Homework Assignment # 10, due Nov. 16

1. (10 points) Let M be a smooth manifold of dimension n. If $f\colon M \to \mathbb{R}$ is a smooth function, then for $p \in M$ its differential

$$df_p\colon T_pM \longrightarrow T_{f(p)}\mathbb{R} = \mathbb{R}$$

is an element of $\mathrm{Hom}(T_pM, \mathbb{R})$. This vector space dual to the tangent space T_pM is called the *cotangent space*, and is denoted T_p^*M.

(a) Let $x^i\colon \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate function, which maps $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ to $x_i \in \mathbb{R}$. Show that for any point $q \in \mathbb{R}^n$ a basis of the cotangent space $T_q^*\mathbb{R}^n$ is given by $\{dx_q^i\}_{i=1,\dots,n}$.

(b) If $M \supset U \xrightarrow{\phi} V \subset \mathbb{R}^n$ is a smooth chart of M, the component functions of ϕ, given by $y^i := x^i \circ \phi$ are called *local coordinates*. Show that for $p \in U$, a basis of the cotangent space T_p^*M is given by $\{dy_p^i\}_{i=1,\dots,n}$.

Hint for part (b): let $(D\phi_p)^*\colon T_q^*\mathbb{R}^n \to T_p^*M$, $q = \phi(p)$ be the linear map dual to the differential $D\phi_p\colon T_pM \to T_q\mathbb{R}^n$ defined by

$$(D\phi_p)^*(\xi)(v) = \xi(D\phi_p(v)) \qquad \text{for } \xi \in T_q^*\mathbb{R}^n \text{ and } v \in T_pM.$$

Show first that $(D\phi_p)^*(dx_q^i) = dy_p^i$.

2. (10 points) Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Show that the differential df is a smooth section of the cotangent bundle T^*M. Hint: smoothness of a section s is a local property and hence to check smoothness it suffices to check that the composition $\Phi_\alpha \circ s$ is smooth for local trivializations Φ_α of the cotangent bundle T^*M.

3. (10 points) We recall that the projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n whose underlying set is the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. In particular, each point $p \in \mathbb{RP}^n$ determines tautologically a 1-dimensional subspace $E_p \subset \mathbb{R}^{n+1}$. Let E be the disjoint union $E = \amalg_{p \in \mathbb{RP}^n} E_p$ of the vector spaces E_p. More explicitly,

$$E = \{([x], v) \mid [x] \in \mathbb{RP}^n, v \in \langle x \rangle\},$$

where $x \in \mathbb{R}^{n+1} \setminus \{0\}$, $\langle x \rangle \subset \mathbb{R}^{n+1}$ is the one-dimensional subspace spanned by x, and $[x] \in \mathbb{RP}^n$ is the corresponding point in the projective space.

(a) Use the Vector Bundle Construction Lemma to show that E is a smooth vector bundle of rank 1 over $\mathbb{RP}^n$ (which is called the *tautological line bundle over* $\mathbb{RP}^n$; *line bundle* is a synonym for *vector bundle of rank* 1). Hint: Construct local trivializations of E restricted to $U_i = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$.

(b) Show that the complement of the zero section in E is diffeomorphic to $\mathbb{R}^{n+1} \setminus \{0\}$.

(c) Show that the line bundle E is not isomorphic to the trivial line bundle. Hint: consider the complement of the zero-section of E and compare it with the complement of the zero-section of the trivial line bundle.

4. (10 points) The goal of this problem is to prove the following construction lemma for vector bundles over topological spaces.

Lemma 0.1. *Let M be a topological space, and let $\{E_p\}$ be a collection of vector spaces parametrized by $p \in M$. Let E be the set given by the disjoint union of all these vector spaces, which we write as*

$$E := \coprod_{p \in M} E_p = \{(p, v) \mid p \in M,\ v \in E_p\}$$

and let $\pi\colon E \to M$ be the projection map defined by $\pi(p, v) = p$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M, and let for each $\alpha \in A$, let $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ be maps with the following properties

(i) *The diagram*

$$\begin{array}{ccc} E_{|U_\alpha} := \pi^{-1}(U_\alpha) & \xrightarrow{\quad\Phi_\alpha\quad} & U_\alpha \times \mathbb{R}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U_\alpha & \end{array} \tag{0.2}$$

is commutative, where π_1 is the projection onto the first factor.

(ii) *For each $p \in U_\alpha$, the restriction of Φ_α to $E_p = \pi^{-1}(p)$ is a vector space isomorphism between E_p and $\{p\} \times \mathbb{R}^k = \mathbb{R}^k$ (which implies that Φ_α is a bijection).*

(iii) *For $\alpha, \beta \in A$, the composition*

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\Phi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\Phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^k \tag{0.3}$$

is continuous.

Then the total space E can be equipped with a topology such that $\pi\colon E \to M$ is a topological vector bundle of rank k with local trivializations Φ_α.

(a) Construct a topology on E by declaring $U \subset E$ to be *open* if $\Phi_\alpha(U \cap E_{|U_\alpha})$ is an open subset of $U_\alpha \times \mathbb{R}^k$ for all $\alpha \in A$. Show that this satisfies the conditions for a topology.

(b) Show that with this topology on E the projection map $\pi\colon E \to M$ is continuous

(c) Show that the map Φ_α is a homeomorphism (for the subspace topology on $E_{|U_\alpha}$). That implies that (U_α, Φ_α) is a bundle atlas for $\pi\colon E \to M$ and which then finishes the proof of the Lemma.

5. (10 points) The goal of this problem is to prove the vector bundle construction lemma from class. We recall that this has slightly stronger assumptions than the lemma of the previous problem, namely M is required to be a smooth manifold, and the transition maps (0.3) are required to be smooth.

(a) Show that E, equipped with the topology E constructed in the previous problem, is a topological manifold of dimension $n+k$ (don't bother to check the technical conditions of being Hausdorff and second countable). Hint: Let $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ be an atlas for M. Show that the bundle chart Φ_α and the manifold chart ψ_β can be used to construct a chart

$$\chi_{\alpha,\beta}\colon E \underset{\text{open}}{\supset} E_{|U_\alpha\cap V_\beta} \longrightarrow \mathbb{R}^{n+k}.$$

(b) Show that the charts $\{(E_{|U_\alpha\cap V_\beta}), \chi_{\alpha,\beta})\}$ for $(\alpha,\beta)\in A\times B$ form a smooth atlas for E.

(c) Show that $\pi\colon E \to M$ is a smooth vector bundle of rank k with local trivializations provided by Φ_α.