Homework Assignment # 11, due Nov. 30

1. (10 points) Let V be a vector space of dimension n with basis $\{b_1, \dots, b_n\}$, and let $\{b^1, \dots, b^n\}$ be the dual basis for the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$.

(a) Let $\mathrm{Mult}^k(V, \mathbb{R})$ be the space of multilinear maps $\omega\colon \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$, and for a multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_s \le n$ let $b^I \in \mathrm{Mult}^k(V, \mathbb{R})$ be defined by

$$b^I(v_1, \dots, v_k) := b^{i_1}(v_1) \cdots b^{i_k}(v_k).$$

Show that the elements b^I form a basis for $\mathrm{Mult}^k(V, \mathbb{R})$. Hint: argue that a k-linear form $\omega \in \mathrm{Mult}^k(V, \mathbb{R})$ is determined by the numbers $\omega(b_J) \in \mathbb{R}$ obtained by evaluating it on the k-tupels $b_J := (b_{j_1}, \dots, b_{j_k})$ with $J = (j_1, \dots, j_k)$ and $1 \le j_s \le n$.

(b) What is the dimension of $\mathrm{Mult}^k(V, \mathbb{R})$?

(c) Let $\mathrm{Alt}^k(V, \mathbb{R})$ be the space of alternating multilinear maps $\omega\colon V^{\times k} \to \mathbb{R}$, and for a multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_s \le n$ let $b^{\wedge I} \in \mathrm{Alt}^k(V, \mathbb{R})$ be defined by

$$b^{\wedge I}(v_1, \dots, v_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) b^{i_1}(v_{\sigma(1)}) \cdots b^{i_k}(v_{\sigma(k)}).$$

Show that $b^{\wedge I}(b_J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$ for multi-indices I, J which are *increasing*, i.e., $1 \le i_1 < i_2 < \cdots < i_k \le n$.

(d) Show that the elements $b^{\wedge I}$ for increasing multi-indices I, form a basis for $\mathrm{Alt}^k(V, \mathbb{R})$. Hint: argue that an alternating form $\omega \in \mathrm{Alt}^k(V, \mathbb{R})$ is determined by the numbers $\omega(b_J) \in \mathbb{R}$ obtained by evaluating it on the k-tupels $b_J := (b_{j_1}, \dots, b_{j_k})$ for an increasing multi-index J. As noted in class, this in particular implies that the dimension of $\mathrm{Alt}^k(V, \mathbb{R})$ is $\binom{n}{k}$.

Note: in class we wrote $b^{i_1} \wedge \cdots \wedge b^{i_k}$ for the element $b^I \in \mathrm{Alt}^k(V, \mathbb{R})$ defined above. This was not a good idea, since it is not clear that b^I is in fact equal to the wedge product of the b^{i_s}, which we defined later in class. This is in fact true, as will be proved as part of the next problem.

2. (10 points) We recall that the *wedge product*

$$\mathrm{Alt}^k(V, \mathbb{R}) \times \mathrm{Alt}^\ell(V, \mathbb{R}) \xrightarrow{\wedge} \mathrm{Alt}^{k+\ell}(V, \mathbb{R})$$

is a bilinear associative product defined by

$$(\omega \wedge \eta)(v_1, \dots, v_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sign}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

for $\omega \in \mathrm{Alt}^k(V, \mathbb{R})$, $\eta \in \mathrm{Alt}^\ell(V, \mathbb{R})$, $v_1, \dots, v_{k+\ell} \in V$.

(a) Show that

$$(b^{i_1} \wedge \cdots \wedge b^{i_k})(b_J) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

where $I = (i_1, \ldots, i_k)$, $J = (j_1, \ldots, j_k)$ are increasing sequence, and we use the same notation and terminology as in problem (1). Hint: Use induction over k.

This shows in particular that the alternating k-form $b^I \in \mathrm{Alt}^k(V, \mathbb{R})$ from 1(c) is in fact equal to $b^{i_1} \wedge \cdots \wedge b^{i_k}$.

(b) Show that the wedge product is *graded commutative*, i.e.,

$$\eta \wedge \omega = (-1)^{k\ell} \omega \wedge \eta \qquad \text{for } \omega \in \mathrm{Alt}^k l(V, \mathbb{R}),\ \eta \in \mathrm{Alt}^\ell(V, \mathbb{R}).$$

Hint: First consider the case $k = \ell = 1$, then argue that it suffices to prove the statement in the case $\omega = b^{i_1} \wedge \cdots \wedge b^{i_k}$, $\eta = b^{j_1} \wedge \cdots \wedge b^{j_\ell}$.

