Recall: M mfd, $\omega \in \Omega^2(M)$ is <u>symplectic form</u> if

- $d\omega = 0$
- $\Gamma(TM) \xrightarrow{\cong} \Gamma(T^*M)$

  $X \longmapsto \iota_X \omega$

<u>Lem A</u>

$$C^\infty(M) \longrightarrow \Gamma(TM)$$
$$f \longmapsto X_f$$

<u>Hamiltonian vector field</u> determined by $\iota_{X_f}\omega = -df$

is a Lie algebra homomorphism w.r.t.

commutator bracket $[\, ,\,]$ on $\Gamma(TM) = \{$vector fields on $M\}$

Poisson bracket $\{\, ,\,\}$ on $C^\infty(M)$ given by $\{f,g\} = X_f g = \omega(X_f, X_g)$

<u>Ex</u>: $M = T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ $\qquad \omega = dp \wedge dq$

base $\qquad$ fiber

q = position $\qquad$ p = momentum $\qquad X_q = -\frac{\partial}{\partial p} \quad X_p = \frac{\partial}{\partial q}$

$$\{p,q\} = \omega(X_p, X_q) = \omega\left(\frac{\partial}{\partial q}, -\frac{\partial}{\partial p}\right) = dp \wedge dq\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right) = 1$$

Structure on $C^\infty(M)$ for a symplectic mfld M:

algebra structure : $\cdot$ multiplication of functions

Lie algebra " : $\{\ ,\ \}$ Poisson bracket

(*) Compatibility : $\{f, -\} : C^\infty(M) \longrightarrow C^\infty(M)$ is a derivation w.r.t. the algebra structure on $C^\infty(M)$,

i.e. $\{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\}$

<u>Def</u>: A <u>Poisson algebra</u> is an algebra A plus a Lie algebra structure $\{\ ,\ \}$ with compatibility (*).

physic description of mechanical systems:

$\left\{\begin{array}{c}\text{states of the}\\ \text{system}\end{array}\right\} = M \leftarrow$ symplectic mfd

(phase space)

$\{\text{classical observables}\} = C^\infty(M)$

$x \in M$ state at time $t=0$ $\Rightarrow$ $\phi_t(x)$ state at time t

describe the dynamic of the physical system
$\left\{\begin{array}{l}\phi_t \text{ is a } \underline{\text{flow}} \text{ on } M \text{ (i.e. } \mathbb{R} \to \mathrm{Diff}(M) \text{ is a homomorphism)} \\ \qquad\qquad\qquad\qquad\qquad\quad t \mapsto \phi_t \\ \text{corresponding vector field } \left.\frac{d\phi_t(x)}{dt}\right|_{t=0} \in \Gamma(TM).\end{array}\right.$

$H \in C^\infty(M)$ <u>Hamiltonian function</u>

$$X_H = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x)$$

$f \in C^\infty(M)$ observable $\left.\frac{d\phi_t^*(f)}{dt}\right|_{t=0} = X_H f = \{H, f\}$

<u>Ex</u>: particle moving in a Riem. mfd X

phase space $= T^*X = M$ symplectic mfd $\omega = d\theta$

Hamiltonian function $H(x, \xi) = \frac{1}{2}\|\xi\|^2$

$x \in X$, $\xi \in T_x^*X$

X_H = generator of the geodesic flow on T^*X.

specialize: $X = \mathbb{R}$, $M = T^*\mathbb{R}$

$H = \frac{1}{2}p^2$

$\frac{dp}{dt} = ?$ $\frac{dq}{dt} = ?$ $\Rightarrow$ $\boxed{q(t) = q(0) + tp}$

expectation: $= 0$ proportional to p

$\frac{dp}{dt} = \{H, p\} = \{\frac{1}{2}p^2, p\} = -\frac{1}{2}\{p, p^2\} = 0$

$\frac{dq}{dt} = \{H, q\} = \{\frac{1}{2}p^2, q\} = -\frac{1}{2}\{q, p^2\} = -\frac{1}{2}(\{q, p\}\cdot p + p\{q, p\})$
$= p$

digression on Lie derivatives

M mfd, vector field X, associated flow $\phi_t : M \to M$

$f \in C^\infty(M)$ $\quad \frac{d}{dt}(\phi_t^* f)(x) = Xf = df(X) \in C^\infty(M)$

$L_X f :=$

$Y \in \Gamma(TM)$ $\quad L_X Y := \frac{d}{dt}(\phi_t)_*(Y) = [X, Y]$

$\alpha \in \Omega^k(M)$ $\quad L_X \alpha := \frac{d}{dt}\phi_t^*(\alpha) \underset{\text{Cartan formula}}{=} d\, L_X \alpha + L_X d\alpha \in \Omega^k(M)$

Lie derivative of α in direction X

$L_X : \Omega^k(M) \to \Omega^k(M)$

$L_X(\alpha \wedge \beta) = (L_X \alpha) \wedge \beta + \alpha \wedge (L_X \beta)$

Def: (M, ω) symplectic mfd

$X \in \Gamma(TM)$ is a **symplectic vector field**

if $\mathcal{L}_X \omega = 0$ $\quad (\Leftrightarrow \quad \phi_t^* \omega = \omega)$.

Proof of Lem A, i.e. $[X_f, X_g] = X_{\{f,g\}}$

claim1: X_f is symplectic

Pf: $$\mathcal{L}_{X_f} \omega \underset{CF}{=} d\iota_{X_f} \omega + \underbrace{\iota_{X_f} d\omega}_{=0} = d(-df) = 0$$

claim2: X, Y symplectic $\Rightarrow$ $\iota_{[X,Y]} \omega = -d\,\omega(X,Y)$

Pf: two ways to calculate $\mathcal{L}_X \iota_Y \omega$:

$$\mathcal{L}_X(\iota_Y \omega) = \iota_{[X,Y]} \omega + \underbrace{\iota_Y \mathcal{L}_X \omega}_{0} = \iota_{[X,Y]} \omega$$

product rule

$$L_X(\iota_Y\omega) \underset{CF}{=} \iota_X d\iota_Y\omega + d\iota_X\iota_Y\omega = d\iota_X\omega(Y,-)$$

$$= d\,\omega(Y,X)$$

$$= -d\,\omega(X,Y)$$

$$\iota_X d\iota_Y\omega = \iota_X(d\iota_Y + \iota_Y d)\omega \underset{CF}{=} \iota_X L_Y\omega \overset{\leftarrow Y \text{ sympl.}}{=} 0$$

apply claim 2 to the sympl. v.f. X_f & X_g

$$\iota_{[X_f,X_g]}\omega = -d\,\omega(X_f,X_g) = -d\{f,g\}$$

$$\Updownarrow$$

$$X_{\{f,g\}} = [X_f,X_g]$$

$\square$

X_f is defined by

$$\iota_{X_f}\omega = -df$$