Recall: $\mathcal{C}$ category with pullbacks

$c \in \mathcal{C}$ $\quad \underline{C} = \{c_a \xrightarrow{f_a} c\}_{a\in A}$ $\quad \subset$ full subcat. $\quad \mathcal{C}/c$

associate sieve $\quad \hat{\underline{C}} = \{d \xrightarrow{f} c \mid \exists \text{ factorization}\} \subset \mathcal{C}/c$

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[dr] & c \\ & c_a \arrow[u, "f_a"'] \end{tikzcd}$$

Functor: $\quad P_{fin}(A) \xrightarrow{\alpha} \hat{\underline{C}}$

$$([n] \xrightarrow{a} A) \longmapsto (c_{a_0} \times_c \cdots\cdots \times_c c_{a_n} \xrightarrow{f_a} c)$$

$$\begin{tikzcd} P_{fin}(A) \arrow[r, "\alpha"] & \hat{\underline{C}} \arrow[r, phantom, "\subset"] \arrow[rrr, bend left, "F"] & \mathcal{C}/c \arrow[r] & \mathcal{C} \arrow[r, "F"] & Ch \end{tikzcd}$$

Prop A: $\quad \underset{P_{fin}(A)}{\operatorname{hocolim}} F \circ \alpha \xrightarrow[\sim]{w.e.} \underset{\hat{\underline{C}}}{\operatorname{hocolim}} F$

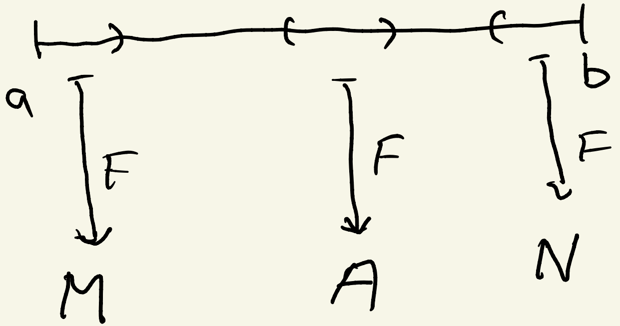
Addendum: If $\underline{C}' = \{c_{a_0} \times_c \cdots\cdots \times_c c_{a_n} \xrightarrow{f} c\}$ then

$$\underset{P_{fin}(A)}{\operatorname{hocolim}} F \circ \alpha \xrightarrow[\sim]{w.e.} \underset{\underline{C}'}{\operatorname{hocolim}} F$$

Recall: A R-algebra, $M \in \mathrm{Mod}_A$, $N \in {}_A\mathrm{Mod}$

(A, M, N are free $/R$)

Let $F = F_{M,R,N}$ be a fact. algebra of $M = [a,b]$ with values in Ch



simplicial object in Ch

$$\downarrow$$

Prop: $F([a,b]) \overset{w.e.}{\sim} \underbrace{M \overset{h}{\underset{A}{\otimes}} N}_{\text{derived tensor product}} := \operatorname*{hocolim}_{\Delta^{op}} B_\bullet(M,A,N)$

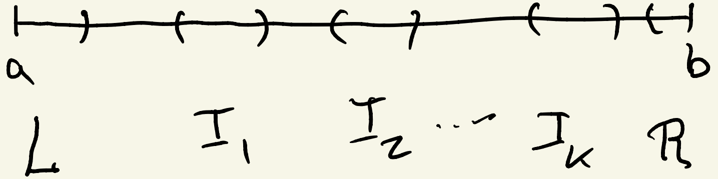
$$\parallel$$

$$B_\bullet(M,A,N)^{alt}$$

Proof: use a Weiss ~~sieve~~ cover on Open(M) + compute F(M) using "abstract" descent property.

Let $\mathrm{Disk}_1(M) \subset \mathrm{Open}(M)$

$\mathcal{U} \in \mathrm{Disk}_1(M) \iff \mathcal{U}$ is a finite union of intervals, $a, b \in \mathcal{U}$



$$\mathcal{U} = L \cup I_1 \cup \dots \cup I_k \cup R$$

Note: $\mathrm{Disk}_1(M)$ is a Weiss ~~sieve~~ cover closed under pullbacks

$$\xRightarrow[\text{addendum Prop A}]{} \operatorname{hocolim}_{\mathrm{Disk}_1(M)} F \overset{w.e.}{\simeq} \operatorname{hocolim}_{P_{fin}(A)} F \circ \alpha \underset{\text{Prop B}}{\overset{w.e.}{\simeq}} \check{C}(\mathrm{Disk}_1(M), F) \overset{w.e.}{\simeq} F([a,b])$$

$$\mathrm{Disk}_1(M) \subset \mathrm{Open}(M) \xrightarrow{F} Ch$$

$$L \sqcup I_1 \sqcup \dots \sqcup I_k \sqcup R \longmapsto M \otimes \underbrace{A \otimes \dots \otimes A}_{k} \otimes N$$

recall : $B_\bullet(M,A,N) : \Delta^{op} \longrightarrow Ch$

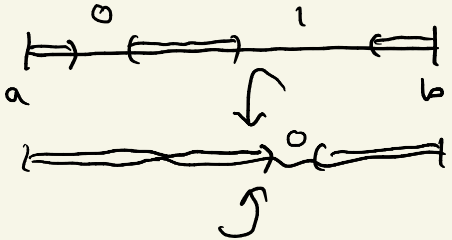
bar construction $[n] \longmapsto M \otimes \underbrace{A \otimes \dots \otimes A}_{n} \otimes N$

idea: factor F through a comm. diagram of functors

$$\begin{tikzcd} \mathrm{Disk}_1(M) \arrow[rr, "\gamma"] \arrow[dr, "F"'] & & \Delta^{op} \arrow[dl, "B_\bullet(M,A,N)"] \\ & Ch & \end{tikzcd}$$

