Recall: $S: X \longrightarrow \mathbb{R}$ action functional

↖ space of field (in our "toy case", $\dim X < \infty$)

classical observables: $Obs^{cl} := \underbrace{(\Gamma(\mathrm{Sym}^\bullet(TX[1])), Q)}_{\text{classical BV-complex}}$, $Q = -\iota_{dS} = \{S, \}$

structure of Obs^{cl}: shifted Poisson algebra, i.e

i) Obs^{cl} is a graded algebra with differential Q

ii) Obs^{cl} is a graded Lie algebra with $\{ , \}$ of deg $+1$

($\{ , \}$ ultimately comes from the Liebracket of vectorfields)

+ compatibility relations

quantum observables: $Obs^{q} := (\Gamma(\mathrm{Sym}^\bullet(TX[1]))\otimes\mathbb{R}[[\hbar]], Q + \hbar\Delta_{BV})$

quantum BV-complex

Δ_{BV} = BV-Laplacian

properties of Δ_{BV}: i) $^2 = 0$

ii) $[\Delta_{BV}, Q] = 0$ ($[\Delta_{BV}, Q] = \Delta_{BV}Q + Q\Delta_{BV}$ ← graded commutator

iii) $\Delta_{BV}(ab) = \Delta_{BV}(a)b + (-1)^{|a|}a\Delta_{BV}(b) + \{a, b\}$

(Δ_{BV} corresponds to the de Rham differential via the isomorphism m_{μ_0} from measure μ_0)

Observations: i) A dga $\Rightarrow$ $H^*(A)$ is a graded algebra

e.g. $A = \mathrm{Obs}^{cl}$ with multiplication $[a]\cdot[b] := [a\cdot b]$

$$d \text{ derivation} \Rightarrow \left(\begin{array}{l} da = 0 \\ db = 0 \end{array} \Rightarrow d(ab) = 0 \right)$$

ii) Obs^q is not a dga, since

$$\underset{\substack{\uparrow \\ \text{derivation}}}{Q} + \hbar \underset{\substack{\uparrow \\ \text{not a derivation} \\ \text{due to term } \{a,b\} \text{ in (iii) above}}}{\Delta_{BV}} \quad \text{is not a derivation}$$

Special case: free field theory

X vector space $\qquad S(x) = \frac{1}{2}\langle Ax, x\rangle$

$\langle \, , \rangle : X \times X \to \mathbb{R}$ non-deg. symm. bilinear form

$A : X \to X$ self-adjoint

$$T^*X[-1] \underset{\substack{\uparrow \\ \text{via } \langle \, , \rangle}}{=} TX[-1] = X \oplus X[-1] =: \mathcal{E} \quad \text{graded vector space}$$

$$\Rightarrow \underset{\cup}{\mathrm{Obs}}^{cl} = \mathcal{O}(\mathcal{E}) = \mathrm{Sym}^\bullet(\mathcal{E}^\vee) \qquad \qquad \begin{array}{l} \mathcal{E}^\vee = (X \oplus X[-1])^\vee = X^\vee[1] \oplus X^\vee \\ \qquad \qquad \qquad \text{deg} \quad\ -1 \qquad\ 0 \end{array}$$

linear observables $= \mathcal{E}^\vee$

<u>Recall</u>: Q and $\{ \, , \}$ (in each slot) are <u>derivations</u>; hence they are determined by specifying them on $\mathcal{E}^\vee = X^\vee[1] \oplus X^\vee$

$$\mathcal{E}^\vee : \quad \begin{array}{ccc} X^\vee[1] & \xrightarrow{Q = A^\vee} & X^\vee \\ \text{deg} \ -1 & & 0 \end{array} \qquad \qquad \begin{array}{rcl} \{ \, , \} : X^\vee[1] \times X^\vee & \longrightarrow & \mathbb{R} \\ (a[1], b) & \longmapsto & \langle a, b\rangle \end{array}$$

This completely describes the shifted Poisson structure on Obs^{cl}

what is Δ_{BV} ? $\Delta_{BV}\Big|_{\mathrm{Sym}^i(\mathcal{E}^\vee)} \equiv 0$ for $i = 0,1$

<u>recall</u>: $\Delta_{BV}(ab) = \Delta_{BV}(a)\, b + (-1)^{|a|} \Delta_{BV}(b) + \{a,b\}$ $(*)$

for $a, b \in \mathrm{Sym}^1$ $\Longrightarrow$ $\Delta_{BV}(\underset{\mathrm{Sym}^2}{\underset{\cap}{ab}}) = \{a,b\}$

$\xRightarrow[(*)]{}$ $\Delta_{BV} : \mathrm{Sym}^k(Y^\vee) \longrightarrow \mathrm{Sym}^{k-2}(Y^\vee)$ is determined by $\{\,,\,\}$.

