Homework Assignment # 2, due Sept. 7, 2023

1. (10 points) Show that a closed subspace C of a compact topological space X is compact.

2. (10 points) Let X be a topological space which is the union of two subspaces X_1 and X_2. Let $f\colon X \to Y$ be a (not necessarily continuous) map whose restriction to X_1 and X_2 is continuous.

(a) Show f is continuous if X_1 and X_2 are open subsets of X.

(b) Show f is continuous if X_1 and X_2 are closed subsets of X.

(c) Give an example showing that in general f is *not continuous*.

3. (10 points) Use the Heine-Borel Theorem to decide which of the topological groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$ are compact. Provide proofs for your statements. Hint: A strategy often useful for proving that a subset C of $\mathbb{R}^n$ is closed is to show that C is of the form $f^{-1}(C')$ for some closed subset $C' \subset \mathbb{R}^k$ (often C' consists of just one point) and some continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^k$.

4. (10 points) Let M be a manifold of dimension m and let N be a manifold of dimension n. Show that the product $M \times N$ is a manifold of dimension $m+n$. Don't forget to check the technical conditions (Hausdorff and second countable) for $M \times N$.

5. (10 points) Show that the real projective space $\mathbb{RP}^n$ is a manifold of dimension n. Don't forget to check that $\mathbb{RP}^n$ is second countable (we have proved in class that the projective space is Hausdorff). Hint: to prove that $\mathbb{RP}^n$ is locally homeomorphic to $\mathbb{R}^n$ suitably modify the method we used for the sphere S^n. For showing that $\mathbb{RP}^n$ is second countable, recall from class that if X is second countable, and $p\colon X \twoheadrightarrow Y$ is an open surjection, then Y is second countable.