“gap”

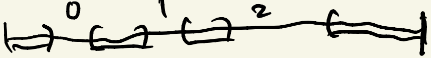
the functor gamma:



$\longmapsto [1] = \{0,1\}$

$\longmapsto [0] = \{0\}$

$$\begin{tikzcd} 1 \\ 0 \arrow[u] \end{tikzcd}$$



$\longmapsto [2] = \{0,1,2\} \ni 2$

Hope: the functor γ is homotopy final, i.e. for every $[n_0] \in \Delta^{op}$ the nerve $N([n_0]/\gamma)$ is contractible.

objects: $\mathrm{Disk}_1(M) \ni U$, $\quad [n_0] \xleftarrow{f} \gamma(U)$ map in Δ

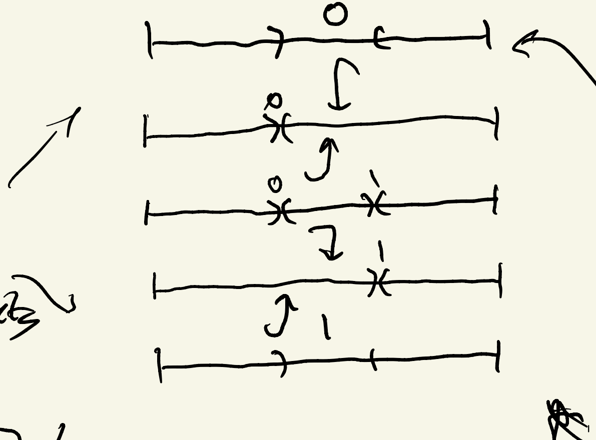
morphisms:

$$\begin{tikzcd} U \arrow[d, hook] \\ U' \end{tikzcd} \qquad \begin{tikzcd} \gamma(U) \arrow[r, "f"] & {[n_0]} \\ \gamma(U') \arrow[u] \arrow[ur, "f'"'] & \end{tikzcd}$$

pictures showing that I can construct a path in $N([n_0]/\gamma)$ between two particular objects.

note: an object $(U, \gamma(U) \xrightarrow{f} [n_0])$ consists of U, and a labeling of the gaps of U by $0<1<\dots<n_0$

two objects in $[1]/\gamma$



$[n_0] = [1]$

$[1]$

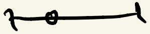
problem: can't have just 2 morphisms from one object of $[1]/\gamma$ to both

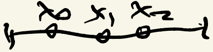
---

Def: Let $F_{M,A,N} = F$ be the fact. alg. on $[a,b]$ as before.

recall: $\check{C}([a,b], F_{M,A,N})$ Čech complex associated to the Weiss cover $\{U_x := [a,b] \setminus x\}_{x \in (a,b)}$.

explicitly:





$\check{C}([a,b],F)$: "bloated bar complex"

$$\begin{tikzcd}
\Big(\bigoplus_{x_0} F(\mathcal{U}_x) \arrow[d, equal] & \bigoplus_{x_0<x_1} F(\mathcal{U}_{x_0}\cap\mathcal{U}_{x_1}) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] & \bigoplus_{x_0<x_1<x_2} F(\mathcal{U}_{x_0}\cap\mathcal{U}_{x_1}\cap\mathcal{U}_{x_2}) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, equal] & \cdots \\
\bigoplus_{x_0} M\otimes N & \bigoplus_{x_0<x_1} M\otimes A\otimes N & \bigoplus_{x_0<x_1<x_2} M\otimes A\otimes A\otimes N &
\end{tikzcd}$$

bar complex:

$B_\bullet(M,A,N)$

$$\begin{tikzcd}
M\otimes N & M\otimes A\otimes N \arrow[l, shift left] \arrow[l, shift right] & M\otimes A\otimes A\otimes N \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

Prop: $\underbrace{\check{C}([a,b],F_{A,A,N})}^{\text{alt}}$ is a resolution of N by free A-modules.

simplicial left A-module

chain cx. of left A-modules

Note:

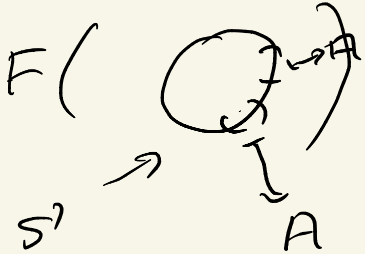
$$\check{C}([a,b],F_{M,A,N})^{\text{alt}} \cong M\otimes_A \check{C}([a,b],F_{A,A,N})^{\text{alt}}$$

$$M\otimes\underbrace{A\otimes\cdots\otimes A}_{k}\otimes N \cong (M\otimes_A A)\otimes\underbrace{A\otimes\cdots\otimes A}_{k}\otimes N$$

Cor: $\check{C}([a,b], F_{M,A,N})^{alt} \overset{w.e.}{\sim} \underbrace{M \overset{h}{\otimes}_A N}_{\text{derived tensor product.}}$

and hence $F([a,b]) \overset{w.e.}{\sim} M \overset{h}{\otimes}_A N$.

Relationship to Hochschild homology.



$= ?$