

Recall: diagram  $F: I \rightarrow \mathcal{C}$

limit diagram = terminal object in  $\varprojlim \text{cone}(F)$   
 colimit " " initial " "  $\varinjlim \text{cone}(F)$

colim:  $\text{Fun}(I, \text{Vect}) \xrightarrow{\quad} \text{Vect}$   
 $\downarrow F \quad \quad \quad \downarrow \omega$   
 $\quad \quad \quad \quad \quad \quad \text{colim } F$  } some kind of summing/integrating

analogy:  $I$  finite set  
 (or finite measure space)

$\mathcal{O}(I, V) \xrightarrow{\quad} V$   
 $f \mapsto \sum_{i \in I} f(i)$  (or  $\int_I f \mu$ )

$V$  vector space  
 $\mathcal{O}(I, V) = \{f: I \rightarrow V\}$   
 (or  $\{f: I \rightarrow V\}$  measurable)

map  $I \xrightarrow{g} J$   
 $\mathcal{O}(I, V) \xleftarrow{g^*} \mathcal{O}(J, V)$   
 $g^* f = f \circ g \quad \quad \quad f$

$$\begin{array}{ccc} \mathcal{O}(I, V) & \xrightarrow{g!} & \mathcal{O}(J, V) \\ f & \longmapsto & (g!f)(j) = \sum_{i \in g^{-1}(j)} f(i) \end{array}$$

properties: (i)  $I \xrightarrow{g} *$  ← one point set

$$(g!f) = \sum_{i \in I} f(i)$$

$$f \in \mathcal{O}(I, V)$$

$$(ii) \quad I \xrightarrow{g} J \xrightarrow{h} K$$

$$(h \circ g)! = h! \circ g! : \mathcal{O}(I, V) \rightarrow \mathcal{O}(K, V)$$

goal: do this for diagrams, i.e.

given functor  $G: I \rightarrow J$

$$\begin{array}{ccc} \text{Fun}(I, \text{Vect}) & \xleftarrow{G^*} & \text{Fun}(J, \text{Vect}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Fun}(I, \text{Vect}) & \xleftarrow{F \circ G} & \text{Fun}(J, \text{Vect}) \end{array}$$

want to define  $G_! : \text{Fun}(I, \text{Vect}) \rightarrow \text{Fun}(J, \text{Vect})$

$$G_! F : J \rightarrow \text{Vect} \quad \text{Left Kan extension}$$

$$G_! F(j) := \text{colim}_{i \in \underbrace{G \downarrow j}_{\text{or } G/j}} F(i) \leftarrow \text{analog of } g^{-1}(j)$$

Def:  $G : I \rightarrow J, j \in \text{ob } J$

category

$G \downarrow j :$

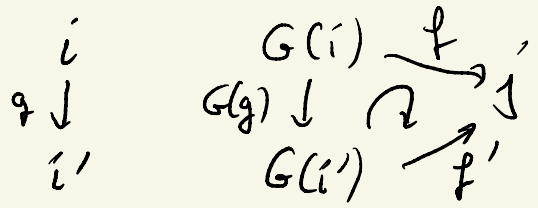
morphism in  $J$

objects:

$(i \in \text{ob}(I), G(i) \xrightarrow{f} j)$

$G \downarrow j$  is called over category or slice category  $G/j$

morphisms:



(or comma category)

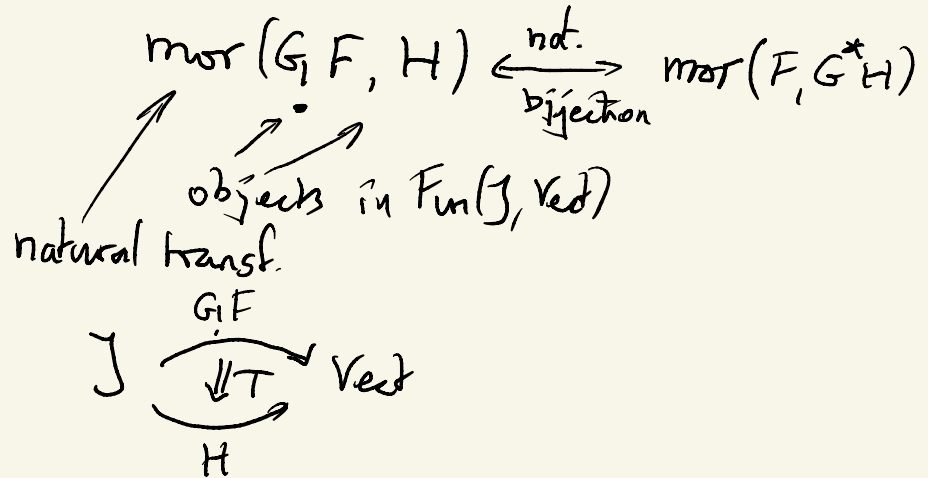
$$G: \mathcal{I} \rightarrow \mathcal{J}$$

$$\text{Fun}(\mathcal{I}, \text{Vect}) \begin{array}{c} \xleftarrow{G^*} \\ \xrightarrow{G_!} \end{array} \text{Fun}(\mathcal{J}, \text{Vect})$$

$G_!$  is the left adjoint of  $G^*$ , i.e.

$$F \in \text{Fun}(\mathcal{I}, \text{Vect})$$

$$H \in \text{Fun}(\mathcal{J}, \text{Vect})$$



Hence  $G_!$  is called left Kan extension.

