Homework Assignment # 9, due Nov. 9

1. (10 points) Let M, N be smooth manifolds, and let $\pi_1\colon M\times N\to M$ and $\pi_2\colon M\times N\to N$ be the projection maps. Show that for any $(x,y)\in M\times N$ the map

$$\alpha\colon T_{(x,y)}(M\times N)\longrightarrow T_xM\oplus T_yN$$

defined by

$$\alpha(v)=(D\pi_1(v),D\pi_2(v))$$

is an isomorphism. Here we suppress the subscripts of the differentials that indicate the point of the domain, i.e., we write $D\pi_1$ instead of $(D\pi_1)_{x,y}$. Hint: To prove this, it is unnecessary to "unpack" the definition of the tangent space of manifolds by using either the geometric or algebraic definition. Rather, only the *functorial properties* of the tangent space, i.e., the chain rule, is needed, applied to suitable projection/inclusion maps. Remark: Using this isomorphism, we will routinely identify T_xM and T_yN with subspaces of $T_{(x,y)}(M\times N)$.

2. (10 points) Let M be a smooth n manifold. For a point $p\in M$ let

$$D^M\colon T_p^{\mathrm{geo}}M\ \longrightarrow\ T_p^{\mathrm{alg}}M=\mathrm{Der}(C_p^\infty(M),\mathbb{R})$$

be the map that sends $[\gamma]\in T_p^{\mathrm{geo}}M$ to the derivation D_γ. More explicitly, if f is (the germ of) a function $f\colon M\to\mathbb{R}$ then $D_\gamma f\in\mathbb{R}$ is the directional derivative of f in the direction of γ defined by

$$D_\gamma f:=(f\circ\gamma)'(0)=\lim_{t\to 0}\frac{f(\gamma(t))-f(p)}{t}.$$

(a) Show that the geometric and the algebraic definition of the differential of a smooth map $F\colon M\to N$ are compatible in the sense that for $p\in M$ the following diagram is commutative:

$$\begin{array}{ccc} T_p^{\mathrm{geo}}M & \xrightarrow{DF_p^{\mathrm{geo}}} & T_{F(p)}^{\mathrm{geo}}N \\ {\scriptstyle D^M}\downarrow & & \downarrow{\scriptstyle D^N} \\ T_p^{\mathrm{alg}}M & \xrightarrow{DF_p^{\mathrm{alg}}} & T_{F(p)}^{\mathrm{alg}}N \end{array}$$

(b) Show that the map D^M is a bijection for any manifold M. Hint: use a chart for M and part (a) to reduce to the case of open subsets $U\subset\mathbb{R}^n$; we have proved in class that the map $D^U\colon T_p^{\mathrm{geo}}U\to T_p^{\mathrm{alg}}U$ is a bijection in that case.

3. (10 points) Let M, N be smooth manifolds, let $F\colon M\to N$ be a smooth map, and $p\in M$.

(a) Show that the map

$$F_p^*\colon C_{F(p)}^\infty(N)\longrightarrow C_p^\infty(M)\qquad\text{given by}\qquad [f]\mapsto[f\circ F]$$

is a well-defined algebra homomorphism.

(b) Show that if $D\colon C_p^\infty(M) \to \mathbb{R}$ is a derivation, then the composition

$$C_{F(p)}^\infty(N) \xrightarrow{F_p^*} C_p^\infty(M) \xrightarrow{D} \mathbb{R}$$

is a derivation. In particular, we can define the (algebraic) differential

$$D^{\text{alg}}F_p\colon T_p^{\text{alg}}M = \text{Der}(C_p^\infty(M), \mathbb{R}) \longrightarrow T_{F(p)}^{\text{alg}}N = \text{Der}(C_{F(p)}^\infty(M), \mathbb{R})$$

by $DF_p^{\text{alg}}(D) := D \circ F_p^*$.

(c) Show that the differential DF_p^{alg} is a linear map.

(d) If $G\colon N \to Q$ is a smooth map, show that

$$D(G \circ F)_p^{\text{alg}} = DG_{F(p)}^{\text{alg}} \circ DF_p^{\text{alg}}.$$

We note that this statement is the *chain rule* (for the algebraic construction of the tangent space).

