# Recall:

**Lem:** $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ vector bdl. Then $\hat{\mathcal{O}}(X) := \Gamma\left(S^\bullet(V^\vee) \otimes \Lambda^\bullet(V^\vee[1])\right)$

with differential $1 \otimes \xi[1] \xrightarrow{d} \xi \otimes 1 \xrightarrow{d} 0$

is a free resolution of $\mathcal{O}(X)$ by free $\mathcal{O}(V) = \Gamma(S^\bullet(V^\vee))$-modules.

**Note:** $S^\bullet(V^\vee) \otimes \Lambda^\bullet(V^\vee[1])$ is a cochain complex of vector bdls.

In particular, for each $x \in X$ it suffices to show

$$H^k(\underbrace{S^\bullet(V_x^\vee) \otimes \Lambda^\bullet(V_x^\vee[1])}_{\text{cochain complex of vector spaces}}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$\Rightarrow$ long exact sequence of vector bdls

| deg | $-2$ | $-1$ | $0$ |
|---|---|---|---|

$$\to S^\bullet(V_x^\vee) \otimes \Lambda^2(V_x^\vee[1]) \xrightarrow{d} S^\bullet(V_x^\vee) \otimes V_x^\vee[1] \xrightarrow{d} S^\bullet(V_x^\vee) \xrightarrow{\varepsilon} \mathbb{R}$$

exact sequence of vector spaces

$\Rightarrow$ exact sequence of vector bdls

$$\to S^\bullet(V^\vee)\otimes \Lambda^2(V^\vee[1]) \to S^\bullet(V^\vee)\otimes V^\vee[1] \to S^\bullet(V^\vee) \to \underbrace{\mathbb{R}}_{\text{trivial bdl.}}$$

$\Rightarrow$

$$\underbrace{\to \Gamma(\quad\quad {}^{\shortparallel}\quad\quad) \to \Gamma(\quad\quad\quad\quad) \to \Gamma(\quad)}_{\Gamma(S^\bullet(V^\vee)\otimes\Lambda^\bullet(V^\vee[1]))} \xrightarrow{\varepsilon} C^\infty(X)$$

⚠ This exactness when passing to sections holds in the smooth setting!

So it suffices to show:

$$H^k(\underbrace{S^\bullet(V_x^\vee)\otimes\Lambda^\bullet(V_x^\vee[1])}_{K(V)}) = \begin{cases}\mathbb{R} & k=0\\ 0 & k\neq 0\end{cases}$$

chain homotopy $H$ between $0$ and a map $\Phi$ to be determined.

$H : K(V^\vee) \longrightarrow K(V^\vee)$ of degree $-1$, a graded derivation, hence determined on generators

$$\begin{array}{c} K(V^\vee) \\ \shortparallel \\ S^\bullet(V^\vee) \otimes \Lambda^\bullet(V^\vee[1]) \end{array}$$

$\xi \in V^\vee$ $\quad H(\xi \otimes 1) := 1 \otimes \xi[1]$

$H(1 \otimes \xi[1]) := 0$

$\Phi := dH + Hd$ $\qquad$ calculate, using the fact that

$= [d, H]$

graded commutator; hence since $d$ & $H$ are graded derivations (of deg $+1$ resp. $-1$)

$\Phi = [d, H]$ graded derivation of deg $0$.

$\Phi : S^k(V^\vee) \otimes \Lambda^l(V^\vee[1]) \circlearrowleft$

is multiplication by $(k+l)$.

$\Rightarrow \Phi$ is chain hmtp to $0$; hence it induces same map on cohomology.

Hence cohomology is concentrated on $l=0, k=0$ $\square$

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Recall: Cor: $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ vector bdl., section $\alpha$, zero section $z$

$$\mathcal{O}(X) \overset{h}{\underset{z^*, \mathcal{O}(V), \alpha^*}{\otimes}} \mathcal{O}(X) \cong \left(\Gamma(\Lambda^\bullet(V^\vee[1])), \iota_\alpha\right)$$

$$\iota_\alpha : \Lambda^k(V^\vee[1]) \to \Lambda^{k-1}(V^\vee[1])$$

in particular:

$$\mathcal{O}(\underbrace{d\mathrm{Crit}(S)}_{\text{derived critical locus}}) \cong \mathcal{O}(X \overset{h}{\cap} \mathrm{graph}(dS)) = (\underbrace{\Gamma(X, \Lambda^\bullet(TX[1])), \iota_{dS}}_{\text{diff. graded algebra of functions on } d\mathrm{Crit}(S).})$$

