$S : X \to \mathbb{R}$ action functional of a classical field theory

old point of view: classical observables

$$\mathrm{old}\,\mathrm{Obs}^{cl} := \mathcal{O}(\mathrm{Crit}(S)) \text{ is a Poisson algebra}$$

new (derived) point of view

$$\mathrm{Obs}^{cl} := \mathcal{O}(\underbrace{d\mathrm{Crit}(S)}_{\text{derived critical pts}}) = \underbrace{\big(\overbrace{\Gamma(X, \Lambda^\bullet(TX[1]))}^{\text{algebra of polyvector fields}}, -\iota_{dS}\big)}_{\text{dga (differential graded algebra)}}$$

classial BV-complex

BV = Batalin & Vilkovisky

change of convention to be motivated later

relationship: if dS is transversal to 0-section of T^*X, then $\mathrm{old}\,\mathrm{Obs}^{cl} \overset{w.e.}{\sim} \mathrm{Obs}^{cl}$, i.e.

$$H^k(\mathrm{Obs}^{cl}) \cong \begin{cases} \mathrm{old}\,\mathrm{Obs}^{cl} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

isom. of algebras

Example: suppose $S : X \to \mathbb{R}$ is the action functional of a free field theory, i.e. X is a vector space and S is a quadratic function,

i.e. $S(x) = \frac{1}{2} \langle Ax, x \rangle$ $\qquad \langle \, , \, \rangle$ inner product on X

$A : X \to X$ linear map self-adjoint

$x \in X$

$v \in T_x X = X$

$$dS_x(v) = \frac{d}{dt}\Big|_{t=0} S(x+tv) = \frac{d}{dt}\Big|_{t=0} \frac{1}{2} \langle A(x+tv), x+tv \rangle$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{1}{2} \left(\langle Ax, tv \rangle + \langle A(tv), x \rangle \right)$$

$$= \langle Ax, v \rangle$$

$$\left(\begin{aligned} \langle Ax, x \rangle &= \frac{1}{2} \left(\langle Ax, x \rangle + \langle x, Ax \rangle \right) \\ &= \frac{1}{2} \left(\langle Ax, x \rangle + \langle A^* x, x \rangle \right) \\ &= \frac{1}{2} \langle (A + A^*) x, x \rangle \end{aligned} \right)$$

self-adjoint

$\Rightarrow \quad dS_x = \langle Ax, - \rangle$

dS is transversal to 0-section $\Leftrightarrow$ A is an isom.

if A is iso $\Rightarrow$ $\mathrm{Crit}(S) = \{0\}$ $\Rightarrow$ $\mathrm{old\,Obs}^{cl} = \mathbb{R}$.

compare with derived picture:

$$\mathrm{Obs}^{cl} = \Gamma(X, \Lambda^\bullet(TX[1])) = \mathcal{O}(X) \otimes \Lambda^\bullet(X[1])$$

X vector space

$$= S^\bullet(X^\vee) \otimes \Lambda^\bullet(X[1])$$

$\leftarrow$ with differential Q acting on this as derivation of degree $+1$

determined on generators:

$\xi \in X^\vee \qquad Q\xi = 0$

$v \in X \qquad \underset{-\iota_{dS}}{Q} : X[1] \to X^\vee$

$$\underbrace{v[1]}_{\deg = -1} \mapsto -\langle A\,\underline{\ \ }\,, v\rangle.$$

goal: calculate $H^*(\mathrm{Obs}^{cl})$.

<u>digression</u>: algebra of functions on graded vector spaces + chain cxs.

<u>recall</u>: V vector space $\Rightarrow$ $\mathcal{O}(V) = S^\bullet(V^\vee)$

more generally: V dg vs, i.e. V is a cochain cx.

goal: define the dga $\mathcal{O}(V)$ of function on V.

