathbb{R}^k}$, see (10.4), is the homeomorphism $\chi_{\alpha,\beta}$ which is part of our smooth atlas for E and hence a diffeomorphism. It follows that the restriction of Φ_α to $E_{U_\alpha \cap V_\beta}$ is a diffeomorphism. □

11 Homework Assignment # 11

1. (10 points) Let V be a vector space of dimension n with basis $\{b_1, \dots, b_n\}$, and let $\{b^1, \dots, b^n\}$ be the dual basis for the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$.

(a) Let $\mathrm{Mult}^k(V, \mathbb{R})$ be the space of multilinear maps $\omega\colon \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$, and for a multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_s \le n$ let $b^I \in \mathrm{Mult}^k(V, \mathbb{R})$ be defined by

$$b^I(v_1, \dots, v_k) := b^{i_1}(v_1) \cdots b^{i_k}(v_k).$$

Show that the elements b^I form a basis for $\mathrm{Mult}^k(V, \mathbb{R})$. Hint: argue that a k-linear form $\omega \in \mathrm{Mult}^k(V, \mathbb{R})$ is determined by the numbers $\omega(b_J) \in \mathbb{R}$ obtained by evaluating it on the k-tupels $b_J := (b_{j_1}, \dots, b_{j_k})$ with $J = (j_1, \dots, j_k)$ and $1 \le j_s \le n$.

(b) What is the dimension of $\mathrm{Mult}^k(V, \mathbb{R})$?

(c) Let $\mathrm{Alt}^k(V, \mathbb{R})$ be the space of alternating multilinear maps $\omega\colon V^{\times k} \to \mathbb{R}$, and for a multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_s \le n$ let $b^{\wedge I} \in \mathrm{Alt}^k(V, \mathbb{R})$ be defined by

$$b^{\wedge I}(v_1, \dots, v_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) b^{i_1}(v_{\sigma(1)}) \cdots b^{i_k}(v_{\sigma(k)}).$$

Show that $b^{\wedge I}(b_J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$ for multi-indices I, J which are *increasing*, i.e., $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

(d) Show that the elements $b^{\wedge I}$ for increasing multi-indices I, form a basis for $\mathrm{Alt}^k(V, \mathbb{R})$. Hint: argue that an alternating form $\omega \in \mathrm{Alt}^k(V, \mathbb{R})$ is determined by the numbers $\omega(b_J) \in \mathbb{R}$ obtained by evaluating it on the k-tupels $b_J := (b_{j_1}, \ldots, b_{j_k})$ for an increasing multi-index J. As noted in class, this in particular implies that the dimension of $\mathrm{Alt}^k(V, \mathbb{R})$ is $\binom{n}{k}$.

Note: in class we wrote $b^{i_1} \wedge \cdots \wedge b^{i_k}$ for the element $b^I \in \mathrm{Alt}^k(V, \mathbb{R})$ defined above. This was not a good idea, since it is not clear that b^I is in fact equal to the wedge product of the b^{i_s}, which we defined later in class. This is in fact true, as will be proved as part of the next problem.

Proof. **Part (a).** For $\omega \in \mathrm{Mult}^k(V, \mathbb{R})$ consider $\omega(v_1, \ldots, v_k)$ for $v_i \in V$. Writing each v_i as a linear combination of the basis elements $b_1, \ldots, b_n$, and using linearity of each slot of ω, we can write $\omega(v_1, \ldots, v_k)$ as a linear combination of the number $\omega(b_J)$ where J ranges over all multi-indices $J = (j_1, \ldots, j_k)$ for $1 \leq j_s \leq n$.