$dS : X \to T^*X$

If $dS$ is transversal to zerosection, then

$$H^k(\Gamma(\Lambda^\bullet(TM[1])), \iota_{dS}) = \begin{cases} C^\infty(\mathrm{Crit}(S)) & k=0 \\ 0 & k\neq 0 \end{cases}$$

originally, we defined $\mathrm{Obs}^{cl} := C^\infty(\underbrace{\mathrm{Crit}(S)}_{\text{space of classical fields}})$

from the derived point of view
we redefine

$$\underbrace{\mathrm{Obs}^{cl}}_{\text{classical observables}} := \mathcal{O}(d\,\mathrm{Crit}(S)) = (\Gamma(\Lambda^\bullet(TX[1])), \iota_{dS}).$$

more explicitely: $\Gamma(\Lambda^\bullet(TX[1]))$ ← a lot of structure, dga + (shifted) Liealgebra

$$\begin{array}{lcccccccc} \deg & -3 & & -2 & & -1 & & 0 \\ & \Gamma(\Lambda^3 TX) & \xrightarrow{dS} & \Gamma(\Lambda^2 TX) & \xrightarrow{dS} & \Gamma(TX) & \xrightarrow{dS} & C^\infty(X) \end{array}$$

poly vector field vector fields

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digression on symplectic manifolds
(classical mechanics)

Def: $M$ mfd. A symplectic structure on $M$

is $\omega \in \Omega^2(M)$ s.t.

(i) $d\omega = 0$

(ii) $\Gamma(TM) \xrightarrow{\cong} \Omega^1(M)$

$X \longmapsto \iota_X \omega$ $\quad (\iota_X \omega)(Y) = \omega(X,Y)$

vector fields

Rem: $\left\{\begin{array}{l}\text{physical states}\\ \text{of a mechanical}\\ \text{system}\end{array}\right\}$ is typically a symplectic mfd.

Ex: a point particle moving in a Riem. mfd $X$

$\{\text{states}\} = \{\text{geodesics in } X\} \cong TX \underset{\text{Riem. metric}}{\cong} T^*X$

for any mfd $X$ (without any structure), there is a preferred symplectic structure on $T^*X$ :

$$\omega = d\Theta \in \Omega^2(T^*X)$$

$$\Theta\big((\underset{\substack{\cap \\ X}}{x}, \underset{\substack{\cap \\ T^*_x X}}{\xi}), \underset{\substack{\cap \\ T_{(x,\xi)}(T^*X)}}{v}\big) := \xi(\pi_*(v)) \in \mathbb{R}$$

$\pi : T^*X \to X$ differential $\pi_* : T_{(x,\xi)} T^*X \to T_x X$

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we are interested in the time development of $\phi \in M$ (symplectic mfd).

<u>Def</u>: given $f \in C^\infty(M)$, the <u>associated Hamiltonian vector field</u> $X_f$ is determined by $\iota_{X_f}\omega = -df$

**Ex:** $M = T^*\mathbb{R} = \overset{\text{base}}{\mathbb{R}} \times \overset{\text{fiber}}{\mathbb{R}}$ ← convention to be motivated later.

coordinate functions: $q$, $p$

"position" "momentum"

homework: i) $\theta = p\,dq$ and hence $\omega = d\theta = dp \wedge dq$

ii) $X_q = -\frac{\partial}{\partial p}$, $X_p = \frac{\partial}{\partial q}$.

Think of functions on a symplectic mfld $M$ as "observables".

**Def:** Poisson bracket: $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$

$M$ symplectic

$$\{f,g\} := X_f\, g = dg(X_f) = -(\iota_{X_g}\omega)(X_f) = -\omega(X_g, X_f)$$
$$= \omega(X_f, X_g)$$

<u>Lem</u>: the Poisson bracket gives $C^\infty(M)$ the structure of a Lie algebra, i.e.

- (skew-symmetry) $\{f,g\} = -\{g,f\}$
- (Jacobi identity): $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

$$\Updownarrow$$

$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$$

i.e. $\{f,-\}$ is a derivation w.r.t. to the Lie bracket.

<u>Pf</u>: homework

<u>Lem</u>:

$$C^\infty(M) \longrightarrow \Gamma(TM)$$
$$f \longmapsto X_f$$

Lie alg. given by $\{\ ,\ \}$

Lie bracket of vector fields

is a Lie algebra homomorphism,

i.e. $[X_f, X_g] = X_{\{f,g\}}$