There is also a right adjoint to  $G^*$ , called the right Kan extension; it can be constructed via limits.

Properties of Kan extension:

i) if  $G: \mathbb{I} \rightarrow \ast$  then  $G_! F = \text{colim}_{\mathbb{I}} F$

ii)  $\mathbb{I} \xrightarrow{G} \mathbb{J} \xrightarrow{H} \mathbb{K}$   
 $H_! \circ G_!$

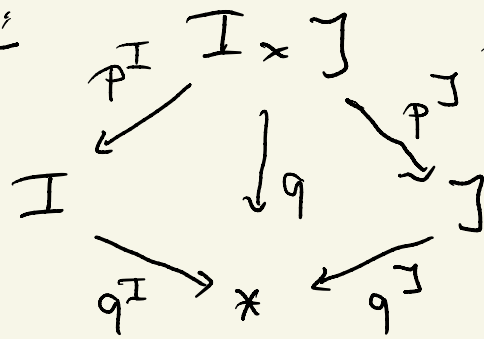
$$\text{Fun}(\mathbb{I}, \text{Vect}) \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \cong \\ \xrightarrow{\quad} \end{array} \text{Fun}(\mathbb{K}, \text{Vect})$$

$(H \circ G)_!$

invariable nat. transf.

Special case:

$F: \mathbb{I} \times \mathbb{J} \rightarrow \text{Vect}$



product category

comm. diagram of functors

$$q_!^{\mathbb{I}} \circ p_!^{\mathbb{I}} \cong q_! F \cong q_!^{\mathbb{J}} (p_!^{\mathbb{I}})$$

$$\begin{array}{ccc} \parallel & & \parallel \\ \operatorname{colim}_{i \in I} \operatorname{colim}_{j \in J} F(i, j) & \operatorname{colim}_{I \times J} F & \end{array}$$

"Fubini Theorem" for colimits

slogan: colimits commute.

$$\begin{array}{c} \parallel \\ \operatorname{colim}_{j \in J} P_i^J F \end{array}$$

$$\parallel \\ \operatorname{colim}_{j \in J} (P_i^J F)(j)$$

$$\parallel \\ \operatorname{colim}_{i \in I} \operatorname{colim}_{j \in J} F(i, j)$$

digression: homotopy colimits in Top and Ch

Example of a colimit (a pushout) in Top:

$$\operatorname{colim} \left( \begin{array}{ccc} S^1 & \hookrightarrow & D^2 \\ \downarrow & & \\ D^2 & & \end{array} \right) = D^2 \cup_{S^1} D^2 \approx S^2$$

↑

$$\operatorname{colim} \left( \begin{array}{ccc} S^1 & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} \right) = * \quad \leftarrow \quad \begin{array}{l} \text{not homotopy} \\ \text{equivalent!} \end{array}$$

↳ "homotopy equivalent diagram"

upshot: colim construction is not compatible with the natural notion of (weak) homotopy.

We want:

$$X \xrightarrow{f} Y$$

map in  $\operatorname{Top}$  is a weak hmtp. equiv. if it induces isomorphisms on  $\pi_x$ .

if  $X', X : I \rightarrow \operatorname{Top}$

$$I \begin{array}{ccc} & \xrightarrow{X} & \operatorname{Top} \\ & \Downarrow T & \\ & \xrightarrow{X'} & \operatorname{Top} \end{array}$$

are diagrams,

$T$  is a weak equivalence of diagrams if

$$T(i) : X(i) \rightarrow X'(i) \text{ is a w.e.}$$

The homotopy colimit  $\text{hocolim}_{\mathcal{I}} X \in \text{Top}$  for all  $i \in \text{ob}(\mathcal{I})$

has the property that if  $T: X' \rightarrow X$  is a weak equiv. of diagrams, then the induced map  $\text{hocolim}_{\mathcal{I}} X \rightarrow \text{hocolim}_{\mathcal{I}} X'$

is a weak equivalence.

In addition, it allows the construction of a homotopy left Kan extension with all the same properties as before.

We will black box "hocolims" and only give an explicit description for



simplicial spaces, i.e.  $X_\bullet : \Delta^{op} \rightarrow \text{Top}$   
and " chains.  $C_\bullet : \Delta^{op} \rightarrow \text{Ch}$ .