Let I be a small category, i.e. the objects of I form a set

Def: A <u>diagram of shape I in a category $\mathcal{C}$</u> is a functor $F : I \to \mathcal{C}$

Ex: $I = $
$$\begin{tikzcd} \bullet_1 \arrow[d, shift right] \arrow[d, shift left] \\ \bullet_2 \end{tikzcd}$$

$$\begin{tikzcd} F(1) = X_1 \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ F(2) = X_2 \end{tikzcd}$$

$\mathrm{Fun}(I, \mathcal{C})$ = functor category, i.e.

obj: functors $I \xrightarrow{F} \mathcal{C}$

mor: natural transformations:
$$\begin{tikzcd} I \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, phantom, "\Downarrow T"] \end{tikzcd}$$

<u>diagram category</u>.

Def: left cone of I : $I^{\triangleleft}$ ← category

$$ob(I^{\triangleleft}) = ob(I) \sqcup \{\phi\}$$

↑ is an initial object in $I^{\triangleleft}$; all other morphisms are the morphisms in I

Ex: $I =$
$$\begin{tikzcd} 1 \arrow[d, shift left] \arrow[d, shift right] \\ 2 \end{tikzcd}$$

$I^{\triangleleft}$:
$$\begin{tikzcd} & 1 \arrow[dd, shift left] \arrow[dd, shift right] \\ \phi \arrow[ur] \arrow[dr] & \\ & 2 \end{tikzcd}$$

Def: right cone : $I^{\triangleright}$

$$ob(I^{\triangleright}) = ob(I) \sqcup \{*\}$$ ← terminal object in $I^{\triangleright}$

$$I \underset{\text{full subcat}}{\subset} I^{\triangleright}$$

Ex: $I^{\triangleright}$:
$$\begin{tikzcd} 1 \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr] & \\ & * \\ 2 \arrow[ur] & \end{tikzcd}$$

Def: A left (right) cone over a diagram $F : I \to \mathcal{C}$ is a functor $G : I^{\triangleleft} \to \mathcal{C}$ s.t. $G|_I = F$

$G : I^{\triangleright} \to \mathcal{C}$ s.t. $G|_I = F$

Ex:

$$\underbrace{\begin{tikzcd} & X_1 \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\ X = G(\emptyset) \arrow[ur] \arrow[dr] & \\ & X_2 \end{tikzcd}}_{\text{left cone}} \qquad \underbrace{\begin{tikzcd} X_1 \arrow[dd, shift right] \arrow[dd, shift left] \arrow[dr] & \\ & G(*) = X \\ X_2 \arrow[ur] & \end{tikzcd}}_{\text{right cone}}$$

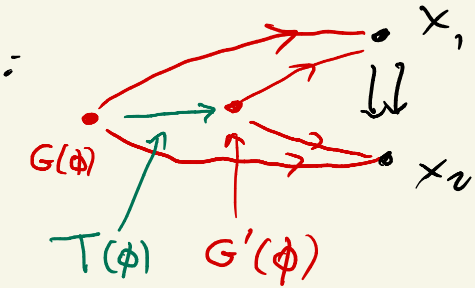
Def: left cones over F are the objects in the category $\mathrm{lcone}(F)$.

morphisms $G \to G'$:

$$\begin{tikzcd} I^{\triangleleft} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow T"] & \mathcal{C} \end{tikzcd}$$

$T(i) = \mathrm{id}_{F(i)}$ if $i \in I$,

Ex:



$y, x \in I^{\triangleleft}$

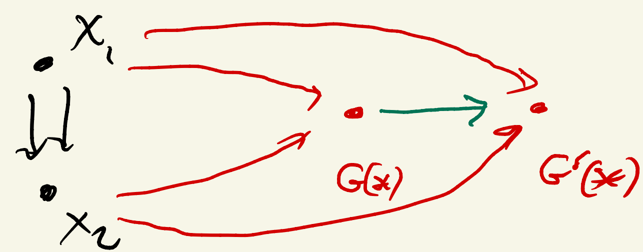
$y \xrightarrow{f} x$

$$\begin{tikzcd} G(y) \arrow[r, "G(f)"] \arrow[d, "T(y)"'] \arrow[dr, phantom, "\circlearrowright"] & G(x) & \in \mathcal{C} \\ G'(y) \arrow[r, "G'(f)"'] & G'(x) \arrow[u, leftarrow, "T(x)"] & \in \mathcal{C} \end{tikzcd}$$

naturality of T means that all these diagrams commute.

similarly, we define the category $r\mathrm{cone}(F)$ of <u>right cones over F</u>.

