<u>digression:</u> dg vector spaces + algebras

<u>recall:</u> dg (= differential graded) vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$

+ differential d of degree $+1$

aka cochain complex

dga = dg algebra $A = \bigoplus_{k \in \mathbb{Z}} A_k$

$$A_k \cdot A_\ell \subset A_{k+\ell}$$

$d : A \to A$ is a graded derivation, i.e.

$$d(ab) = (da)\, b + (-1)^{|a|} a\, db$$

<u>Exs of dga's:</u> (i) de Rham complex $\Omega^*(X)$

mult. = wedge product

(ii) $C^*(X)$ singular cochain cx.

mult. = cup product

$V = \bigoplus_{k \in \mathbb{Z}} V_k$ graded vs

$n \in \mathbb{Z}$ $\qquad \underbrace{(V[n])_k}_{\text{space of deg } k \text{ elements}} := V_{n+k}$ ← cohomological shift convention

e.g. V just a vs (concentrated in degree 0)

$$(V[n])_k = V_{n+k} = \begin{cases} V & k = -n \\ 0 & k \neq -n \end{cases}$$

$V[n]$ is concentrated in degree $-n$!

in other word, $V[n] = V$ shifted <u>down</u> by n

<u>Exs:</u> V vectorspace

a) $S^\bullet(V^\vee) = \mathcal{O}(V) =$ polynomial functions on V (concentrated in degree 0)

b) $S^\bullet(V^\vee[1])$ symm. algebra, generated by elements $\xi[1]$ for $\xi \in V^\vee$

$\xi_1[1] \cdots \xi_k[1] \in S^\bullet(V^\vee[1])$

has deg $= -k$

$\xi[1] \in V^\vee[1]$, degree $= -1$

c) $\Lambda^\bullet(V^\vee[1])$ exterior algebra

graded algebra

d) $v \in V \rightsquigarrow$ graded derivation of deg $+1$

$$\iota_v : \Lambda^\bullet(V^\vee[1]) \longrightarrow \Lambda^\bullet(V^\vee[1])$$

enough to specify ι_v on generators $\xi[1]$

$$\iota_v(\xi[1]) = \xi(v) \in \mathbb{k} \leftarrow \text{ground field.}$$

deg: $\underbrace{+1 \quad -1}_{0}$

e) dg algebra : Koszul algebra

$K(V^\vee) \underset{\circ}{=} S^\circ(V^\vee) \otimes \Lambda^\circ(V^\vee[1])$ with differential

$$d(\xi \otimes 1) = 0 \qquad \xi, \eta \in V^\vee$$

$$d(\underbrace{1 \otimes \eta[1]}_{\deg: -1}) = \eta \otimes 1$$

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all of these constructions work replacing vector spaces V by vector bdls V.

Eg. in example d): $V \to X$ vector bdl, $v: X \to V$ is a section

$\Gamma(X, \underbrace{\Lambda^\circ}_{\text{graded algebra}}(\underset{\uparrow\ \text{bdl}}{V^\vee}[1]))$ graded algebra

graded vb concentrated in degree -1

differential $\iota_v : \Lambda^\bullet(V^\vee[1]) \longrightarrow \Lambda^\bullet(V^\vee[1])$

map of graded vector bdl.
of degree $+1$

get induced map $(\iota_v)_* : \underbrace{\Gamma(X, \Lambda^\bullet(V^\vee[1]))}_{\text{dga.}}$

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vector bdl.

$$\begin{tikzcd} V \arrow[d, "\pi"] \\ X \arrow[u, bend left, "\alpha"] \arrow[u, bend right, "z"'] \end{tikzcd}$$

section

zero section

$$\begin{tikzcd} \mathcal{O}(V) = PF(V) = \{f : V \to \mathbb{R} \mid \text{fiberwise polynomial}\} = \Gamma(X, S^\bullet(V^\vee)) \arrow[d, shift right, "\alpha^*"'] \arrow[d, shift left, "z^*"] \\ \mathcal{O}(X) = C^\infty(X) \end{tikzcd}$$

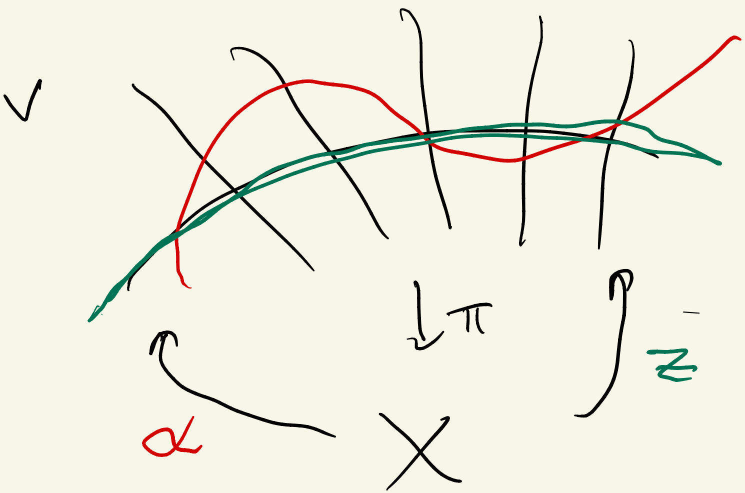
$z^* : \Gamma(X, S^\bullet(V^\vee)) \to C^\infty(X)$ projection to constant term.

