Homework Assignment # 1, due Aug. 31, 2023

1. (10 points) Let $GL_n(\mathbb{R})$ be the set of invertible $n \times n$ matrices.

(a) Show that $GL_n(\mathbb{R})$ is an open subset of the topological space $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ of all $n \times n$ matrices. Hint: use the map $A \mapsto \det(A)$.

(b) Show that the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is a continuous map.

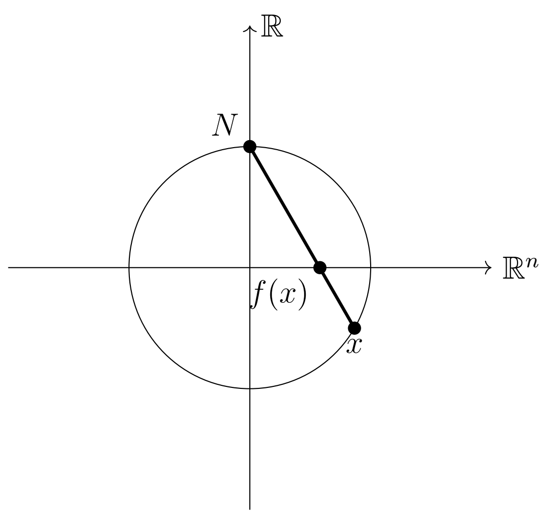
2. (10 points) The point of this problem is to show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology (where each factor is equipped with the metric topology). Since both, the metric topology and the product topology, are defined via a basis, it is good to know how to compare two topologies given in terms of bases. This is provided by the statement of part (a).

(a) Let X be a set, and let $\mathcal{T}$, $\mathcal{T}'$ be topologies generated by a basis $\mathcal{B}$ resp. $\mathcal{B}'$. Show that $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $B \in \mathcal{B}$ and $x \in B$ there is some $B' \in \mathcal{B}'$ with $x \in B'$ and $B' \subset B$.

(b) Show that the products of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^m \times \mathbb{R}^n$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $s, r > 0$ form a basis for the product topology on $\mathbb{R}^m \times \mathbb{R}^n$.

(c) Show that the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ agrees with the product topology. Hint: it might be helpful to draw pictures of a ball around $(x, y) \in \mathbb{R}^{m+n}$ and a product of balls $B_r(x) \times B_s(y) \subset \mathbb{R}^{m+n}$ for $m = n = 1$.

3. (10 points) Let $N \in S^n$ be the "north pole" of S^n, i.e., $N = (0, \ldots, 0, 1) \in S^n$. The *stereographic projection* is the map $f\colon S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$ which sends a point $x \in S^n \setminus \{N\}$ to the intersection point of the straight line L_x in $\mathbb{R}^{n+1}$ with endpoint N and x with $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Here is a picture of the situation for $n = 1$.



The map $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a bijection. Explicitly, the map f and its inverse are given by the explicit formulas

$$f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \qquad f^{-1}(y_1, \ldots, y_n) = \frac{1}{||y||^2 + 1}(2y_1, \ldots, 2y_n, ||y||^2 - 1)$$

for $(x_1, \dots, x_{n+1}) \in S^n$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$. Provide a careful argument for the continuity of f and f^{-1} (you can use freely that recognize that certain maps $\mathbb{R} \supset U \to \mathbb{R}$ are continuous, but each time you use one of our "continuity criteria" for maps involving sub-spaces, products and quotients, you should be explicit about it).

4. (10 points) Do the first step towards proving that the quotient space D^n/S^{n-1} is homeomorphic to the sphere S^n by constructing a continuous bijection from one of these spaces to the other (a result we'll cover in class next week will make it easy to conclude that this is in fact a homeomorphism). Hint: produce a bijective map f relating these spaces by writing down an explicit formula, paying attention to have this map go the "natural direction" to make proving its continuity simple.

5. (10 points) Consider the following topological spaces

- The subspace $T_1 := \{v \in \mathbb{R}^3 \mid \operatorname{dist}(v, K) = r\} \subset \mathbb{R}^3$ equipped with the subspace topology, where $K = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $0 < r < 1$.
- The product space $T_2 := S^1 \times S^1$ equipped with the product topology.
- The quotient space $T_3 := ([-1, 1] \times [-1, 1])/\sim$ equipped with the quotient topology, where the equivalence relation is generated by $(s, -1) \sim (s, 1)$ and $(-1, t) \sim (1, t)$.

Construct two bijective continuous maps between these spaces such that each of these three spaces features in at least one of these (once we have the convenient continuity criterion for the inverse of a continuous bijection, this easily implies that these three spaces are all homeomorphic). Hint: as in the previous problem, pick your maps to go in a direction that makes it easy to verify continuity using the Continuity Criterions for maps to/from subspaces, product spaces resp. quotient spaces.