Homework Assignment # 6, due Oct. 5

1. (10 points) Let (X, x_0), (Y, y_0) be pointed spaces. We recall that writing $f\colon (X, x_0) \to (Y, y_0)$ means that f is a map from X to Y which is basepoint-preserving in the sense that $f(x_0) = y_0$. Maps $f_0, f_1\colon (X, x_0) \to (Y, y_0)$ *basepoint-preserving homotopic*, notation $f_0 \sim_{\mathrm{bp}} f_1$, if there is a homotopy $H\colon X \times I \to Y$ from f_0 to f_1 which is *basepoint-preserving* in the sense that $H(x_0, t) = y_0$ for all $t \in I$. A map $f\colon (X, x_0) \to (Y, y_0)$ is a *basepoint-preserving homotopy equivalence* if there is a map $g\colon (Y, y_0) \to (X, x_0)$ such that $g \circ f \sim_{\mathrm{bp}} \mathrm{id}_X$ and $f \circ g \sim_{\mathrm{bp}} \mathrm{id}_Y$.

(a) Show that if $f, g\colon (X, x_0) \to (Y, y_0)$ are basepoint-preserving homotopic, then the induced homomorphisms $f_*, g_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ are equal.

(b) Show that if $f\colon (X, x_0) \to (Y, y_0)$ is a basepoint-preserving homotopy equivalence, then the induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

2. (10 points) In this problem you are ask to show that an object X in a category $\mathcal{C}$ is the categorical product of two other objects. Recall that this means that you need to construct morphisms $p_1\colon X \to X_1$ and $p_2\colon X \to X_2$ and show that the following diagram in $\mathcal{C}$ has the property of being a *product diagram* discussed in the lectures:

$$X_1 \xleftarrow{p_1} X \xrightarrow{p_2} X_2$$

(a) Show that the cartesian product $G_1 \times G_2$ of two groups G_1, G_2, equipped with the usual multiplication given by $(g_1, g_2) \cdot (h_1, h_2) := (g_1 h_1, g_2 h_2)$ is the categorical product of G_1 and G_2 in the category Grp of groups and group homomorphisms.

(b) Let (X_1, x_1), (X_2, x_2) be pointed topological spaces. Show that the pointed space $(X_1 \times X_2, (x_1, x_2))$ is the categorical product of (X_1, x_1) and (X_2, x_2) in the category Top_* of pointed topological spaces and basepoint-preserving maps.

3. (10 points)

(a) Show that the free product $G_1 * G_2$ of groups G_1, G_2 is the coproduct of G_1 and G_2 in the category Grp of groups and group homomorphisms. Hint: proving this amounts to constructing homomorphisms $i_1\colon G_1 \to G_1 * G_2$ and $i_2\colon G_2 \to G_1 * G_2$ and verifying that the diagram

$$G_1 \xrightarrow{i_1} G_1 * G_2 \xleftarrow{i_2} G_2$$

is a coproduct diagram.

(b) Let $G_1 \xleftarrow{j_1} H \xrightarrow{j_2} G_2$ be a diagram of groups and homomorphisms. Show that the free product with amalgamation $G_1 *_H G_2$ is a pushout of the diagram above in the category of groups. Hint: Showing that $G_1 *_H G_2$ is a pushout of the diagram above means that there are homomorphisms $k_1 \colon G_1 \to G_1 *_H G_2$ and $k_2 \colon G_2 \to G_1 *_H G_2$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{j_1} & G_1 \\ \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle k_1} \\ G_2 & \xrightarrow[k_2]{} & G_1 *_H G_2 \end{array}$$

is commutative, and has the property of being a *pushout diagram.*

4. (10 points) Let M, N be path-connected manifolds of dimension $n \geq 3$. The goal of this problem is to compute the fundamental group of their connected sum $M\#N$ in terms of the fundamental groups of M and N. We provide an alternative description of the connected sum $M\#N$, which is easier for the problem at hand, works for smooth manifolds, and uses pushout diagrams (it is not hard to show that this version of $M\#N$ is homeomorphic to the version presented in class).

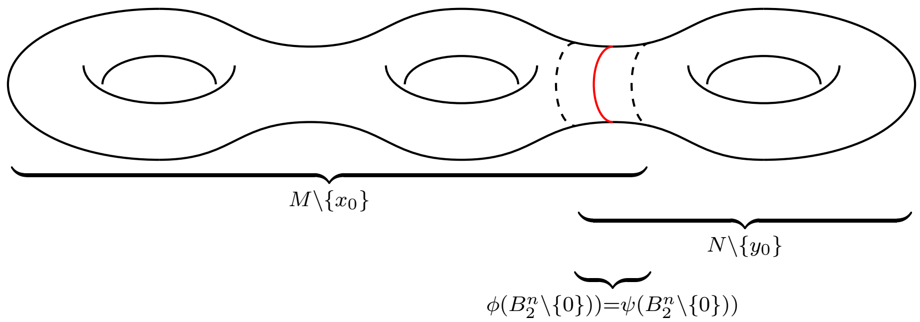
For the construction of the connected sum we pick points $x_0 \in M$, $y_0 \in N$ and maps $\phi \colon B^n_2 \to M$, $\psi \colon B^n_2 \to N$ which are are homeomorphisms onto their image with $\phi(0) = x_0$, $\psi(0) = y_0$; here $B^n_2 = \{v \in \mathbb{R}^n \mid ||v|| < 2\} \subset \mathbb{R}^n$ is the open ball of radius 2. Let α be the homeomorphism

$$\alpha \colon S^{n-1} \times (-1,1) \xrightarrow{\approx} B^n_2 \setminus \{0\} \qquad \text{given by} \qquad (v,t) \mapsto (1-t)v,$$

and let $g \colon S^{n-1} \times (-1,1) \xrightarrow{\approx} S^{n-1} \times (-1,1)$ be the homeomorphism given by $g(v,t) = g(v,-t)$. Let $M\#N$ be the space determined by the pushout diagram

$$\begin{array}{ccc} S^{n-1} \times (-1,1) & \xrightarrow{\psi\circ\alpha} & N \setminus \{y_0\} \\ {\scriptstyle \phi\circ\alpha\circ g}\downarrow & & \downarrow \\ M \setminus \{x_0\} & \longrightarrow & M\#N. \end{array}$$

In other words, $M\#N = (M \setminus \{x_0\}) \cup_{S^{n-1}\times(-1,1)} (N \setminus \{y_0\})$ is obtained from the disjoint union $(M \setminus \{x_0\}) \amalg (N \setminus \{y_0\})$ by identifying the point $\phi \circ \alpha \circ g(v,t) \in M \setminus \{x_0\}$ with the point $\psi \circ \alpha(v,t) \in N \setminus \{y_0\}$ for $(v,t) \in S^{n-1} \times (-1,1)$. Here is a picture of $M\#N$, where the red circle is the image of $S^{n-1} \times \{0\} \subset S^{n-1} \times (-1,1)$ under either map in the commutative diagram above.



(a) Determine the fundamental group of $M \setminus \{x_0\}$ in terms of the fundamental group of M. Hint: use the Seifert van Kampen Theorem.

(b) Determine the fundamental group of $M \# N$ in terms of the fundamental groups of M and N.

5. (10 points) Let X be the subspace of $\mathbb{R}^3$ given by the union of the 2-sphere S^2 and the segment S of the x-axis given by $S = \{(t, 0, 0) \in \mathbb{R}^3 \mid -1 \leq t \leq 1\}$. Calculate the fundamental group of X. Hint: use the Seifert van Kampen Theorem.