Homework Assignment # 7, due Oct. 26

1. (10 points) We recall that if $G \times X \to X$ is the action of a group G on a set X, then the subgroup $G_x := \{g \in G \mid gx = x\} \subseteq G$ is the *isotropy subgroup* of the point $x \in X$. The action is called *free* if the isotropy subgroup G_x is the trivial group for all $x \in X$. If X is a topological space, the action is called *continuous* if for every $g \in G$ the map $X \to X$ given by $x \mapsto gx$ is continuous.

(a) Show that if G is a finite group which acts freely and continuously on a Hausdorff space X, then the projection map $p \colon X \to G\backslash X$ to the orbit space $G\backslash X$ is a covering map. Hint: Use the assumptions that the action is free and X is Hausdorff to show that for every $x \in X$ there is an open neighborhood U such that the subsets $gU \subset X$ for $g \in G$ are mutually disjoint.

(b) Show that if X is a manifold of dimension n, then also the orbit space $G\backslash X$ is a manifold of dimension n (it is true, but in order to make this problem a little shorter, don't worry about proving that $G\backslash X$ is Hausdorff and second countable).

(c) Show that the map $\mathbb{Z}/2 \times S^n \to S^n$ given by $(m, v) \mapsto (-1)^m v$ is a continuous free action. We note that the orbit space $\mathbb{Z}/2\backslash S^n$ is the real projective space $\mathbb{RP}^n$, and hence part (b) of this problem provides a different way to show that $\mathbb{RP}^n$ is a manifold of dimension n.

(d) Show that the map

$$\mathbb{Z}/k \times S^{2n-1} \longrightarrow S^{2n-1} \qquad \text{given by} \qquad (m, v) \mapsto e^{2\pi i m/k} v$$

is a continuous free action of the cyclic group $\mathbb{Z}/k$ on the sphere $S^{2n-1} \subset \mathbb{C}^n$. By part (b) the orbit space $\mathbb{Z}/k\backslash S^{2n-1}$ is then a manifold of dimension $2n-1$, which is known as a *lens space*. Note that for $k = 2$, this is the real projective space $\mathbb{RP}^{2n-1}$.

2. (10 points) Let $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. Let Y be a path-connected and let $f \colon (Y, y_0) \to (X, x_0$ be a map such that the image $f_*\pi_1(Y, y_0)$ is contained in the image $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$. We proved in class that then there exists a unique (not necessarily continuous) map $\widetilde{f}$ making the diagram

$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{x}_0) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

commutative. We constructed $\widetilde{f}(y)$ by picking a path $\gamma \colon I \to Y$ from y_0 to y, composed with the map $f \colon Y \to X$ to obtain the path $f \circ \gamma \colon I \to X$, and defined $\widetilde{f}(y) := \widetilde{f \circ \gamma}(1)$, where $\widetilde{f \circ \gamma} \colon I \to \widetilde{X}$ is the unique lift of $f \circ \gamma$ with starting point $\widetilde{x}_0$.

Show that $\widetilde{f}$ is continuous under the additional assumption that Y is locally path-connected. Hint: It suffices to show that $\widetilde{f}$ is continuous in some open neighborhood V of every point $y \in Y$. Show that the assumption that Y is locally path-connected can be used to choose for every point $y \in Y$ a path-connected neighborhood V such that $f(V)$ is contained in a evenly covered open subset $U \subset X$. To analyze $\widetilde{f}(y')$ for $y' \in V$, use the concatenation $\gamma * \delta$ of a path γ from y_0 to y and $\delta \colon I \to V$ from y to y'.

3. (10 points) Let $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering of a path-connected and locally path-connected space X.

(a) It follows from the General Lifting Criterion that for $g \in G := \pi_1(X, x_0)$ there is a unique map $\phi_g \colon (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, g\widetilde{x}_0)$ making the diagram

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{x}_0) & \xrightarrow{\phi_g} & (\widetilde{X}, g\widetilde{x}_0) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & (X, x_0) & \end{array}$$

commutative. Here $g\widetilde{x}_0 := \widetilde{\gamma}(1)$ is the endpoint of a lift $\widetilde{\gamma} \colon I \to \widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ of any based loop γ in (X, x_0) which represents $g \in \pi_1(X, x_0)$ (we have shown that $\widetilde{\gamma}(1)$ depends only on $[\gamma] \in \pi_1(X, x_0)$, not on the particular loop γ). Show that the map

$$G \times \widetilde{X} \longrightarrow \widetilde{X} \qquad (g, \widetilde{x}) \mapsto \phi_g(\widetilde{x})$$

is an action map.

(b) Show that the action is free, i.e., for every $\widetilde{x} \in \widetilde{X}$, the only element of $g \in G$ with $g\widetilde{x} = \widetilde{x}$ is the identity element.

(c) Show that the action is transitive on the fiber $p^{-1}(x)$ for all $x \in X$, i.e., for $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$ there is some $g \in G$ such that $g\widetilde{x} = \widetilde{x}'$.

4. (10 points) Let (X, x_0) be a pointed space which is path-connected, locally path-connected, and semilocally simply connected. The goal of this assignment is the classification of isomorphism classes of objects of the category $\mathrm{Cov}_*(X, x_0)$ of based path connected covering spaces $p \colon (E, e_0) \to (X, x_0)$. More precisely, the goal is to show that there is a bijection Ψ between

$$\{\text{based coverings } p \colon (E, e_0) \to (X, x_0) \text{ with } E \text{ path-connected}\}/\text{isomorphism}$$

and

$$\{\text{subgroups of } \pi_1(X, x_0)\}.$$

It is given by sending a covering p to the subgroup $p_* \pi_1(E, e_0) \subset \pi_1(X, x_0)$.

(a) Show that Ψ is injective. Hint: use the general lifting criterion to show that any two path-connected based coverings $p\colon (E, e_0) \to (X, x_0)$ and $p'\colon (E', e_0') \to (X, x_0)$ are isomorphic.

(b) Show that Ψ is surjective by the following construction. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal covering of X. By problem # 3, the fundamental group $G = \pi_1(X, x_0)$ acts freely on $\widetilde{X}$ by covering maps; this action is transitive on all fibers $p^{-1}(x)$ for $x \in X$. Let H be a subgroup of G and let $H\backslash\widetilde{X}$ be the orbit space of action of the subgroup H and let $p^H\colon (H\backslash\widetilde{X}, [\widetilde{x}_0]) \to (X, x_0)$, $[\widetilde{x}] \mapsto p(\widetilde{x})$ be the projection map. Here $[\widetilde{x}] = H\widetilde{x}$ denotes the orbit through the point $\widetilde{x}$. Show that p^H is a covering and that $p^H_*\pi_1(H\backslash\widetilde{X}, [\widetilde{x}_0]) \subset G$ is the subgroup H.