Recall: diagram $F : I \to \mathcal{C}$

limit diagram = $\underbrace{\text{terminal}}$ object in $\underbrace{l}$ cone(F)

colimit " initial " " r cone(F)

$$\text{colim} : \begin{array}{ccc} \text{Fun}(I, \text{Vect}) & \longrightarrow & \text{Vect} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ F & \longmapsto & \text{colim}\, F \end{array}$$

} some kind of summing/integrating

<u>analogy</u>: I finite set
(or finite measure space)
V vector space
$\mathcal{O}(I, V) = \{ f : I \to V \}$
(or $\{ f : I \xrightarrow[\text{measurable}]{} V$)

$$\begin{array}{ccc} \mathcal{O}(I, V) & \longrightarrow & V \\ f & \longmapsto & \sum_{i \in I} f(i) \end{array} \quad \left(\text{or } \int_I f \mu\right)$$

map $I \xrightarrow{g} J$

$$\begin{array}{ccc} \mathcal{O}(I, V) & \xleftarrow{g^*} & \mathcal{O}(J, V) \\ g^* f = f \circ g & \longleftarrow & f \end{array}$$

$$\mathcal{O}(I,V) \xrightarrow{g_!} \mathcal{O}(J,V)$$
$$f \longmapsto (g_!f)(j) = \sum_{i \in g^{-1}(j)} f(i)$$

<u>properties</u>: i) $I \xrightarrow{g} *$ ← one point set

$$(g_!f) = \sum_{i \in I} f(i)$$

$f \in \mathcal{O}(I,V)$

ii) $I \xrightarrow{g} J \xrightarrow{h} K$

$(h \circ g)_! = h_! \, g_! \; : \; \mathcal{O}(I,V) \to \mathcal{O}(K,V)$

<u>goal</u>: do this for diagrams; i.e.

given functor $G: I \to J$

$$\begin{array}{ccc} \mathfrak{Z} & & \\ \mathrm{Fun}(I, \mathrm{Vect}) & \xleftarrow{G^*} & \mathrm{Fun}(J, \mathrm{Vect}) \\ F \circ G & \longleftarrow & F \end{array}$$

want to define $G_! : \mathrm{Fun}(I, \mathrm{Vect}) \to \mathrm{Fun}(J, \mathrm{Vect})$

$$F \in \mathrm{Fun}(I,\mathrm{Vect})$$

$G_! F : J \to \mathrm{Vect}$ Left Kan extension

$$G_! F(j) := \operatorname*{colim}_{i \in \underbrace{G\downarrow j}_{\text{or } G/j}} F(i) \leftarrow \text{analog of } g^{-1}(j)$$

Def. $G : I \to J$, $j \in \mathrm{ob}\, J$

category $G\downarrow j$:

objects : $\left(\underset{\in \mathrm{ob}(I)}{i}, \; G(i) \xrightarrow{f} j \right)$ ← morphism in J

morphisms :

$$\begin{tikzcd} i \arrow[d, "g"'] \\ i' \end{tikzcd} \qquad \begin{tikzcd} G(i) \arrow[d, "G(g)"'] \arrow[r, "f"] & j \\ G(i') \arrow[ur, "f'"'] & \end{tikzcd}$$

$G\downarrow j$ is called over category or slice category G/j (or comma category)

$G : I \to J$ $\qquad$ $\mathrm{Fun}(I, \mathrm{Vect}) \underset{G_!}{\overset{G^*}{\rightleftarrows}} \mathrm{Fun}(J, \mathrm{Vect})$

$G_!$ is the <u>left adjoint</u> of G^*, i.e.

$F \in \mathrm{Fun}(I, \mathrm{Vect})$
$H \in \mathrm{Fun}(J, \mathrm{Vect})$

$$\mathrm{mor}(G_! F, H) \xleftrightarrow[\text{bijection}]{\text{nat.}} \mathrm{mor}(F, G^* H)$$

objects in $\mathrm{Fun}(J, \mathrm{Vect})$

natural transf.

$$\begin{tikzcd} J \arrow[r, bend left, "G_! F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow T"] & \mathrm{Vect} \end{tikzcd}$$

Hence $G_!$ is called
<u>left</u> Kan extension.

There is also a right adjoint to G^*, called the <u>right Kan extension</u>; it can be constructed via <u>limits</u>.

Properties of Kan extension:

i) if $G : I \to *$ then $G_! F = \operatorname{colim}_I F$

ii) $I \xrightarrow{G} J \xrightarrow{H} K$

$$\begin{tikzcd}
\mathrm{Fun}(I, \mathrm{Vect}) \arrow[rr, bend left, "H_! \circ G_!"] \arrow[rr, bend right, "(H \circ G)_!"'] \arrow[rr, phantom, "T \Downarrow \cong"] & & \mathrm{Fun}(K, \mathrm{Vect})
\end{tikzcd}$$

invertible nat. transf.

<u>Special case:</u>

$F : I \times J \to \mathrm{Vect}$

$$\begin{tikzcd}
 & I \times J \arrow[dl, "p^I"'] \arrow[dd, "q"] \arrow[dr, "p^J"] & \\
I \arrow[dr, "q^I"'] & & J \arrow[dl, "q^J"] \\
 & * &
\end{tikzcd}$$

product category

comm. diagram of functors

$$q^I_! \circ p^I_! \cong q_! F \cong q^J_! (\circ p^J_!)$$

$$\operatorname*{colim}_{i\in I}\ \operatorname*{colim}_{j\in J} F(i,j) = \operatorname*{colim}_{I\times J} F = \operatorname*{colim}_{J}\ p_!^J F = \operatorname*{colim}_{j\in J} (p_!^J F)(j) = \operatorname*{colim}_{j\in J}\ \operatorname*{colim}_{i\in I} F(i,j)$$

"Fubini theorem" for colimits

slogan: colimits commute.

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digression: homotopy colimits in Top and Ch

Example of a colimit (a pushout) in Top:

$$\operatorname{colim}\begin{pmatrix} S^1 & \hookrightarrow & D^2 \\ \downarrow & & \\ D^2 & & \end{pmatrix} = D^2 \cup_{S^1} D^2 \approx S^2$$

$\uparrow$

$$\operatorname{colim}\begin{pmatrix} S^1 \dashrightarrow * \\ \downarrow \\ * \end{pmatrix} = * \longleftarrow$$ not homotopy equivalent!

"homotopy equivalent diagram"

<u>upshot:</u> colim construction is <u>not</u> compatible with the natural notion of (weak) homotopy.

<u>We want:</u> $X \xrightarrow{f} Y$ map in Top is a weak hmtp. equiv. if it induces isomorphisms on π_*.

if $X', X : I \to Top$ are diagrams,

$$\begin{tikzcd} I \arrow[r, bend left, "X"] \arrow[r, bend right, "X'"'] \arrow[r, phantom, "\Downarrow T"] & Top \end{tikzcd}$$

T is a <u>weak equivalence of diagrams</u> if $T(i) : X(i) \to X'(i)$ is a w.e.

the <u>homotopy colimit</u> $\operatorname{hocolim}_I X \in Top$ for all $i \in ob(I)$

has the property that if $T : X' \to X$ is a weak equiv. of diagrams then the induced map $\operatorname{hocolim}_I X \longrightarrow \operatorname{hocolim}_I X'$ is a weak equivalence.

In addition, it allows the construction of a homotopy left Kan extension with all the same propertie as before.

We will black box "hocolims" and only give an explicit description for simplicial spaces, i.e. $X_\bullet : \Delta^{op} \to Top$

and " chaincx. $C_\bullet : \Delta^{op} \to Ch$.