<u>Upshot</u>: $\mathcal{E} := \left(X^\vee[1] \xrightarrow{Q = A^\vee} X^\vee \right)$ dg vector space

$$\mathrm{Obs}^{cl} = \left(\mathrm{Sym}^\bullet(\mathcal{E}^\vee), Q \right)$$

determined by $\{\,,\,\}$ on $\mathcal{E}^\vee$.

$$\mathrm{Obs}^{q} = \left(\mathrm{Sym}^\bullet(\mathcal{E}^\vee) \otimes \mathbb{R}[[\hbar]], Q + \hbar \Delta_{BV} \right)$$

First real example: free scalar field theory

- spacetime: M Riem. mfd, e.g. $M=\mathbb{R}^n$
- space of fields in spacetime region $U \subset M$: $\mathcal{E}(U) = C^\infty(U,\mathbb{R})$

Laplace operator ↓

- action functional: $S : \mathcal{E}(U) \to \mathbb{R}$ (energy), $S(\phi) = \frac{1}{2} \langle \Delta\phi, \phi \rangle$

$\Delta : C^\infty(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d^*} C^\infty(U)$ $\quad \langle f, g \rangle := \int_U f \cdot g \, \mathrm{vol}$ for $f, g \in C^\infty(U)$

Subtlety: ⚠ $S(\phi)$ isn't really defined for all $\phi \in C^\infty(U)$; ok for $\phi \in C^\infty_c(U)$.

unlike $S(\phi)$ for $\phi \in \mathcal{E}(U)$, $dS_\phi(\psi)$ makes sense for $\phi \in C^\infty(U)$, $\psi \in C^\infty_c(U)$

since
$$\underbrace{S(\phi+t\psi) - S(\phi)} = \frac{1}{2} \langle \Delta(\phi+t\psi), \phi+t\psi \rangle - \frac{1}{2} \langle \Delta\phi, \phi \rangle$$
$$= \frac{1}{2} t \big(\langle \Delta\phi, \psi \rangle + \langle \Delta\psi, \phi \rangle + \frac{t^2}{2} \langle \Delta\psi, \psi \rangle$$
$$= t \langle \Delta\phi, \psi \rangle + \frac{t^2}{2} \langle \Delta\psi, \psi \rangle$$

(these terms make sense since $\psi \in C^\infty_c(U)$)

$$dS_\phi(\psi) = \frac{d}{dt}\Big|_{t=0} \qquad = \langle \Delta\phi, \psi \rangle$$

Solution: define $\mathrm{Crit}(S) := \{ \phi \in \mathcal{E}(U) \,/\, dS_\phi(\psi) = 0 \ \ \forall \psi \in C^\infty_c(U) \}$

Like the toy example, except $X := C^\infty(U)$ has $\dim X = \infty$

$$\mathcal{E} := TX[-1] = C^\infty(U) \xrightarrow{\Delta} C^\infty(U)[-1]$$

$$\mathcal{E}^\vee = C^\infty(U)^\vee[1] \xrightarrow{\Delta^\vee} C^\infty(U)^\vee$$

$$\mathcal{E}^! := C_c^\infty(U)[1] \xrightarrow{Q=\Delta} C_c^\infty(U)$$ ↙ linear observables

$$\mathrm{Obs}^{cl}(U) := (\mathrm{Sym}^\bullet(\mathcal{E}^!), Q)$$

$$\mathrm{Obs}^{q}(U) := (\mathrm{Sym}^\bullet(\mathcal{E}^!) \otimes \mathbb{R}[[\hbar]], Q + \hbar\Delta_{BV})$$

↑ determined by pairing on $\mathcal{E}^!$

$$\{\underset{\substack{\cap \\ C_c^\infty(U)[1]}}{f}, \underset{\substack{\cap \\ C_c^\infty(U)}}{g}\} := \langle f, g\rangle$$

problem: $\langle\,,\rangle$ on $C^\infty(U)$ <u>does not</u> determine inner product on $C^\infty(U)^\vee = \{\text{distributions}\}$.

solution: replace by

$$C_c^\infty(U) \subset C^\infty(U)^\vee$$

$$f \longmapsto \left(g \mapsto \langle f, g\rangle = \int f g \,\mathrm{vol}\right)$$

