Homework Assignment # 5, due Sep. 28

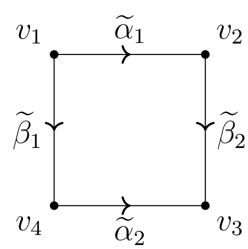
1. (10 points) A subspace $A \subset X$ of a topological space X is called a *retract of* X if there is a continuous map $r\colon X \to A$ whose restriction to A is the identity.

(a) Show that S^1 is not a retract of D^2. Hint: Show that the assumption that there is a continuous map $r\colon D^2 \to S^1$ which restricts to the identity on S^1 leads to a contradiction by contemplating group homomorphisms $r_*\colon \pi_1(D^2, x_0) \to \pi_1(S^1, x_0)$ and $i_*\colon \pi_1(S^1, x_0) \to \pi_1(D^2, x_0)$ induced by the retraction f resp. the inclusion map $i\colon S^1 \hookrightarrow D^2$.

(b) Brouwer's Fixed Point Theorem states that every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e., a point x with $f(x) = x$. Prove this for $n = 2$. Hint: show that if f has no fixed point, then a retraction map $r\colon D^2 \to S^1$ can be constructed out of f.

2. (10 points) The goal of this problem is to construct elements in the fundamental group of the torus T, the Klein bottle K and the projective plane $\mathbb{RP}^2$ and to show that they satisfy certain relations.

(a) Recall that T is homeomorphic to $\Sigma(aba^{-1}b^{-1})$, the quotient of the polygon P_4 with four edges (also known as "square") according using the edge identification determined by the word $aba^{-1}b^{-1}$. Let $p\colon P_4 \to \Sigma(aba^{-1}b^{-1})$ be the projection map. Let $\widetilde{\alpha}_i, \widetilde{\beta}_i\colon I \to P_4$ be the linear edge paths as shown in the the picture below (the edge $\widetilde{\alpha}_1$ is identified with $\widetilde{\alpha}_2$ and $\widetilde{\beta}_1$ is identified with $\widetilde{\beta}_2$ to obtain $\Sigma(aba^{-1}b^{-1}) \approx P_4/\sim$)



Let $\alpha = p \circ \widetilde{\alpha}_i$, $\beta = p \circ \widetilde{\beta}_i$ be the based loops in $(\Sigma(aba^{-1}b^{-1}), v)$, with $v := p(v_i) \in \Sigma(aba^{-1}b^{-1})$, and let $a := [\alpha]$, $b := [\beta]$ be the elements of the fundamental group $\pi_1(\Sigma(aba^{-1}b^{-1}), v)$ represented by these based loops. Show that these elements satisfy the relation $aba^{-1}b^{-1} = 1 \in \pi_1(\Sigma(aba^{-1}b^{-1}), v)$.

(b) Similarly, construct elements a, b in the fundamental group of the Klein bottle $K \approx \Sigma(aba^{-1}b)$ and show that they satisfy the relation $aba^{-1}b = 1$.

(c) Similarly, construct an element a in the fundamental group of the real projective plane $\mathrm{RP}^2 \approx \Sigma(aa)$ and show that it satisfies the relation $a^2 = 1$.

(d) Recall that $\mathbb{RP}^2$ is also homeomorphic to $\Sigma(abab)$. Can we use the same techniques as above to construct elements a, b in the fundamental group of $\Sigma(abab)$ which satisfy the relation $abab = 1$? If yes, construct these elements and prove the relation; if no, explain the difference to the previous cases.

3. (10 points) Two topological spaces X, Y are *homotopy equivalent* if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f\colon X \to X$ is homotopic to id_X and $f \circ g\colon Y \to Y$ is homotopic to id_Y. Show that the following five topological spaces are all homotopy equivalent:

(1) the circle S^1,

(2) the open cylinder $S^1 \times \mathbb{R}$,

(3) the annulus $A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$,

(4) the solid torus $S^1 \times D^2$,

(5) the Möbius strip

Hint: A subspace $A \subset X$ is a *retract of* X if there is map $r\colon X \to A$ which restricts to the identity on A. It is a *deformation retract* of X if in addition the composition $X \xrightarrow{r} A \overset{i}{\hookrightarrow} X$ with the inclusion map i is homotopic to the identity on X. Note that if A is a deformation retract of X, then $r \circ i = \mathrm{id}_A$ and $i \circ r \sim \mathrm{id}_X$. In particular, A is homotopy equivalent to X. Show that each of the spaces (2)-(5) contains a subspace S homeomorphic to the circle S^1 which is a deformation retract of the bigger space it is contained in.

4. (10 points) Let $f\colon (S^1, 1) \to (S^1, 1)$ be the basepoint preserving map defined by $f(z) = z^n$ for some $n \in \mathbb{Z}$ and let

$$f_*\colon \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$$

be the induced homomorphism on the fundamental group. Calculate explicitly the group homomorphism f_*. By this, we mean the following: the winding number gives an explicit isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$. Via this isomorphism, the automorphism f_* of the group $\pi_1(S^1, 1)$ corresponds to an automorphism of the group $\mathbb{Z}$. Any automorphism of $\mathbb{Z}$ is of the form $\mathbb{Z} \to \mathbb{Z}$, $m \mapsto km$, i.e., is given by multiplication by some integer $k \in \mathbb{Z}$. In other words, "calculate explicitly" means "determine the integer $k \in \mathbb{Z}$ such that the following diagram commutes":

$$\begin{tikzcd}
\pi_1(S^1, 1) \arrow[r, "f_*"] \arrow[d, "W"', "\cong"] & \pi_1(S^1, 1) \arrow[d, "\cong"', "W"] \\
\mathbb{Z} \arrow[r, "k"] & \mathbb{Z}
\end{tikzcd}$$

5. (10 points) A *d-fold covering map* is a covering map $p\colon \widetilde{X} \to X$ such that for each point $x \in X$, the fiber $p^{-1}(x)$ consists of d points.

(a) Let X be compact 2-manifold and let $p\colon \widetilde{X} \to X$ be a d-fold covering map. Show that $\chi(\widetilde{X}) = d \cdot \chi(X)$. Hint: Choose a pattern of polygons Γ on X such that each polygon is contained in some evenly covered subset $U \subset X$. Argue that Γ determines a compatible pattern of polygons $\widetilde{\Gamma}$.

(b) Let $\widetilde{X} \to X$ is a d-fold covering of orientable compact connected 2-manifolds. Give a formula expressing the genus $\widetilde{g}$ of $\widetilde{X}$ in terms of the genus g of X.