To show that the elements b^I span $\mathrm{Mult}^k(V, \mathbb{R})$, we claim that for $\omega \in \mathrm{Mult}^k(V, \mathbb{R})$

$$\omega = \sum_I \omega(b_I) b^I, \tag{11.1}$$

where the sum ranges over all multi-indices $I = (i_1, \ldots, i_k)$, $1 \leq i_s \leq n$. To prove the claim, it suffices by our earlier observation to evaluate both sides on b_J. We note that $b^I(b^J) = 0$ unless $I = J$, and hence evaluating the right hand side of (11.1) on b^J the right hand side simplifies to $\omega(b_J) b^J(b_J) = \omega(b_J)$, which agrees with the left hand side evaluated on b_J.

To show that the elements b^I are linearly independent, suppose $0 = \sum_I k_I b^I$ with $k_I \in \mathbb{R}$. Evaluating on b_J, the right hand side simplifies to $k_J b^J(b_J) = k_J$, which shows that the coefficients k_J are all zero.

Part (b). Part (a) shows that the dimension of $\mathrm{Mult}^k(V, \mathbb{R})$ is the cardinality of set of multi-indices $I = (i_1, \ldots, i_k)$ with $1 \leq i_s \leq n$. Since each index i_s can take on n values, and there are k indices, there are n^k multi-indices. This shows $\dim \mathrm{Mult}^k(V, \mathbb{R}) = n^k$.

Part (c).

$$b^{\wedge I}(b_J) = b^{\wedge I}(b_{j_1}, \ldots, b_{j_k}) = \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) b^{i_1}(b_{j_{\sigma(1)}}) \cdots b^{i_k}(b_{j_{\sigma(k)}}).$$

We note that the summand $\mathrm{sign}(\sigma) b^{i_1}(b_{j_{\sigma(1)}}) \cdots b^{i_k}(b_{j_{\sigma(k)}})$ is zero, unless the multi-indices $I = (i_1, \ldots, i_k)$ and $(j_{\sigma(1)}, \ldots, j_{\sigma(k)})$ are the same. Since I and J are both increasing, the summand must be trivial unless σ is the identity and $I = J$. If $\sigma = \mathrm{id}$ and $I = J$, then the summand is $b^I(b_J) = 1$, which proves part (c).

Part (d). The argument here is entirely analogous to the argument in part (a). We first observe that $\omega \in \mathrm{Alt}^k(V,\mathbb{R})$ is determined by $\omega(b_J)$ for *increasing* indices J, i.e., $J = (j_1,\dots,j_k)$ with $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. This is the case, since after applying a permutation $\sigma \in S_k$ to a multi-index J, we can obtain a multi-index J' that is increasing, but $\omega(b_{J'}) = \mathrm{sign}(\sigma)\omega(b_J)$.

To show that the elements $b^{\wedge I}$ span $\mathrm{Alt}^k(V,\mathbb{R})$, we claim that

$$\omega = \sum_I \omega(b_I) b^{\wedge I},$$

where the sum is taken over increasing multi-indices I. To show this, we evaluate both sides on b_J for increasing multi-indices J. On the right side, using part (c), we obtain

$$\sum_I \omega(b_I) b^{\wedge I}(b_J) = \sum_I \omega(b_I)\delta_{I,J} = \omega(b_J),$$

which agrees with evaluating the left hand side on b_J, thus proving the claim.

To prove linear independence of the elements $b^{\wedge I}$, suppose $\sum_I k_I b^I = 0$ for $k_I \in \mathbb{R}$. Then evaluating on b_J, the right hand side simplifies to $k_J b^{\wedge J}(b_J) = k_J$, which shows that the coefficients k_J are all zero. □

2. (10 points) We recall that the *wedge product*

$$\mathrm{Alt}^k(V,\mathbb{R}) \times \mathrm{Alt}^\ell(V,\mathbb{R}) \xrightarrow{\wedge} \mathrm{Alt}^{k+\ell}(V,\mathbb{R})$$

is a bilinear associative product defined by

$$(\omega \wedge \eta)(v_1,\dots,v_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)},\dots,v_{\sigma(k)})\eta(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)})$$

for $\omega \in \mathrm{Alt}^k(V,\mathbb{R})$, $\eta \in \mathrm{Alt}^\ell(V,\mathbb{R})$, $v_1,\dots,v_{k+\ell} \in V$.

