Recall: $S : X \to \mathbb{R}$ action functional

classical paradigm: The only fields $\phi \in X$ that occur in nature belong to $\mathrm{Crit}(S) = \{\text{critical points of } S\} \subset X$

quantum paradigm: all fields $\phi \in X$ are possible in nature; ϕ further away from $\mathrm{Crit}(S)$ are just less likely.

quantity of interest in QFT (= Quantum Field Theory):

Expectation value $\langle f \rangle := \underset{\phi \in X}{\mathrm{Average}}\, f(\phi)$ for $f \in \mathcal{O}(X)$

mathematically, we need a measure μ on X

$$\boxed{\langle f\rangle_\mu := \frac{1}{Z_\mu}\int_X f(\phi)\,\mu\ , \qquad \mu = e^{-S/\hbar}\underbrace{\mathcal{D}\phi}_{\mu_0} \text{ "Lebesgue measure"}}$$

$$Z_\mu = \int_X \mu$$

probability measure $\frac{\mu}{Z_\mu}$

<u>note</u>: for $\hbar \to 0$: the measure get more + more concentrated near $\mathrm{Crit}(S)$.

<u>goal of perturbative QFT</u>: make sense of rhs. as a power series in $\hbar$.

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toy example: X is a finite dimensional mfd.

<u>Def</u>: A volume form μ on X^n is $\mu \in \Omega^n(X)$ s.t. $\underset{\neq 0}{\mu(x)} \in \Lambda^n(T^*_x X)$ $\forall\, x \in X$

<u>note</u>: μ determines an orientation on X

$b_1, \dots, b_n$ is an <u>oriented</u> basis for $T_x X$ $\iff \mu(b_1, \dots, b_n) > 0$

$\rightsquigarrow$ measure $\hat{\mu}$ defined by $\hat{\mu}(\mathcal{U}) := \int_{\mathcal{U}} \mu$.

Exs of volume forms:

i) $X = \mathbb{R}^n$ $\quad \mu = dx_1 \wedge \dots \wedge dx_n \Rightarrow \hat{\mu} =$ Lebesgue measure

ii) X Riemann. mfd oriented $\rightsquigarrow$ Riem. volume form $\mathrm{vol} \in \Omega^n(X)$

char. by $\mathrm{vol}(\underbrace{b_1, \dots, b_n}_{\text{oriented orthonormal basis of } T_x X}) = 1$

<u>homological approach to integration</u>

X n-mfd, oriented + compact

$$H^n_{dR}(X) = \frac{\ker\left(\Omega^n(X) \xrightarrow{d} \Omega^{n+1}(X)\right)}{\mathrm{im}\left(\underset{\shortparallel}{\Omega^{n-1}(X)} \xrightarrow{d} \Omega^n(X)\right)} \xrightarrow{\cong} \mathbb{R}$$

$$\frac{\Omega^n(X)}{d\,\Omega^{n-1}(X)} \ni [\omega] \longmapsto \int_X \omega$$

$f \in C^\infty(X)$

μ volume form

$$\langle f \rangle_\mu := \frac{\int_X f\mu}{\int_X \mu} = \frac{[f\mu]}{[\mu]} \quad \text{i.e. } [f\mu] = \langle f \rangle_\mu [\mu]$$

de Rham complex:

$$\begin{tikzcd}
C^\infty(X) \arrow[r, "d"] & \Omega^1(X) \arrow[r] & \cdots \arrow[r] & \Omega^{n-2}(X) \arrow[r, "d"] & \Omega^{n-1}(X) \arrow[r, "d"] & \Omega^n(X) \ni f\mu \\
\Gamma(\Lambda^n TX) \arrow[u, "m_\mu", "\cong"'] \arrow[r, "\Delta_\mu"] & \Gamma(\Lambda^{n-1} TM) \arrow[u, "m_\mu", "\cong"'] & \cdots \arrow[r, "\Delta_\mu"] & \Gamma(\Lambda^2 TX) \arrow[u, "m_\mu", "\cong"'] \arrow[r, "\Delta_\mu"] & \Gamma(TX) \arrow[u, "m_\mu", "\cong"'] \arrow[r, "\Delta_\mu"] & C^\infty(X) \ni f \arrow[u, "m_\mu", "\cong"']
\end{tikzcd}$$