4. (10 points) Let $M_{n\times k}(\mathbb{R})$ be the vector space of $n \times k$-matrices. For $A \in M_{n\times k}(\mathbb{R})$ let $A^t \in M_{k\times n}(\mathbb{R})$ be the transpose of A, and let $\mathsf{Sym}(\mathbb{R}^k) = \{B \in M_{k\times k}(\mathbb{R}) \mid B^t = B\}$ be the vector space of *symmetric* $k \times k$*-matrices*.

(a) Show that the map $\Phi\colon M_{n\times k}(\mathbb{R}) \to \mathsf{Sym}(\mathbb{R}^k)$, $A \mapsto A^tA$ is smooth, and that its differential

$$D\Phi_A\colon T_AM_{n\times k}(\mathbb{R}) = M_{n\times k}(\mathbb{R}) \longrightarrow T_{\Phi(A)}\mathsf{Sym}(\mathbb{R}^k) = \mathsf{Sym}(\mathbb{R}^k)$$

is given by $D\Phi_A(C) = C^tA + A^tC$. Hint: Use the geometric description of tangent spaces. More explicitly, the tangent space $T_A^{\text{geo}}M_{n\times k}(\mathbb{R})$ can be identified with $M_{n\times k}(\mathbb{R})$ by sending a matrix $C \in M_{n\times k}(\mathbb{R})$ to the path $\gamma(t) := A + tC$.

(b) Show that the identity matrix is a regular value of the map Φ. This implies in particular that the level set Φ^{-1}(identity matrix) is a smooth manifold. We recall that we showed in class that Φ^{-1}(identity matrix) is the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$. Hint: to show that $D\Phi_A\colon T_AM_{n\times n}(\mathbb{R}) \to T_e\mathsf{Sym}(\mathbb{R}^k)$ is surjective for $e =$ identity matrix, compute $D\Phi_A(C)$ for $C = AB$ for $B \in \mathsf{Sym}(\mathbb{R}^k)$.

(c) What is the dimension of $V_k(\mathbb{R}^n)$?

We remark that identifying $M_{n\times k}(\mathbb{R})$ in the usual way with the vector space $\text{Hom}(\mathbb{R}^k, \mathbb{R}^n)$ of linear maps $f\colon \mathbb{R}^k \to \mathbb{R}^n$, a matrix belongs to $V_k(\mathbb{R}^n)$ if and only if the corresponding linear map f is an *isometry*, that is, if f preserves the length of vectors in the sense that $||f(v)|| = ||v||$, or equivalently, if f preserves the scalar product in the sense that

$$\langle f(v), f(w)\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in \mathbb{R}^k.$$

The manifold $V_k(\mathbb{R}^n)$ is called the *Stiefel manifold*. We observe that $V_n(\mathbb{R}^n)$ is the orthogonal group $O(n)$ of isometries $\mathbb{R}^n \to \mathbb{R}^n$.

5. (10 points) Recall that the special linear group $SL_n(\mathbb{R})$ and the orthogonal group $O(n)$ are both submanifolds of the vector space $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices. In particular, the tangent spaces $T_ASL_n(\mathbb{R})$ for $A \in SL_n(\mathbb{R})$ and $T_AO(n)$ for $A \in O(n)$ are subspaces of the tangent space $T_AM_{n\times n}(\mathbb{R})$, which can be identified with $M_{n\times n}(\mathbb{R})$, since $M_{n\times n}(\mathbb{R})$ is a vector space.

(a) Show that $T_eSL_n(\mathbb{R}) = \{C \in M_{n\times n} \mid \mathrm{tr}(C) = 0\}$, where e is the identity matrix, and $\mathrm{tr}(C)$ denotes the trace of the matrix C.

(b) Show that $T_eO(n) = \{C \in M_{n\times n} \mid C^t = -C\}$.

Hint for parts (a) and (b): $SL_n(\mathbb{R})$ and $O(n)$ can be both be described as level sets $F^{-1}(c)$ of a regular value c for a suitable smooth map F (as we did in class for $SL_n(\mathbb{R})$ and you did for $O(n)$ in problem 4 of this homework assignment; note that $O(n)$ is equal to the Stiefel manifold $V_n(\mathbb{R}^n)$).

Remark: A *Lie group* is a group G which also is a smooth manifold and these structures are compatible in the sense that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are smooth maps. The tangent space T_eG at the identity element $e \in G$ is called the *Lie algebra of* G. In other words, this problem asks you to calculate the Lie algebra for the Lie groups $SL_n(\mathbb{R})$ resp. $O(n)$.