Recall: classical field theory consists of data:

- M mfd (spacetime mfd)
- a sheaf $M \underset{\text{open}}{\supset} \mathcal{U} \longmapsto \mathcal{E}(\mathcal{U})$ = space of fields in the spacetime region $\mathcal{U}$
- an action functional

$$S : \mathcal{E}(\mathcal{U}) \longrightarrow \mathbb{R}$$

Then $\mathrm{Crit}(S) = \{\text{critical points of } S\} = \{\text{classical fields}\}$

$$\boxed{\text{e.g. } \mathcal{E}(\mathcal{U}) = \{\underset{\mathbb{R}}{\mathcal{U}} \xrightarrow{\gamma} \underset{\substack{\text{"target mfd"} \\ \text{Riemannian mfd}}}{X}\} \quad S(\gamma) = \text{energy of } \gamma \quad \{\text{geodesics } \mathcal{U} \xrightarrow{\gamma} X\}}$$

$\mathrm{Crit}(S) = \{\text{fields which satisfy Euler-Lagrange equations}\} = EL(\mathcal{U})$

$\{\text{classical observables in } \mathcal{U}\} = \mathrm{Obs}^{cl}(\mathcal{U}) = \mathcal{O}(EL(\mathcal{U}))$

functions on $EL(\mathcal{U})$

$M \overset{\text{open}}{\supset} \mathcal{U} \longmapsto \mathrm{Obs}^{cl}(\mathcal{U}) \in \mathrm{Vect}$

factorization algebra (should be)

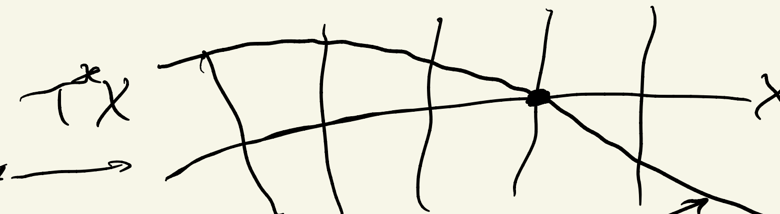
<u>Q:</u> Why should we consider Ch-valued factorization algebras in this context?

<u>assume</u> $X = \mathcal{E}(\mathcal{U})$ mfd $\dim X < \infty$

$S : X \to \mathbb{R}$ smooth

$\mathrm{Crit}(S) = \{ x \in X \,/\, dS_x = 0 \}$ $\quad dS \in \Omega^1(X) = \Gamma(X, T^*X)$

$= X \cap \mathrm{graph}(dS).$



X = zero section $\longrightarrow$

$\{(x, dS_x) \in T^*X\} = \mathrm{graph}(dS)$

if X & $\mathrm{graph}(dS)$ intersect transversally in T^*X

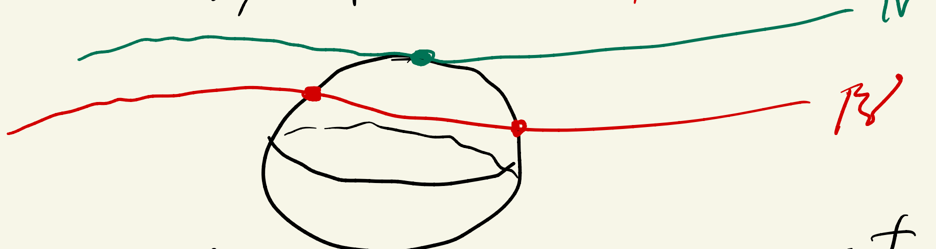
$\Rightarrow X \cap \mathrm{graph}(dS)$ is a submfd of T^*X

Def: M, N submflds of W

$M \pitchfork N$ i.e. M is **transversal** to N

if for all $x \in M \cap N$, $\quad T_xM + T_xN = T_xW$

Ex: $W = \mathbb{R}^3$, $M = S^2$



~~$M \pitchfork N'$~~

$M \pitchfork N$

Idea: define the **derived intersection** $M \overset{h}{\cap} N$

or rather $C^\infty(M \overset{h}{\cap} N)$

Q: Can $C^\infty(M \pitchfork N)$ be expressed in terms of $C^\infty(M)$, $C^\infty(N)$ & $C^\infty(W)$?

$$M \underset{i_M}{\hookrightarrow} W \rightsquigarrow C^\infty(W) \xrightarrow{i_M^*} C^\infty(M)$$

algebra hom.

$$C^\infty(W) \to C^\infty(N)$$

Lem:

$$\Phi : C^\infty(M) \otimes_{C^\infty(W)} C^\infty(N) \xrightarrow{\cong} C^\infty(M \pitchfork N)$$

$$\Rightarrow C^\infty(M), C^\infty(N) \text{ algebras over } C^\infty(W)$$

$$f \otimes g \longmapsto (f|_{M\cap N}) \cdot (g|_{M \cap N})$$

is an isomorphism of algebras.