Δ_μ makes diagram commutative, i.e. $\Delta_\mu := m_\mu^{-1} \circ d \circ m_\mu$

$$m_\mu : \Gamma(\Lambda^k(TX)) \xrightarrow{\cong} \Omega^{n-k}(X)$$

$$V = V_1 \wedge \dots \wedge V_k \longmapsto \iota_V \mu = \mu(V_1, \dots, V_k, \underbrace{\dots}_{n-k \text{ slots}})$$

V_i vector fields

Δ_μ BV-Laplacian

$$\underbrace{(\Gamma(\Lambda^\bullet(TX)), \Delta_\mu)}_{\text{quantum BV-complex}}$$

$\in H^0_{BV}$

note: for $f \in C^\infty(X)$ $\quad \langle f \rangle_\mu = \dfrac{[f]_{BV}}{[1]_{BV}}$

goal: understand relationship between quantum BV-complex and classical "

Def: V vector field on X, volume form μ

lie der. $\quad L_V \mu = \mathrm{div}_\mu(V)\mu \in \Omega^n(X)$

$\uparrow$ <u>divergence</u> of V w.r.t. μ

<u>Lem:</u> $\Delta_\mu(V) = \mathrm{div}_\mu(V)$ for $V \in \Gamma(TX)$.

<u>Pf:</u>
$$\Delta_\mu(V) = m_\mu^{-1}\circ d \circ m_\mu(V) = m_\mu^{-1} d(\iota_V \mu)$$
$$= m_\mu^{-1}\underbrace{(d\iota_V + \underbrace{\iota_V d}_{0})}_{L_V}\mu = m_\mu^{-1}\,\mathrm{div}_\mu(V)\mu$$
$$= \mathrm{div}_\mu(V) \qquad \square$$

$\mu = e^{-S/\hbar}\mu_0$ Q: how is Δ_μ related to Δ_{μ_0} and S?

<u>Lem:</u> $\Delta_\mu = -\frac{1}{\hbar}\iota_{dS} + \Delta_{\mu_0}$

<u>Pf:</u> $V \in \Gamma(\Lambda^k(TX))$
$$m_\mu(V) = \iota_V \mu = \iota_V e^{-S/\hbar}\mu_0 = e^{-S/\hbar}\iota_V \mu_0 = e^{-S/\hbar} m_{\mu_0}(V)$$

$$\Delta_\mu = m_\mu^{-1}\, d\, m_\mu = e^{S/\hbar} m_{\mu_0}^{-1} d\left(e^{-S/\hbar} m_{\mu_0}\right)$$

$$= e^{S/\hbar} m_{\mu_0}^{-1}\left(-\frac{1}{\hbar} e^{-S/\hbar} \cdot dS\, m_{\mu_0} + e^{-S/\hbar} d\, m_{\mu_0}\right)$$

$$= -\frac{1}{\hbar} dS + \Delta_{\mu_0} \qquad \square$$

<u>Cor</u>: $\Gamma(\Lambda^\bullet(TX), \Delta_\mu)$

multiple by $\hbar^k$ is degree k $\big\downarrow$ $\cong$

$(\Gamma(\Lambda^\bullet(TX)), -\iota_{dS} + \hbar\Delta_{\mu_0})$

$$\boxed{\begin{array}{ll} Obs^q := (\Gamma(\Lambda^\bullet(TX)) \otimes_{\mathbb{R}} \mathbb{R}[[\hbar]],\ -\iota_{dS} + \hbar\Delta_{\mu_0}) & \text{quantum BV complex} \\ Obs^{cl} = (\Gamma(\Lambda^\bullet(TX)),\ -\iota_{dS}) & \text{classical BV complex} \end{array}}$$

$\iota_{ds} : \Gamma(\Lambda^\bullet(TX)) \to \Gamma(\Lambda^\bullet(TX))$ graded derivation

so suffices to define $\iota_{ds}(V) = ds(V)$

$\uparrow$ vector field

Observation: We recover Obs^{cl} from Obs^{q} by setting $\hbar = 0$.

Formal properties of Δ_μ:

(1) $\Delta_\mu^2 = 0$

(2) $$\Delta_\mu(V \wedge W) = \Delta_\mu(V) \wedge W + (-1)^{|V|} V \wedge \Delta_\mu(W) + \{V, W\}$$

$V, W \in \Gamma(\Lambda^\bullet(TX))$