$V^\vee$ is defined by: $(V^\vee)_k = (V_{-k})^\vee$

want evaluation: $V^\vee \otimes V \to \mathbb{k}$ ← field

$V_k \xrightarrow{Q} V_{k+1}$ dualize $\rightsquigarrow$ $V_k^\vee \xleftarrow{Q^\vee} V_{k+1}^\vee$

$$\begin{array}{ccc} V_k^\vee & \xleftarrow{Q^\vee} & V_{k+1}^\vee \\ \| & & \| \\ (V^\vee)_{-k} & \xleftarrow{q} & (V^\vee)_{-(k+1)} \end{array}$$

q of degree $+1$

V, W dg vector spaces $\Rightarrow$ we know $V \otimes W$ dg vector space

$$\mathrm{Sym}^k(V) := \underbrace{(V \otimes \dots \otimes V)}_{k}{}_{\Sigma_k}$$

$\leftarrow$ coinvariants

$\Sigma_k \leftarrow$ symmetric group

$$\sigma(v_1 \otimes \dots \otimes v_k) := (-1)^{|v_i||v_{i+1}|} v_1 \otimes \dots \otimes v_{i+1} \otimes v_i \otimes \dots \otimes v_k$$

$\sigma \in \Sigma_k$

$\sigma = \tau_{i,i+1}$ transposition

$$\mathrm{Sym}^\bullet(V) := \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V)$$

exponential property:

$$\mathrm{Sym}^\bullet(V \oplus W) \cong \mathrm{Sym}^\bullet(V) \otimes \mathrm{Sym}^\bullet(W)$$

iso of dg vs

$V \cong V_{ev} \oplus V_{odd}$ as graded vs $\quad V_{ev} = \bigoplus_{k \text{ even}} V_k \quad V_{odd} = \bigoplus_{k \text{ odd}} V_k$

$$\mathrm{Sym}^\bullet(V) \underset{\substack{\uparrow \\ \text{as graded vs}}}{=} \mathrm{Sym}^\bullet(V_{ev} \oplus V_{odd}) = \mathrm{Sym}^\bullet(V_{ev}) \otimes \mathrm{Sym}^\bullet(V_{odd})$$

$$= S^\bullet(V_{ev}) \otimes \Lambda^\bullet(V_{odd})$$

note: $H^\bullet(V \otimes W) \underset{\text{Künneth}}{=} H^\bullet(V) \otimes H^\bullet(W)$

$$H^\bullet(\mathrm{Sym}^k(V)) = H^\bullet\Big((\underbrace{V \otimes \cdots \otimes V}_{k})_{\Sigma_k} \Big) = H^\bullet(\underbrace{V \otimes \cdots \otimes V}_{k})_{\Sigma_k}$$

finite group

so $H^\bullet(\Sigma_k; \mathbb{k}) = 0$ for $\mathrm{char}(\mathbb{k}) = 0$

$$\simeq \mathrm{Sym}^k(H^\bullet(V)).$$

$$H^\bullet(\mathrm{Sym}(V)) \simeq \mathrm{Sym}(H^\bullet(V)).$$

Back to our example:

$$\mathrm{Obs}^{cl} = S^\bullet(X^\vee) \otimes \Lambda^\bullet(X[1]) = \mathrm{Sym}^\bullet\big(\underbrace{X[1] \xrightarrow{Q} X^\vee}_{V}\big)$$

deg -1 $\qquad$ 0

$$Q(x[1]) = (v \longmapsto -\langle Ax, v\rangle)$$

$$H^\bullet(\mathrm{Obs}^{cl}) = \mathrm{Sym}^\bullet(H^\bullet(V))$$

$$H^k(V) = \begin{cases} \ker Q & k=-1 \\ \mathrm{coker}\, Q & k=0 \end{cases}$$

$$= \mathrm{Sym}^\bullet\big((\ker Q)[1] \oplus \mathrm{coker}\, Q\big)$$

$$= \Lambda^\bullet(\ker Q) \otimes S^\bullet(\mathrm{coker}\, Q) = \mathbb{R}.$$

if A is isom. then Q is isom