Homework Assignment # 8, due Nov. 2

1. (10 points) A standard atlas for the sphere S^n is provided by the hemisphere atlas, given by the open subsets $U_i^\epsilon := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \epsilon x_i > 0\} \subset S^n$ for $i = 0, \dots, n$, $\epsilon \in \{\pm 1\}$, and the homeomorphism

$$\phi_i^\epsilon \colon U_i^\epsilon \xrightarrow{\approx} B_1^n \qquad \text{given by} \qquad \phi_i^\epsilon(x_0, \dots, x_n) = (x_0, \dots, \widehat{x}_i, \dots, x_n)$$

(a) Show that $\{(U_i^\epsilon, \phi_i^\epsilon)\}$ is a smooth atlas for S^n. You can use your calculus knowledge about smooth functions on open subsets of $\mathbb{R}$. Beware that the function $\sqrt{x}$ is defined, but not smooth at $x = 0$.

(b) Show that with respect to the smooth structure on S^n given by the smooth atlas from part (a), the inclusion map $i \colon S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.

2. (10 points) We recall that the stereographic projection provides a homeomorphism between the open subsets $U_\pm := S^n \setminus \{(\mp 1, 0, \dots, 0)\}$ of S^n and $\mathbb{R}^n$. More explicitly, the stereographic projection is the map

$$\psi_\pm \colon U_\pm \longrightarrow \mathbb{R}^n \qquad \text{is defined by} \qquad \psi_\pm(x_0, \dots, x_n) := \frac{1}{1 \pm x_0}(x_1, \dots, x_n),$$

and its inverse $\psi_\pm^{-1} \colon \mathbb{R}^n \to U_\pm$ is given by the formula

$$\psi_\pm^{-1}(y_1, \dots, y_n) = \frac{1}{||y||^2 + 1}(\pm(1 - ||y||^2), 2y_1, \dots, 2y_n).$$

In particular, the two charts (U_+, ψ_+), (U_-, ψ_+) form an atlas for S^n.

(a) Show that $\{(U_+, \psi_+), (U_-, \psi_-)\}$ is a *smooth* atlas for S^n.

(b) Show that the atlas above determines the same smooth structure on S^n as the smooth atlas of the previous problem.

3. (10 points) Let $\mathbb{RP}^n$ be the real projective space of dimension n, which can be defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by identifying x with λx for a non-zero $\lambda \in \mathbb{R}$.

(a) Show that the atlas $\{(U_i, \phi_i)\}_{i=0,\dots,n}$ with $U_i := \{[x_0, \dots, x_n] \mid x_i \neq 0\} \subset \mathbb{RP}^n$ and the homeomorphism $\phi_i \colon U_i \to \mathbb{R}^n$ given by

$$\phi_i([x_0, \dots, x_n]) = \frac{1}{x_i}(x_0, \dots, \widehat{x}_i, \dots, x_n)$$

is a smooth atlas.

(b) Show that the function $h\colon \mathbb{RP}^n \to \mathbb{R}$ defined by

$$h([x]) = \frac{1}{||x||^2} \sum_{\ell=0}^{n} \ell x_\ell^2 \qquad \text{for } x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$$

is smooth.

4. (10 points) Show that the Cartesian product of $M \times N$ of smooth manifolds of dimension m resp. n is a smooth manifold of dimension $m + n$.

5. (10 points) We recall that for an open subset $U \subset \mathbb{R}^m$ and $p \in U$ the map

$$D^U \colon T_p^{\text{geo}} U \longrightarrow \mathbb{R}^m \qquad \text{given by} \qquad [\gamma] \mapsto \gamma'(0)$$

is a bijection. Show that for a smooth map $\mathbb{R}^m \underset{\text{open}}{\supset} U \xrightarrow{F} V \underset{\text{open}}{\subset} \mathbb{R}^n$ and $p \in U$, the diagram

$$\begin{array}{ccc} T_x^{\text{geo}} U & \xrightarrow{DF_x^{\text{geo}}} & T_{F(p)}^{\text{geo}} V \\ {\scriptstyle D^U}\downarrow{\scriptstyle \cong} & & {\scriptstyle D^V}\downarrow{\scriptstyle \cong} \\ \mathbb{R}^m & \xrightarrow{dF_x} & \mathbb{R}^n \end{array}$$

is commutative. We note that this expresses the compatibility of the Jacobian dF_x (the traditional calculus definition of the derivative of the map F) and DF_x^{geo} (the new definition of the derivative, which generalizes to manifolds).