Ex:



Def: $F : I \to \mathcal{C}$ is a diagram in $\mathcal{C}$

A *limit diagram* is a terminal object G in the category $\ell\mathrm{cone}(F)$.

Then $G(\phi)$ is called *the limit of F*,
$\phi \in I^{\triangleleft}$

notation: $G(\phi) = \lim\limits_{I} F$

or $\lim(F : I \to \mathcal{C})$

or $\lim\limits_{i \in I} F(i)$

⚠ $\lim\limits_{I} F$ is well defined up to unique isomorphisms.

A *colimit diagram* is an initial object G in $r\mathrm{cone}(F)$.

$G(x) \in \mathcal{C}$ is called the colimit of F over I.

Notation: $\operatorname{colim}_{I} F$ or $\operatorname{colim}_{i \in I} F(i)$ or $\operatorname{colim}(F : I \to \mathcal{C})$.

<u>Def</u>: If all limits in $\mathcal{C}$ exist, then $\mathcal{C}$ is called <u>complete</u>.

" co " " " <u>cocomplete</u>.

Explicit construction in $\mathcal{C} = \text{Set}, \text{Top}$:

$X : I \to \text{Set}$

$$\lim_{I} X = \left\{ (x_i) \in \prod_{i \in I} X(i) \;\middle|\; \begin{array}{l} f : i \to j \text{, i.e. } f \in I(i,j) \\ X(i) \xrightarrow{X(f)} X(j) \\ \;\;\rotatebox{90}{$\in$} \\ x_i \longmapsto X(f)(x_i) = x_j \end{array} \right\}$$

requirement $\forall$ morphisms f

$$\operatorname*{colim}_I X = \coprod_{i \in I} X(i) \Big/ \sim$$

$\sim$ equiv. relation gen. by $x \sim X(f)(x)$, $x \in X(i)$, $X(f)(x) \in X(j)$

$f \in I(i,j)$, $X(i) \xrightarrow{X(f)} X(j)$

in particular Set, Top are complete + cocomplete.

for $\mathcal{C} = \mathrm{Vect}$: limit description is as for Set, Top.

$V : I \to \mathrm{Vect}$ $\qquad \operatorname*{colim}_I V = \bigoplus_{i \in I} V(i) \Big/ v \sim V(f)(x)$, $v \in V(i)$, $V(f)(x) \in V(j)$.

<u>Rem:</u> if I has an initial object $\phi \in I$

then $\lim_I F \cong F(\phi)$ $\qquad$ HW!

$* \in I$ terminal object $\Rightarrow \operatorname*{colim}_I F \cong F(*)$

$V : I \to \mathrm{Vect}$	$\lim_{I} V$	$\mathrm{colim}_{I} V$
I discrete category, i.e. only identity morphisms.	$\prod_{i \in I} V(i)$	$\bigoplus_{i \in I} V(i)$
$V_1 \overset{f}{\underset{g}{\rightrightarrows}} V_2$	$\mathrm{eq}\left(V_1 \overset{f}{\underset{g}{\rightrightarrows}} V_2\right)$	$\mathrm{coeq}\left(V_1 \overset{f}{\underset{g}{\rightrightarrows}} V_2\right)$
$U \xrightarrow{f} V$, $U \xrightarrow{g} W$	U	$V \oplus W \big/ f(u) \sim g(u)$ for all $u \in U$ **pushout**
$V \xrightarrow{f} U$, $W \xrightarrow{g} U$	$\{(v,w) \in V \times W \,/\, f(v) = g(w)\}$ **pullback**	U
$\rho : G \to \mathrm{Aut}(W)$ representation $*/\!/G = I \longrightarrow \mathrm{Vect}$ __obj:__ one object $* \longmapsto W$ $\mathrm{mor}(*,*) \cong G \ni g \longmapsto \rho(g)$	$W^G = \{w \in W \,/\, \rho(g)w = w\}$ G-invariants	$W_G = W \big/ w \sim \rho(g)w$ G-coinvariants.