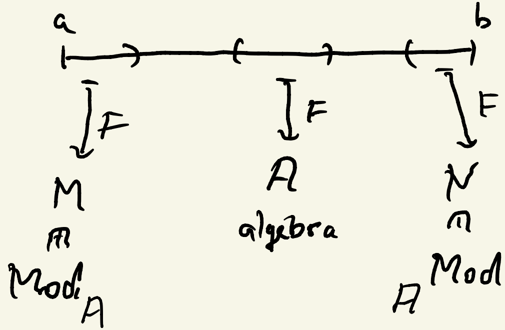
__Recall:__ locally constant factorization algebra $\mathcal{F} = \mathcal{F}_{M,A,N}$ on $[a,b]$



Then $\mathcal{F}([a,b]) \overset{\text{w.e.}}{\sim} M \overset{h}{\underset{A}{\otimes}} N \overset{\text{w.e.}}{\sim} B_\bullet(M,A,N)$

__Recall:__ A R-algebra $\quad B$ A-A-bimodule $\quad \left(\begin{array}{c} A, B \text{ free} \\ \text{over } R \end{array}\right)$

Hochschild homology $HH_k(A;B) := H_k(\underbrace{HHC_\bullet(A;B)}_{\text{Hochschild complex}})$

special case: $HH_k(A) = HH_k(A;A)$

$HHC_\bullet(A;B)$ is the chain cx. associated to the simplicial object

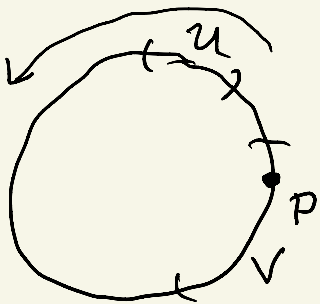
$$B \leftleftarrows B\otimes A \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; B\otimes A\otimes A \;\cdots$$

$$\begin{array}{lll} ba \longleftarrow\!\shortmid\; b\otimes a & & ba_1\otimes a_2 \longleftarrow\!\shortmid\; b\otimes a_1\otimes a_2 \\ ab \longleftarrow\!\shortmid & & b\otimes a_1a_2 \longleftarrow\!\shortmid \\ & & a_2b\otimes a_1 \longleftarrow\!\shortmid \end{array}$$

Q: Can the Hochschild complex be obtained via factorization algebras?

Construction: A R-algebra, B A-A-bimodule (free as R-modules).

$\rightsquigarrow$ factorization algebra $F = F_{A,B}$ on S^1.



$$F(U) = \begin{cases} A & \text{if } p \notin U \\ B & \text{if } p \in U \end{cases}$$

U: segment of S^1, i.e. a connected proper open subset

need to specify structure maps:

$U_1 \sqcup \dots \sqcup U_k \hookrightarrow V \rightsquigarrow m^V_{U_1,\dots,U_k} : F(U_1) \otimes \dots \otimes F(U_k) \to F(V)$

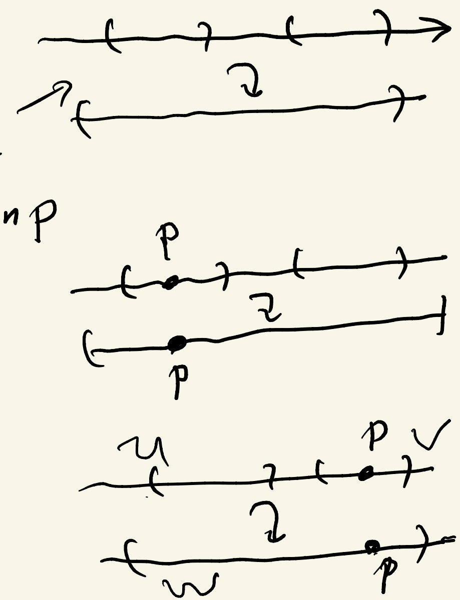
$$\begin{array}{ccc} \emptyset & & \\ \downarrow & \rightsquigarrow & \\ U & & \end{array} \qquad p \notin U: \begin{array}{ccc} F(\emptyset) = R & \ni & 1 \\ \downarrow & & \downarrow \\ F(U) = A & \ni & 1 \end{array} \qquad p \in U: \begin{array}{ccc} F(\emptyset) = R & \ni & 1 \\ \downarrow & & \downarrow \\ F(U) = B & \ni & b_0 \end{array}$$

note: So we need to specify an element $b_0 \in B$ as datum.

Rem: If F is any factorization algebra, then $F(\emptyset) = R$ and hence for any $\mathcal{U} \subset M$ open, there is a distinguished element

$$R = F(\emptyset) \longrightarrow F(\mathcal{U})$$
$$1 \longmapsto v_0 \quad \text{"vacuum".}$$

• The structure maps for $F_{A,B}$:

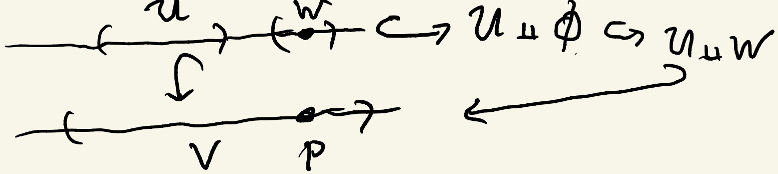


These don't contain p

$$\rightsquigarrow \quad A \otimes A \to A \quad \leftarrow \text{mult}^{\text{n}}\text{ol.}$$

$$\rightsquigarrow \quad B \otimes A \to B, \qquad b \otimes a \mapsto ba$$

$$\rightsquigarrow \quad A \otimes B \to B, \qquad a \otimes b \mapsto ab$$



$$A \longrightarrow A\otimes R \longrightarrow A\otimes B$$
$$a \longmapsto a\otimes 1 \longmapsto a\otimes b_0 \longmapsto ab_0 \in B$$

<u>Prop</u>: $F(S^1) \overset{w.e.}{\sim} HHC_\bullet(A;B)$

<u>proof</u>: use a Weiss cover of S^1 to calculate $F(S^1)$.

