Homework Assignment # 3, due Sept. 14, 2023

1. (10 points) Show that the complex projective space $\mathbb{CP}^1$ is homeomorphic to the 2-sphere S^2. Hint: recall that $\mathbb{CP}^1$ is a quotient of $\mathbb{C}^2 \setminus \{0\}$ and hence a point of $\mathbb{CP}^1$ is an equivalence class $[z_0, z_1]$ of elements $(z_0, z_1) \in \mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\})$. Construct a bijection f between $\mathbb{CP}^1$ with the point $[0,1]$ removed and $\mathbb{C}$. Compose the map f with the map $g\colon \mathbb{C} = \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\}$ which is the inverse of the stereographic projection map (see the formula from the previous homework set). Simplify the explicit formula for $g \circ f\colon \mathbb{CP}^1 \setminus \{[0,1]\} \longrightarrow S^2 \setminus \{(0,0,1)\}$ to show that it extends to a continuous bijection between $\mathbb{CP}^1$ and S^2.

2. (10 points) Which of the topological groups $GL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are connected? Hint: To show that one of these topological groups is connected, it might be easier to show that it is path-connected. Note that to prove this, it suffices to find a path connecting any element with the identity element (why?). Use without proof the fact that every element in $SO(n)$ (the group of linear maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which are isometries with determinant one) for a suitable choice of basis of $\mathbb{R}^n$ is represented by a matrix of block diagonal form whose diagonal blocks are the 1×1 matrix with entry $+1$ and/or 2×2 rotational matrices

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Here "block diagonal" means that all other entries are zero.

3. (10 points) The definition of a manifold involves the technical conditions of being Hausdorff and second countable. Show that these properties are "inherited" by subspaces in the following sense. Let X be a topological space and A a subspace.

(a) Show that if X is Hausdorff, then so is A.

(b) Show that if X is second countable, then so is A.

4. (10 points) Let Σ, Σ' be compact connected 2-manifolds. Show that the Euler characteristic of the connected sum $\Sigma\#\Sigma'$ is given by the following formula:

$$\chi(\Sigma\#\Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2.$$

In your proof, do **not** use the Classification Theorem for compact connected 2-manifolds, or the statement $\Sigma(W_1)\#\Sigma(W_2) \approx \Sigma(W_1 W_2)$ we'll prove in class on Tuesday.

5. (10 points) By the classification theorem for compact connected 2-manifolds, the connected sum $T\#T\#K\#K\#K$ of two copies of the torus T and three copies of the Klein bottle K is homeomorphic to exactly one of the manifolds Σ_g (the surface of genus $g \geq 0$) or X_k (the connected sum of k copies of the real projective plane $\mathbb{RP}^2$). Which one is it? (provide detailed arguments!).