(a) Show that

$$(b^{i_1} \wedge \cdots \wedge b^{i_k})(b_J) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

where $I = (i_1,\dots,i_k)$, $J = (j_1,\dots,j_k)$ are increasing sequence, and we use the same notation and terminology as in problem (1). Hint: Use induction over k.

This shows in particular that the alternating k-form $b^I \in \mathrm{Alt}^k(V,\mathbb{R})$ from 1(c) is in fact equal to $b^{i_1} \wedge \cdots \wedge b^{i_k}$.

(b) Show that the wedge product is *graded commutative*, i.e.,

$$\eta \wedge \omega = (-1)^{k\ell}\omega \wedge \eta \qquad \text{for } \omega \in \mathrm{Alt}^k l(V, \mathbb{R}),\ \eta \in \mathrm{Alt}^\ell(V, \mathbb{R}).$$

Hint: First consider the case $k = \ell = 1$, then argue that it suffices to prove the statement in the case $\omega = b^{i_1} \wedge \cdots \wedge b^{i_k}$, $\eta = b^{j_1} \wedge \cdots \wedge b^{j_\ell}$.

Proof. **Part (a).** Working by induction over k, let us assume that the desired statement

$$(b^{i_1} \wedge \cdots \wedge b^{i_k})(b_J) = \delta_{I,J}$$

has been proved for increasing multi-indices of length k. Then we need to prove the statement for increasing multi-indices I, J of length $k+1$. Using the definition of the wedge product, we compute

$$\begin{aligned}
&((b^{i_1} \wedge \cdots \wedge b^{i_k}) \wedge b^{i_{k+1}})(b_{j_1}, \ldots, b_{j_{k+1}}) \\
=&\frac{1}{k!} \sum_{\sigma \in S_{k+1}} \mathrm{sign}(\sigma)(b^{i_1} \wedge \cdots \wedge b^{i_k})(b_{j_{\sigma(1)}}, \ldots, b_{j_{\sigma(k)}})b^{i_{k+1}}(b_{j_{\sigma(k+1)}})
\end{aligned}$$

By our inductive hypothesis, we know that $b^{i_k})(b_{j_{\sigma(1)}}, \ldots, b_{j_{\sigma(k)}})$ is non-zero if and only if $\{i_1, \ldots, i_k\} = \{j_{\sigma(1)}, \ldots, j_{\sigma(k)}\}$. Moreover, $b^{i_{k+1}}(b_{j_{\sigma(k+1)}}) \neq 0$ if and only if $i_{k+1} = j_{\sigma(k+1)}$. Since I, J are both *increasing* multi-indices, these these conditions imply $I = J$, and $\sigma \in S_k$, the subgroup of S_{k+1} with $\sigma(k+1) = k+1$. Hence the above sum simplifies to

$$\begin{aligned}
&\frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)(b^{i_1} \wedge \cdots \wedge b^{i_k})(b_{i_{\sigma(1)}}, \ldots, b_{i_{\sigma(k)}})b^{i_{k+1}}(b_{i_{k+1}}) \\
=&\frac{1}{k!} \sum_{\sigma \in S_k} (b^{i_1} \wedge \cdots \wedge b^{i_k})(b_{i_1}, \ldots, b_{i_k}) \\
=&(b^{i_1} \wedge \cdots \wedge b^{i_k})(b_{i_1}, \ldots, b_{i_k}) = 1
\end{aligned}$$

Here the first equality holds since $b^{i_1} \wedge \cdots \wedge b^{i_k}$ is an alternating k-form, and $b^{i_{k+1}}(b_{i_{k+1}}) = 1$. The second equality follows, since S_k has cardinality $k!$.