<u>digression</u>: relation between $HHC_\bullet(A;B)$ and $B_\bullet(M,A,N)$ and right/left gymnastics for modules.

A algebra, $A^{op} = A$ as R-module, but equipped with new multiplication:

$$a * a' := a' \cdot a$$

↑ old multiplication.

Left A-module structure on N $\longleftrightarrow$ right A^{op}-module structure on N

$$\Downarrow$$

$$A \otimes N \to N \quad \rightsquigarrow \quad N \otimes A^{op} \to N$$

$$n \otimes a \longmapsto \underset{\text{new}}{n * a} := \underset{\text{old}}{a \cdot n}$$

check associativity of right A^{op}-action on N:

$$(n*a)*a' \overset{?}{=} n*(\underset{\text{mult. in } A^{op}}{a*a'})$$

$$\begin{aligned}
(n*a)*a' &= a'(n*a) = a'(an) = (a'a)n = n*(a'a) \\
n*(a*a') &= n*(a'a)
\end{aligned}$$

A-A-bimodule structures on B $\leftrightarrow$ $A^{op} \otimes A$-right module structure on B

$$\updownarrow$$

$A \otimes A^{op}$ left module struct. on B

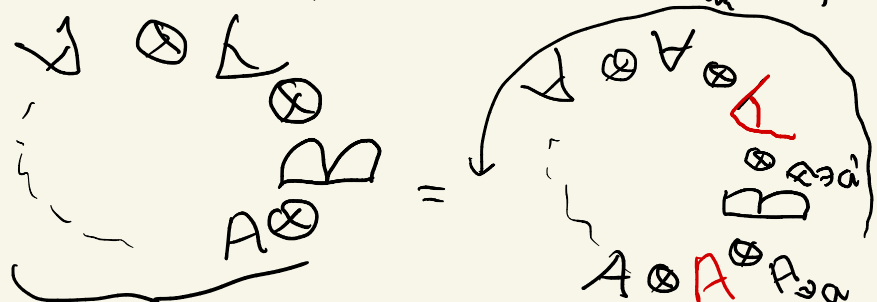
$$b*(\underset{\in A^{op}}{a} \otimes \underset{\in A}{a'}) := aba'$$

$HHC_k(A;B) = B \otimes \underbrace{A \otimes \cdots \otimes A}_{k}$

with differential

$$d(b\otimes a_1\otimes a_2 \otimes \cdots \otimes a_k) = ba_1 \otimes a_2 \cdots \otimes a_k - b \otimes a_1a_2 \otimes \cdots \otimes a_k \cdots (-1)^{k-1} b\otimes a_1 \otimes \cdots a_{k-1}a_k + (-1)^k a_k b \otimes a_1 \otimes \cdots \otimes a_{k-1}$$

$HHC_k(A;B) =$



differential = alternating sum of multiplying conseq. elements

$$= \underbrace{B}_{\text{right } \underset{a}{A^{op}} \otimes \underset{a'}{A}\text{-module}} \underset{A^{op}\otimes A}{\otimes} \underbrace{A \otimes \underbrace{A \otimes \cdots \otimes A}_{k} \otimes A}_{\text{left } A^{op}\otimes A\text{-module}}$$

$$\begin{array}{c} \parallel \\ B \otimes_{A^{op} \otimes A} B_k(A,A,A) \end{array}$$

<u>Upshot:</u> $HHC_\bullet(A, B) \cong B \otimes_{A^{op}\otimes A} B_\bullet(A,A,A)$.

$$e : [0,1] \longrightarrow S^1$$
$$x \longmapsto e^{2\pi i x}$$

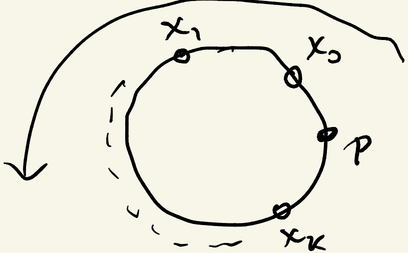
Weiss cover of S^1 : $\underline{U} = \{ U_x := S^1 \setminus \{e(x)\} \}_{x \in (0,1)}$

$$\check{C}(S^1, \underline{U}, F)_k = \bigoplus_{x_0 < \cdots < x_k} F(U_{x_0} \cap \cdots \cap U_{x_k})$$

$$\parallel$$

$$\bigoplus_{x_0 < \cdots < x_k} B \otimes \underbrace{A \otimes \cdots \otimes A}_{k}$$

$\leftarrow$ "bloated version of the bar cx.



$$\|$$

$$\bigoplus_{x_0<\dots<x_k} B \otimes_{A^{op}\otimes A} \underbrace{A\otimes \cdots \otimes A}_{k+2}$$

bloat. bar cx.

$$= B \otimes_{A^{op}\otimes A} \Big(\underbrace{\bigoplus_{x_0<\dots<x_k} \underbrace{A\otimes \cdots \otimes A}_{k+2}}_{\check{C}([0,1],\underline{\mathcal{U}},F_{A,A,A})} \Big)$$

$\Rightarrow$

$$\check{C}(S^1,\underline{\mathcal{U}},F) \cong B \otimes_{A^{op}\otimes A} \big(\check{C}([0,1],\underline{\mathcal{U}},F_{A,A,A})\big)$$

$$\overset{w.e.}{\sim} B \otimes_{A^{op}\otimes A} B_\bullet(A,A,A) \cong HHC_\bullet(A;B)$$

$\square$