$$\alpha^* : \Gamma(X, S^\bullet(V^\vee)) = S^\bullet_{C^\infty(X)}(\Gamma(V^\vee)) \longrightarrow C^\infty(X)$$

$$\xi_1 \cdots \xi_k \longmapsto \alpha^*(\xi_1 \cdots \xi_k) = \alpha^*(\xi_1) \cdots \alpha^*(\xi_k)$$

$$= \xi_1(\alpha) \cdots \xi_k(\alpha) \in C^\infty(X)$$

$\xi_i \in \Gamma(X, V^\vee)$



$$\mathcal{O}(X \overset{h}{\cap} \mathrm{graph}(\alpha))$$

$$= \mathcal{O}(X) \overset{h}{\underset{\mathcal{O}(V)}{\otimes}} \mathcal{O}(\mathrm{graph}(\alpha))$$

$$= \mathcal{O}(X) \overset{h}{\otimes} \mathcal{O}(X)$$

$$\longrightarrow \quad z^*, \mathcal{O}(V), \alpha^*$$

$$\nearrow \quad \hat{\mathcal{O}}(X) \underset{z^*, \mathcal{O}(V), \alpha^*}{\otimes} \mathcal{O}(X)$$

where $\int$ is a resolution of $\mathcal{O}(X)$ by free $\mathcal{O}(V)$-modules

specifies how the two factors $\mathcal{O}(X)$ are modules over $\mathcal{O}(V)$:

$f, g \in \mathcal{O}(X)$
$h \in \mathcal{O}(V)$

$$f \otimes \alpha^*(h) \cdot g = f \cdot z^*(h) \otimes g$$

<u>Lem</u>: $\hat{\mathcal{O}}(X) = \Gamma(X, K(V^\vee)) = \Gamma(X, S^\bullet(V^\vee) \otimes \Lambda^\bullet(V^\vee[1]))$

$$= S^\bullet_R(\Gamma(V^\vee)) \otimes_R \Lambda^\bullet(\Gamma(V^\vee)[1])$$

dga $\nearrow$ $\quad R = C^\infty(X)$

with Koszul differential: $d(\xi \otimes 1) = 0$

$$d(1 \otimes \eta[1]) = \eta \otimes 1$$

In particular, if $V = X \times W$ is the trivial vector bdl, then $\hat{\mathcal{O}}(X) = C^\infty(X) \otimes_{\mathbb{R}} K(W^\vee) = C^\infty(X) \otimes S^\bullet(W^\vee) \otimes \Lambda^\bullet(W^\vee[1])$

$\uparrow$ has homology $\mathbb{R}$ in deg 0, trivial other

$\underline{Q}$: $M, N \underset{\text{submflds}}{\subset} W$, M transverse to N

$$\mathcal{O}(M \cap N) \underset{\text{Lem}}{\cong} \mathcal{O}(M) \otimes_{\mathcal{O}(W)} \mathcal{O}(N) \overset{\text{w.e. ?}}{\simeq}$$

$$\mathcal{O}(M \overset{h}{\cap} N) := \mathcal{O}(M) \overset{h}{\otimes}_{\mathcal{O}(W)} \mathcal{O}(N) = \hat{\mathcal{O}}(M) \otimes_{\mathcal{O}(W)} \mathcal{O}(N)$$

vs ↓

Argument in the linear situation: $N, M \underset{\text{sub vs.}}{\subset} W$

$N \pitchfork M$

$I \oplus M_1 \oplus N_1 = W$

$$\begin{tikzcd}
 & W & \\
I \oplus M_1 = M \arrow[ur] & & N = I \oplus N_1 \arrow[ul] \\
 & M \cap N = I \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\hat{\mathcal{O}}(M) \otimes_{\mathcal{O}(W)} \mathcal{O}(N) = \left(\mathcal{O}(M) \otimes K(N_1^\vee)\right) \otimes_{\mathcal{O}(W)} \mathcal{O}(N)$$

$$= \left(\mathcal{O}(I) \otimes \cancel{\mathcal{O}(M_1)} \otimes K(N_1^\vee)\right) \otimes_{\cancel{\mathcal{O}(I)} \otimes \cancel{\mathcal{O}(M_1)} \otimes \cancel{\mathcal{O}(N_1)}} \left(\cancel{\mathcal{O}(I)} \otimes \cancel{\mathcal{O}(N_1)}\right)$$

$$\simeq \mathcal{O}(I) \otimes K(N_1^\vee) \overset{w.e.}{\sim} \mathcal{O}(I) = \mathcal{O}(M \cap N).$$

<u>Cor</u>: $\mathcal{O}(X) \otimes^h_{z^*, \mathcal{O}(V), \alpha^*} \mathcal{O}(X) \cong \Gamma(X, \Lambda^\bullet(V^\vee[1]))$ with differential ι_α

<u>Pf</u>: $\quad \| \text{ Lem.}$

$$(1 \otimes \xi[1]) \otimes 1 \in \Gamma(X, S^\bullet(V^\vee) \otimes \Lambda^\bullet(V^\vee[1])) \otimes_{z^*, \mathcal{O}(V), \alpha^*} \mathcal{O}(X)$$

$$\Gamma(X, S^\bullet(V^\vee)) \otimes_R \Gamma(X, \Lambda^\bullet(V^\vee[1])) \qquad \Gamma(X, \cancel{S^\bullet(V^\vee)}) \qquad \Gamma(X, \Lambda^\bullet(V^\vee[1]))$$

$R = C^\infty(X)$

care with differential:

$$d((1 \otimes \xi[1]) \otimes 1) = (\xi \otimes 1) \otimes 1 = (1 \otimes 1) \otimes \alpha^*(\xi) = (1 \otimes 1) \otimes \iota_\alpha \xi \qquad \square$$