Part (b). For $\omega, \eta \in \mathrm{Alt}^1(V, \mathbb{R})$ and $v_1, v_2 \in V$ we have

$$\begin{aligned}
(\omega \wedge \eta)(v_1, v_2) =&\frac{1}{1!1!} \sum_{\sigma \in S_2} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)})\eta(v_{\sigma(2)}) = \omega(v_1)\eta(v_2) - \omega(v_2)\eta(v_1) \\
(\eta \wedge \omega)(v_1, v_2) =&\eta(v_1)\omega(v_2) - \eta(v_2)\omega(v_1)
\end{aligned}$$

This proves $\omega \wedge \eta = -\eta \wedge \omega$ as it should for $\eta, \omega \in \mathrm{Alt}^1(V, \mathbb{R})$ according to graded commutativity.

According to problem 1(d) and 2(a), the elements $b^{i_1}\wedge\cdots\wedge b^{i_k}$ form a basis for $\mathrm{Alt}^k(V,\mathbb{R})$. So it suffices to prove graded commutativity for basis elements $\omega = b^{i_1}\wedge\cdots\wedge b^{i_k}$, $\eta = b^{j_1}\wedge\cdots\wedge b^{j_\ell}$.

$$\begin{aligned}\omega\wedge\eta =&(b^{i_1}\wedge\cdots\wedge b^{i_k})\wedge(b^{j_1}\wedge\cdots\wedge b^{j_\ell})\\ =&(-1)^k b^{j_1}\wedge(b^{i_1}\wedge\cdots\wedge b^{i_k})\wedge(b^{j_2}\wedge\cdots\wedge b^{j_\ell})\\ =&(-1)^{k\ell}(b^{j_1}\wedge\cdots\wedge b^{j_\ell})\wedge(b^{i_1}\wedge\cdots\wedge b^{i_k})\\ =&(-1)^{k\ell}\eta\wedge\omega\end{aligned}$$

Here the sign $(-1)^k$ comes from permuting b^{j_1} with each of the factors b^{i_s} for $s=1,\dots,k$. Doing the same step for b^{j_2} again gives a factor of $(-1)^k$, e.t.c. Moving all b^j's past the b^i's yields the sign $(-1)^{k\ell}$ □

3. (10 points) Let M, N be smooth manifolds and $F\colon M\to N$ a smooth map. Then a differential form $\omega\in\Omega^k(N)$ leads to a form $F^*\omega\in\Omega^k(M)$, called the *pullback of ω along F* which is defined by

$$(F^*\omega)_p(v_1,\dots,v_k) := \omega_p(DF_p(v_1),\dots,DF_p(v_k)) \qquad \text{for } p\in M,\ v_1,\dots,v_k\in T_pM.$$

In more detail: the k-form $F^*\omega$ is a section of the vector bundle $\mathrm{Alt}^k(TM;\mathbb{R})$, and hence it can be evaluated at $p\in M$ to obtain an element $(F^*\omega)_p$ in the fiber of that vector bundle over p, which is $\mathrm{Alt}^k(T_pM;\mathbb{R})$. In other words, $(F^*\omega)_p$ is an alternating multilinear map

$$(F^*\omega)_p\colon \underbrace{T_pM\times\cdots\times T_pM}_{k}\longrightarrow\mathbb{R},$$

and hence it can be evaluated on the k tangent vectors $v_1,\dots,v_k\in T_pM$ to obtain a real number $(F^*\omega)_p(v_1,\dots,v_k)$. On the right hand side of the equation defining $F^*\omega$, the map $DF_p\colon T_pM\to T_{F(p)}N$ is the differential of F. Hence the alternating multilinear map $\omega_{F(p)}\in\mathrm{Alt}^k(T_{F(p)}N;\mathbb{R})$ can be evaluated on $DF_p(v_1),\dots,DF_p(v_k)$ to obtain the real number $\omega_p(DF_p(v_1),\dots,DF_p(v_k))$.