Def: A *free field theory* on an n-mfd M consists of:

(i) vector bdls $E_k \to M$, $k \in \mathbb{Z}$

(ii) differential operators
$$\overbrace{\to \Gamma(E_{k-1}) \xrightarrow{Q_{k-1}} \Gamma(E_k) \xrightarrow{Q_k} \Gamma(E_{k+1}) \to}^{\mathcal{E} \text{ sheaf of dg vector spaces}}$$

s.t. $Q_k \circ Q_{k-1} = 0$ and the complex is elliptic

(this implies that cohomology groups are finite dim for M compact

Ex: de Rham complex $\to \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to$)

(iii) $E_k^! := E_k^\vee \otimes \Lambda^n T^* M$; pairing $\langle\, , \rangle : \Gamma(E_k) \times \Gamma_c(E_k^!) \to \mathbb{R}$

$$\langle f, g \rangle := \int_M \underbrace{\langle f(x), g(x) \rangle_{loc}}_{\Lambda^n T_x^* M} \in \mathbb{R}$$

$Q_k^* : \Gamma_c(E_{k+1}^!) \to \Gamma_c(E_k^!)$

adjoint to Q_k

$\langle Q_k f, g \rangle = \langle f, Q_k^* g \rangle$

for $f \in \Gamma(E_k)$, $g \in \Gamma_c(E_{k+1}^!)$

$\mathcal{E}_c^!$ ← linear observables

$$\mathcal{E}_c^! : \quad \to \Gamma_c(E_{k+1}^!) \xrightarrow{Q_k^*} \Gamma_c(E_k^!) \to$$

deg $\qquad -(k+1) \qquad -k$

a pairing $\{\, , \} : \mathcal{E}_c^! \times \mathcal{E}_c^! \to \mathbb{R}$ of degree $+1$ compatible with differentials

$$\{Q^* f, g\} + (-1)^{|f|} \{f, Q^* g\} = 0$$

classical + quantum observables of a free field theory

$\mathcal{U} \underset{\text{open}}{\subset} M$ spacetime region

$$Obs^{cl}(\mathcal{U}) := \Big(\mathrm{Sym}^{\bullet}\big(\underbrace{\mathcal{E}_c^{\overset{!}{\bullet}}(\mathcal{U})}_{\text{linear observables}}\big), Q^{*}\Big)$$ classical observables

$$Obs^{q}(\mathcal{U}) := \Big(\mathrm{Sym}^{\bullet}\big(\mathcal{E}_c^{\overset{!}{\bullet}}(\mathcal{U})\big)\otimes\mathbb{R}[[\hbar]], Q^{*}+\hbar\Delta_{BV}\Big)$$ quantum "

$$Obs : \mathrm{Open}(\mathcal{U}) \longrightarrow \mathrm{Ch}$$ functor

$$\mathcal{U}_1, \mathcal{U}_2 \subset V \quad m^{V}_{\mathcal{U}_1,\mathcal{U}_2} : Obs(\mathcal{U}_1)\otimes Obs(\mathcal{U}_2) \longrightarrow Obs(V)$$

$$f\otimes g \longmapsto \underset{\uparrow}{f\cdot g}$$

product in $\mathrm{Sym}^{\bullet}$

Chain map?

for Obs^{cl} ok, since Q^{*} is derivation

for Obs^{q}: $\Delta_{BV}(f\cdot g) = \Delta_{BV}(f)\cdot g + (-1)^{|f|} f\cdot\Delta_{BV}(g) + \{f,g\}$

↲ vanishes if f,g have disjoint support.

<u>Prop</u>: Obs^{cl} + Obs^{q} are factorization algebras

<u>Pf</u>: multiplicative: $\mathrm{Obs}(\mathcal{U}_1 \amalg \mathcal{U}_2) \simeq \mathrm{Obs}(\mathcal{U}_1) \otimes \mathrm{Obs}(\mathcal{U}_2)$

due to exponential property of $\mathrm{Sym}^{\bullet}$

descent property: $\mathcal{U} \longmapsto \mathcal{E}_c^{\overset{!}{\bullet}}(\mathcal{U})$ (homotopy) cosheaf

w.r.t. ordinary covers

$$\Downarrow$$

$\mathcal{U} \longmapsto \mathrm{Sym}^{\bullet}(\mathcal{E}_c^{\overset{!}{\bullet}}(\mathcal{U}))$ " " " Weiss covers