Homework Assignment # 4, due Sept. 21

1. (10 points) Let $\alpha, \beta, \gamma\colon I \to X$ be paths in a topological space X. Assume that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$ which guarantees that the concatenated paths $\alpha*(\beta*\gamma)$ and $(\alpha*\beta)*\gamma$ can be formed. Show that these two paths are homotopic (relative endpoints). Verifying this shows that if α, β, γ are loops based at $x_0 \in X$ representing elements $a = [\alpha]$, $b = [\beta]$, $c = [\gamma]$ in $\pi_1(X; x_0)$, then $a(bc) = (ab)c$. In other words, this proves associativity of multiplication in $\pi_1(X; x_0)$, one of the last things to verify in order to prove that $\pi_1(X; x_0)$ is indeed a group.

Hint: Show that both paths can be written as the composition $\Psi \circ \phi$ of a suitable map $\phi\colon I \to [0, 3]$ and the map

$$[0,3] \xrightarrow{\Psi} X \qquad \text{defined by} \quad \Psi(s) := \begin{cases} \alpha(s) & 0 \le s \le 1 \\ \beta(s-1) & 1 \le s \le 2 \\ \gamma(s-2) & 2 \le s \le 3 \end{cases}$$

Avoid writing down explicit homotopies; instead use the handy fact that any two paths with the same endpoints in a convex subset of $\mathbb{R}^n$ are homotopic relative endpoints.

2. (10 points) Let X be a topological space and let β be a path from x_0 to x_1. Show that the map

$$\Phi_\beta\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \qquad [\gamma] \mapsto [\bar\beta * \gamma * \beta]$$

is an isomorphism of groups. In particular, the isomorphism class of the fundamental group $\pi(X, x_0)$ of a path connected space does not depend on the choice of the base point $x_0 \in X$. Hint: Recall from class that for any path γ in X, there are homotopies

$$\gamma * \bar\gamma \simeq c_{\gamma(0)} \qquad \bar\gamma * \gamma \simeq c_{\gamma(1)} \qquad c_{\gamma(0)} * \gamma \simeq \gamma, \qquad \gamma * c_{\gamma(1)} \simeq \gamma$$

where c_x for $x \in X$ denotes the constant path at x.

3. (10 points) A *pointed topological space* is a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. Let (X, x_0), (Y, y_0) be pointed topological spaces and let $f\colon (X, x_0) \to (Y, y_0)$ be a basepoint preserving map, i.e., a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$.

(a) Show that the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $f_*([\gamma]) = [f \circ \gamma]$ is a well-defined.

(b) Show that f_* is a group homomorphism.

The map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is called the *homomorphism of fundamental groups induced by f*.

4. (10 points) Let (X, x_0), (Y, y_0) be pointed topological spaces. Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to the Cartesian product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of (X, x_0) and (Y, y_0).

Hint: use the base point preserving projection maps $p^X\colon X \times Y \to X$, $p^Y\colon X \times Y \to Y$, and the induced homomorphisms (see the previous problem)

$$p^X_*\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \qquad p^Y_*\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0).$$

5. (10 points) The goal of this problem is to show that the winding number map

$$W\colon \pi_1(S^1, 1) \to \mathbb{Z}$$

is a group isomorphism. We recall that for a based loop γ in $(S^1, 1)$, the winding number $W(\gamma) \in \mathbb{Z}$ is defined by $W(\gamma) := \widetilde{\gamma}(1) \in \mathbb{Z}$, where $\widetilde{\gamma}\colon I \to \mathbb{R}$ is the unique lift of $\gamma\colon I \to S^1$ (i.e., $p \circ \widetilde{\gamma} = \gamma$) with starting point $\widetilde{\gamma}(0) = 0 \in \mathbb{R}$. We assume that W is well-defined, i.e., that the winding number $W(\gamma)$ of a based loop γ depends only on the homotopy class of γ relative endpoints (which will be proved in class on Tuesday, 9-19). Let

$$\Phi\colon \mathbb{Z} \longrightarrow \pi_1(S^1, 1) \qquad \text{be defined by} \qquad \Phi(n) := [\gamma_n],$$

where $\gamma_n\colon I \to (S^1, 1)$ is the based loop $\gamma_n(s) = e^{2\pi i n s}$.

(a) Show that the composition $\mathbb{Z} \xrightarrow{\Phi} \pi_1(S^1, 1) \xrightarrow{W} \mathbb{Z}$ is the identity.

(b) Show that the composition $\pi_1(S^1, 1) \xrightarrow{W} \mathbb{Z} \xrightarrow{\Phi} \pi_1(S^1, 1)$ is the identity. Hint: If $[\gamma] \in \pi_1(S^1, 1)$ and $\Phi(W([\gamma])) = [\gamma']$, you need to show that $\gamma \sim \gamma'$. Let $\widetilde{\gamma}, \widetilde{\gamma}'\colon I \to \mathbb{R}$ be the unique lifts of γ (resp. γ') with $\widetilde{\gamma}(0) = 0 = \widetilde{\gamma}'(0)$. Try to show that $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$. Using the fact that paths with the same endpoints in a convex subspace are homotopic relative endpoints, why does this imply $\gamma \sim \gamma'$?

(c) Show that W is a group homomorphism.