Let $F\colon\mathbb{R}^n\to\mathbb{R}^n$ be a smooth map. Show that

$$F^*(dx^1\wedge\cdots\wedge dx^n) = \det(DF)\ dx^1\wedge\cdots\wedge dx^n \tag{11.2}$$

Here $DF\colon\mathbb{R}^n\to\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)$ is the differential of DF, which maps $x\in\mathbb{R}^n$ to $DF_x\colon\mathbb{R}^n\to\mathbb{R}^n$, the differential of F at the point $x\in\mathbb{R}^n$. Hint: Evaluated at a point $x\in\mathbb{R}^n$ both sides of the equation are vectors of the 1-dimensional vector space $\mathrm{Alt}^n(T_x\mathbb{R}^n,\mathbb{R}) = \mathrm{Alt}^n(\mathbb{R}^n,\mathbb{R})$. Hence it suffices to show equality after evaluating both sides on $(e_1,\dots,e_n)$, where $e_1,\dots,e_n$ is the standard basis of $\mathbb{R}^n$.

Proof. Evaluating the n-forms given by either side of the equation at a point $x \in \mathbb{R}^n$, we obtain a vector in $\mathrm{Alt}^n(T_x\mathbb{R}^n, \mathbb{R})$, the 1-dimensional vector space of multilinear alternating maps

$$\underbrace{T_x\mathbb{R}^n \times \cdots \times T_x\mathbb{R}^n}_{n} \longrightarrow \mathbb{R}.$$

Hence it suffices to show that both sides evaluated on the standard n-tupel $(e_1, \dots, e_n)$ agree. We calculate the left hand side, writing A for the $n \times n$-matrix corresponding to the differential $DF_x \colon T_x\mathbb{R}^n = \mathbb{R}^n \to T_x\mathbb{R}^n = \mathbb{R}^n$ and A_i for the i-th column vector of A:

$$\begin{aligned} F^*(dx^1 \wedge \cdots \wedge dx^n)(e_1, \dots, e_n) =& (dx^1 \wedge \cdots \wedge dx^n)(Df_x(e_1), \dots, Df_x(e_n)) \\ =& (dx^1 \wedge \cdots \wedge dx^n)(Ae_1, \dots, Ae_n) \\ =& (dx^1 \wedge \cdots \wedge dx^n)(A_1, \dots, A_n) \end{aligned}$$

Evaluating $\det(DF_x)dx_1 \wedge \cdots \wedge dx_n$ on $(e_1, \dots, e_n)$ and writing it in terms of A, we have

$$\det(DF_x)dx^1 \wedge \cdots \wedge dx^n(e_1, \dots, e_n) = \det(A)dx^1 \wedge \cdots \wedge dx^n(e_1, \dots, e_n) = \det(A_1, \dots, A_n).$$

We note that the dependence of both sides on the column vectors A_i is *multilinear and alternating*, a well-known property for the determinant. Since the space of such maps is 1-dimensional, we can test whether both functions of the matrix A agree by *evaluating on the identity matrix*, for which the agreement is obvious. □

4. (10 points) For any smooth manifold M the *de Rham differential* (also called *exterior differential*) is the unique map $d \colon \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) d is linear.

(ii) For a function $f \in C^\infty(M) = \Omega^0(M)$ the 1-form $df \in \Omega^1(M) = C^\infty(M, T^*M)$ is the usual differential of f.

(iii) d is a graded derivation with respect to the wedge product; i.e.,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \text{for } \omega \in \Omega^k(M),\ \eta \in \Omega^l(M).$$

(iv) $d^2 = 0$.

We recall that for $M = \mathbb{R}^n$, every k-form $\eta \in \Omega^k(\mathbb{R}^n)$ can be written uniquely in the form

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for smooth functions $f_{i_1,\dots,i_k} \in C^\infty(\mathbb{R}^n)$. The point of this problem is to give an *explicit* formula for d for $M = \mathbb{R}^n$ (which works equally well locally, on a coordinate patch of a smooth n-manifold).

