$S : X \longrightarrow \mathbb{R}$

$\uparrow$ action functional

$\uparrow$ space of fields

<u>Algebraic structure on</u> $\mathrm{Obs}^{cl} = \underbrace{\left(\Gamma(X, \Lambda^{\bullet}(TX[1])), -L_{dS}\right)}_{\text{BV- complex of poly vector fields}}$

$= \mathcal{O}(d\mathrm{Crit}(S))$

- dga (differential graded algebra) w.r.t. multiplication of poly vector fields
  - given by the wedge product
- Lie algebra structure $\{\ ,\ \} : \mathrm{Obs}^{cl} \times \mathrm{Obs}^{cl} \longrightarrow \mathrm{Obs}^{cl}$

  graded

  $\uparrow$ Schouten bracket

  $u, v, w \in \Gamma(X, TX[1])$ vector fields

  $\{u, v\} := [u, v] = L_u v$

  <u>note</u>: $\{u, -\}$ has degree 0.

  so $\{\ ,\ \}$ has degree $+1$

$\{u, f\} = L_u f = df(u) = uf$

$f \in C^\infty(X)$

extend $\{u, -\}$ to polyvector fields $V_1 \wedge \dots \wedge V_k$

by requiring $\{u, -\}$ to be a graded derivation, e.g.

$$\{u, V_1 \wedge V_2\} = \{u, V_1\} \wedge V_2 + \underbrace{(-1)^{|\{u,-\}||V_1|}}_{+1} V_1 \wedge \{u, V_2\}$$

extend to $\{u_1 \wedge \dots \wedge u_n, V\}$ by considering $\{V, \underbrace{u_1 \wedge \dots \wedge u_n}_{\text{already defined}}\}$

extend to polyvector fields in the first slot by derivation property.

<u>properties</u>: (a) $\{\,,\,\}$ is a graded Lie bracket of degree $+1$,

(in particular, $|\{x, -\}| = |x| + 1$)

i.e. i) (graded skew-symmetry):

$$\{x, y\} = -(-1)^{(|x|+1)(|y|+1)} \{y, x\}$$

(a) (graded) Jacobi relation:

$\{x, -\}$ is a graded derivation w.r.t. $\{\,,\}$,

i.e. $\{x, \{y,z\}\} = \{\{x,y\},z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x,z\}\}$.

(b) $\{x, -\}$ is a graded derivation w.r.t. $\cdot$, i.e.

$$\{x, y\cdot z\} = \{x,y\}\cdot z + (-1)^{(|x|+1)|y|}\, y\cdot\{x,z\}$$

(c) compatibility of $\{\,,\}$ and differential Q:

$$Q\{x,y\} = \{Qx,y\} + (-1)^{|x|}\{x, Qy\}$$

<u>Def:</u> A dga A with a map $\{\,,\}: A\otimes A \to A$ is a <u>Poisson algebra with bracket of degree +1</u> if it satisfies properties (a), (b), (c).

Q: What kind object is $d\mathrm{Crit}(S)$?

We know: $\mathcal{O}(\underbrace{d\mathrm{Crit}(S)}_{\text{derived critical pts.}}) = \Gamma(X, \Lambda^{\bullet}(TX[1]))$

<u>Recall</u>: $V \to X$ vector bundle $\mathcal{O}(V) = \Gamma(X, S^{\bullet}(V^{\vee}))$

<u>Def</u>: A <u>$\mathbb{Z}$-graded manifold</u> is a mfd X equipped with a $\mathbb{Z}$-graded vector bdl. $V = \bigoplus_{k \in \mathbb{Z}} V_k$

$$\mathcal{O}(V) := \Gamma(X, \mathrm{Sym}^{\bullet}(V^{\vee}))$$

where $V^{\vee}$ and $\mathrm{Sym}^{\bullet}(V^{\vee})$ are the vector bdl operations that in each fiber are what we did before.

$$\Gamma(X, \Lambda^{\bullet}(TX[1])) = \Gamma(X, \mathrm{Sym}^{\bullet}(TX[1])) = \Gamma(X, \mathrm{Sym}^{\bullet}((T^*X)^{\vee}[1]))$$
$$= \Gamma(X, \mathrm{Sym}^{\bullet}((T^*X[-1])^{\vee}))$$

$\underline{\text{Upshot}}$: $d\mathcal{C}rit(S)$ is the $\mathbb{Z}$-graded mfd $T^*X[-1]$ $\overset{\mathcal{O}(T^*X[-1])}{\parallel}$ ← shifted cotangent bdl

$\underline{\text{Recall}}$: $\mathrm{Sym}^\bullet(V) = \mathrm{Sym}^\bullet(V^{ev} \oplus V^{odd}) = \mathrm{Sym}^\bullet(V^{ev}) \otimes \mathrm{Sym}^\bullet(V^{odd})$

$= S^\bullet(V^{ev}) \otimes \Lambda^\bullet(V^{odd})$

Still missing: the differential $Q = -\iota_{dS}$

algebraically: $Q : \mathcal{O}(T^*X[-1]) \circlearrowleft$ derivation of degree $+1$

geometrically: vector field of degree $+1$

$\underline{Q}$: Is this a Hamiltonian vector field, i.e. $Q = X_H = \{H, -\}$ for some function $H \in \mathcal{O}(T^*X[-1])$?

$\underline{A}$: yes, in fact $Q = \{S, -\}$; recall: $Q := -\iota_{dS}$

suffices to check $\quad\quad\quad\quad \overset{\deg +1}{} \quad\quad\quad\quad +2$

$$\{S, u\} \quad \text{for} \quad u \in \mathcal{O}(T^*X[-1])$$

$\uparrow$ vector field

$$\{S, u\} = -\{u, S\} = -L_u S = -dS(u) \doteq Q(u)$$

Quantum observables

<u>classical paradigm</u>: only fields ϕ "in nature" are $\phi \in \mathrm{Crit}(S)$

<u>quantum paradigm (Feynman)</u>: any field is possible, but $\phi \notin \mathrm{Crit}(S)$ is less likely.

<u>Cartoon of a physics experiment</u>:

$\boxed{\text{creation of a field } \phi \in X} \longrightarrow \boxed{\text{measuring a particular observable } f \in \mathcal{O}(X)} \longrightarrow f(\phi) \in \mathbb{R}$

When quantum effects are there, the same experiment leads to different results $f(\phi)$.

What physicists are interested in are expectation values $\langle f \rangle$ = average of $f(\phi)$ over many experiments.