__Pf:__ Using partitions of unity it suffices to prove this locally, i.e. can assume $W \supseteq M, N$ are vector spaces/subspaces.

$$W = M\cap N \oplus M' \oplus N'$$

$$\begin{tikzcd}
 & W \arrow[dl, shift right, "p_M"'] \arrow[dr, shift left, "p_N"] & \\
M \arrow[ur, shift right] \arrow[dr, shift right, "q_M"'] & & N \arrow[ul, shift left] \arrow[dl, shift left, "q_N"] \\
 & M\cap N \arrow[ul, shift right] \arrow[ur, shift left] &
\end{tikzcd}$$

$$M\cap N \oplus M' = M \qquad N = M \cap N \oplus N'$$

<u>Surj</u>: $\quad f \in C^\infty(M \cap N)$

$$\Phi\Big(\underset{\underset{C^\infty(M)}{\cap}}{(q_M^* f)} \otimes \underset{\underset{C^\infty(N)}{\cap}}{1} \Big) = (q_M^* f)|_{M\cap N} \cdot 1 = f \quad \checkmark$$

<u>Inj</u>: $$\underset{\underset{C^\infty(M) \otimes_{C^\infty(W)} C^\infty(N)}{\ni}}{\sum_{i=1}^{K}} \underset{\underset{C^\infty(M)}{\cap}}{f_i} \otimes \underset{\underset{C^\infty(N)}{\cap}}{g_i} = \sum f_i \otimes (p_N^* g_i)|_N = \underset{\underset{f \otimes 1}{\|}}{\sum f_i (p_N^* g_i)|_M \otimes 1}$$

where $f = \sum f_i (p_N^* g_i)|_M$

$$f \otimes 1 = (p_N^* f)|_M \otimes 1 = 1 \otimes (p_M^* f)|_N = 1 \otimes q_N^*(f|_{M\cap N}).$$

in particular, $\quad 0 = \Phi(f \otimes 1) = f|_{M\cap N} \implies f \otimes 1 = 0 \quad \square$

The lemma above suggests:

<u>Def</u>: for any submflds $N, M \subset W$

define $C^\infty(\underbrace{N \overset{h}{\cap} M}_{\text{derived intersection}}) := C^\infty(N) \overset{h}{\underset{C^\infty(W)}{\otimes}} C^\infty(M)$

derived tensor product.

this is a derived mfd, i.e. a mfd equipped with a sheaf of dg algebras.

<u>Note</u>: if $M \pitchfork N$, then $C^\infty(N \cap M) = C^\infty(N) \underset{C^\infty(W)}{\otimes} C^\infty(M)$

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special case: $W = \begin{array}{c} V \\ \downarrow \\ X \end{array} \Big\}$ vector bdl.

$M = \text{graph}(\alpha)$

$\alpha \in \Gamma(X, V)$

$N = X \subset V$

to calculate $C^\infty(X \overset{h}{\cap} \text{graph}(\alpha))$, we need a free resolution of $C^\infty(X)$ as $C^\infty(V)$-module.

"polynomial" $C^\infty(V) \supset \{ f : V \to \mathbb{R} \mid f|V_x$ is a polynomial function for each $x \in X \}$

$$\begin{array}{c} V \\ \downarrow \\ X \end{array}$$

$\to PF(V) = \Gamma(X, S^\circ(V^\vee))$

$f|V_x \in S^\circ(V_x^\vee)$

symm. power — S°; dual space to V_x — $V_x^\vee$

$$S^\circ(V_x^\vee) = \underbrace{S^\circ}_{\text{symm. power of vector bdl.}}(V^\vee)_x$$

<u>digression:</u>

graded vector space : $V = \bigoplus_{k \in \mathbb{Z}} V_k$

differential graded vector space = dg vector space

= graded vector space + differential d of degree $+1$.

aka cochain complexes.

<u>dg algebra</u> = graded algebra $\left(A = \bigoplus_{k \in \mathbb{Z}} A_k \quad A_k \cdot A_\ell \subset A_{k+\ell} \right)$

+ differential d

compatibility condition :

$$\underbrace{d(ab) = (da)b + (-1)^{|a|} a(db)}_{d \text{ is a } \underline{\text{graded derivation}} \text{ of } A.}$$

$a, b \in A$