(a) Show that for $f \in C^\infty(\mathbb{R}^n)$ the differential $df \in \Omega^1(\mathbb{R}^n)$ is given by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

(b) Show that for $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(\mathbb{R}^n)$

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Proof. **Part (a).** We recall that the 1-form df evaluated on a tangent vector $v \in T_xM$ is by definition the tangent map $Df_x \colon T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}$ applied to v. Also, the partial derivative $\frac{\partial f}{\partial x^i}(x)$ can be identified with the tangent map Df_x evaluated on e_i (the i-th element of the standard basis of $\mathbb{R}^n$). It follows that

$$(\sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j)_x(e_i) = \sum_{j=1}^n \frac{\partial f}{\partial x^j}(x)(dx^j)_x(e_i) = \frac{\partial f}{\partial x^i}(x) = Df_x(e_i) = df_x(e_i)$$

for all i. Hence the 1-forms $\sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j$ and df evaluated on all tangent vectors $v \in T_x\mathbb{R}^n$ agree, which means that these 1-forms are the same.

Part (b). For general k we calculate

$$\begin{aligned}
d\omega =& d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
=& (df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + f \wedge d(dx^{i_1}) \wedge \cdots \wedge dx^{i_k} + \cdots + \pm f \wedge dx^{i_1} \wedge \cdots \wedge d(dx^{i_k}) \\
=& (df) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\end{aligned}$$

Here the second equality is the derivation property (iii) of d, the third equality is a consequence of $d^2 = 0$. □

5. (10 points) Show that the exterior derivative for differential forms on $\mathbb{R}^3$ corresponds to the classical operations of *gradient* resp. *curl* resp. *divergence*. More precisely, show that there is a commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & \mathcal{V}\mathrm{ect}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\cong"] & \mathcal{V}\mathrm{ect}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\cong"] & C^\infty(\mathbb{R}^3) \arrow[d, "\cong"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

Here $\mathcal{V}\mathrm{ect}(\mathbb{R}^3)$ is the space of vector fields on $\mathbb{R}^3$, and we recall that grad, curl and divergence are given by the formulas

$$\begin{aligned}
\mathrm{grad}(f) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\mathrm{curl}(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
\mathrm{div}(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\end{aligned}$$

Here we identify a vector field on $\mathbb{R}^3$ with a triple (f_1, f_2, f_3) of smooth functions on $\mathbb{R}^3$. The vertical isomorphisms are given by

$$\begin{aligned}
(f_1, f_2, f_3) &\mapsto f_1 dx + f_2 dy + f_3 dz \\
(f_1, f_2, f_3) &\mapsto f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \\
f &\mapsto f dx \wedge dy \wedge dz
\end{aligned}$$

Proof. Let $f \in C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3)$. Then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \Phi_1(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = \Phi_1(\mathrm{grad}(f))$$

For $(f_1, f_2, f_3) \in \mathrm{Vect}(\mathbb{R}^3)$ we calculate:

$$\begin{aligned}
d\Phi_1(f_1, f_2, f_3) =& d\left(f_1 dx + f_2 dy + f_3 dz\right) \\
=& df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz \\
=& \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz\right) \wedge dx + \left(\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz\right) \wedge dy \\
&+ \left(\frac{\partial f_3}{\partial x}dx + \frac{\partial f_3}{\partial y}dy + \frac{\partial f_3}{\partial z}dz\right) \wedge dz \\
=& \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy
\end{aligned}$$

For $(f_1, f_2, f_3) \in \mathrm{Vect}(\mathbb{R}^3)$ we calculate:

$$\begin{aligned}
d\Phi_2(f_1, f_2, f_3) &= d\left(f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy\right) \\
&= df_1 \wedge dy \wedge dz + df_2 \wedge dz \wedge dx + df_3 \wedge dx \wedge dy \\
&= \frac{\partial f_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial f_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial f_3}{\partial z} dz \wedge dx \wedge dy \\
&= (\mathrm{div}(f_1, f_2, f_3) dx \wedge dy \wedge dz = \Phi_3(\mathrm{div}(f_1, f_2, f_3))
\end{aligned}$$

$\square$