3. (10 points) Let M, N be smooth manifolds and $F\colon M \to N$ a smooth map. Then a differential form $\omega \in \Omega^k(N)$ leads to a form $F^*\omega \in \Omega^k(M)$, called the *pullback of* ω *along* F which is defined by

$$(F^*\omega)_p(v_1, \ldots, v_k) := \omega_p(DF_p(v_1), \ldots, DF_p(v_k)) \qquad \text{for } p \in M,\ v_1, \ldots, v_k \in T_pM.$$

In more detail: the k-form $F^*\omega$ is a section of the vector bundle $\mathrm{Alt}^k(TM; \mathbb{R})$, and hence it can be evaluated at $p \in M$ to obtain an element $(F^*\omega)_p$ in the fiber of that vector bundle over p, which is $\mathrm{Alt}^k(T_pM; \mathbb{R})$. In other words, $(F^*\omega)_p$ is an alternating multilinear map

$$(F^*\omega)_p\colon \underbrace{T_pM \times \cdots \times T_pM}_{k} \longrightarrow \mathbb{R},$$

and hence it can be evaluated on the k tangent vectors $v_1, \ldots, v_k \in T_pM$ to obtain a real number $(F^*\omega)_p(v_1, \ldots, v_k)$. On the right hand side of the equation defining $F^*\omega$, the map $DF_p\colon T_pM \to T_{F(p)}N$ is the differential of F. Hence the alternating multilinear map $\omega_{F(p)} \in \mathrm{Alt}^k(T_{F(p)}N; \mathbb{R})$ can be evaluated on $DF_p(v_1), \ldots, DF_p(v_k)$ to obtain the real number $\omega_p(DF_p(v_1), \ldots, DF_p(v_k))$.

Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that

$$F^*(dx^1 \wedge \cdots \wedge dx^n) = \det(DF)\ dx^1 \wedge \cdots \wedge dx^n \tag{0.1}$$

Here $DF\colon \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is the differential of DF, which maps $x \in \mathbb{R}^n$ to $DF_x\colon \mathbb{R}^n \to \mathbb{R}^n$, the differential of F at the point $x \in \mathbb{R}^n$. Hint: Evaluated at a point $x \in \mathbb{R}^n$ both sides of the equation are vectors of the 1-dimensional vector space $\mathrm{Alt}^n(T_x\mathbb{R}^n, \mathbb{R}) = \mathrm{Alt}^n(\mathbb{R}^n, \mathbb{R})$.

Hence it suffices to show equality after evaluating both sides on $(e_1, \dots, e_n)$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

4. (10 points) For any smooth manifold M the *de Rham differential* (also called *exterior differential*) is the unique map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) d is linear.

(ii) For a function $f \in C^\infty(M) = \Omega^0(M)$ the 1-form $df \in \Omega^1(M) = C^\infty(M, T^*M)$ is the usual differential of f.

(iii) d is a graded derivation with respect to the wedge product; i.e.,

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \text{for } \omega \in \Omega^k(M),\ \eta \in \Omega^l(M).$$

(iv) $d^2 = 0$.

We recall that for $M = \mathbb{R}^n$, every k-form $\eta \in \Omega^k(\mathbb{R}^n)$ can be written uniquely in the form

$$\eta = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for smooth functions $f_{i_1, \dots, i_k} \in C^\infty(\mathbb{R}^n)$. The point of this problem is to give an *explicit* formula for d for $M = \mathbb{R}^n$ (which works equally well locally, on a coordinate patch of a smooth n-manifold).

(a) Show that for $f \in C^\infty(\mathbb{R}^n)$ the differential $df \in \Omega^1(\mathbb{R}^n)$ is given by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

(b) Show that for $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(\mathbb{R}^n)$

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

5. (10 points) Show that the exterior derivative for differential forms on $\mathbb{R}^3$ corresponds to the classical operations of *gradient* resp. *curl* resp. *divergence*. More precisely, show that there is a commutative diagram

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathsf{Vect}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & \mathsf{Vect}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\
\Vert & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

Here $\mathsf{Vect}(\mathbb{R}^3)$ is the space of vector fields on $\mathbb{R}^3$, and we recall that grad, curl and divergence are given by the formulas

$$\begin{aligned}
\mathrm{grad}(f) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\mathrm{curl}(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
\mathrm{div}(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}
\end{aligned}$$

Here we identify a vector field on $\mathbb{R}^3$ with a triple (f_1, f_2, f_3) of smooth functions on $\mathbb{R}^3$. The vertical isomorphisms are given by

$$\begin{aligned}
(f_1, f_2, f_3) &\mapsto f_1 dx + f_2 dy + f_3 dz \\
(f_1, f_2, f_3) &\mapsto f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \\
f &\mapsto f dx \wedge dy \wedge dz
\end{aligned}$$