Recall: $\mathcal{C}$ category with pull backs

$c \in \mathcal{C}$ $\quad$ $\underline{C} = \{C_a \xrightarrow{f_a} c\}_{a \in A}$ $\quad \subset$ (full subcat.) $\quad \mathcal{C}/c$

associate sieve $\quad \hat{\underline{C}} = \{d \xrightarrow{f} c \mid \exists \text{ factorization } \} \subset \mathcal{C}/c$

$$\begin{array}{ccc} d & \xrightarrow{f} & c \\ & \searrow & \uparrow f_a \\ & & C_a \end{array}$$

Functor: $\quad P_{fin}(A) \xrightarrow{\alpha} \hat{\underline{C}}$

on objects: $\quad ([n] \xrightarrow{a} A) \longmapsto (C_{a_0} \times_c \cdots \times_c C_{a_n} \xrightarrow{f_a} c)$

morphisms:

$$\begin{array}{ccc} [n] & \xrightarrow{a} & \\ g \uparrow & & A \\ [n'] & \xrightarrow{a'} & \end{array} \quad \longmapsto \quad \begin{array}{ccc} C_{a_0} \times_c \cdots \times_c C_{a_n} & \xrightarrow{f_a} & \\ g^* \downarrow & & c \\ C_{a'_0} \times_c \cdots \times_c C_{a'_{n'}} & \xrightarrow{f_{a'}} & \end{array}$$

$$P_{fin}(A) \xrightarrow{\alpha} \hat{\underline{C}} \subset \mathcal{C}/c \to \mathcal{C} \xrightarrow{F} Ch$$

Prop A: $\quad \operatorname{hocolim}_{P_{fin}(A)} F \circ \alpha \xrightarrow[\sim]{\text{w.e.}} \operatorname{hocolim}_{\hat{\underline{C}}} F$

Thm ("Quillen's Thm. A")

e.g. Riehl: Categorical homotopy theory

Thm. 8.5.6

functor $\mathcal{C} \xrightarrow{\alpha} \mathcal{D}$, $\mathcal{D} \xrightarrow{F} Ch$

$\underset{\mathcal{C}}{\text{hocolim}}\, F\circ\alpha \xrightarrow[\sim]{w.e.} \underset{\mathcal{D}}{\text{hocolim}}\, F$

provided α is homotopy final.

Def. $\mathcal{C} \xrightarrow{\alpha} \mathcal{D}$ is homotopy final if $\forall d \in \mathcal{D}$

Riehl: 8.5.1 the nerve of the slice category d/α is contractible

slice category: d/α

objects: $(c, f: d \to \alpha(c))$, $c \in \mathcal{C}$, $f \in \mathcal{D}(d, \alpha(c))$

topological analog: $\mathcal{C} \xrightarrow{\alpha} \mathcal{D}$ maps between top. spaces

$d \in \mathcal{D}$

$\{(c \in \mathcal{C}$, path from d to $\alpha(c)\}$.

"homotopy fiber over d"

morphisms:

$$\begin{array}{ccccc} c & & & \overset{f}{\nearrow} & \alpha(c) \\ g\downarrow & & d & & \downarrow \alpha(g) \\ c' & & & \underset{f'}{\searrow} & \alpha(c') \end{array}$$

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digression: nerve of a category $\mathcal{C}$

Def: $N(\mathcal{C})$ The **nerve of $\mathcal{C}$** is a simplicial set

$$N(\mathcal{C}) : \Delta^{op} \longrightarrow \mathrm{Set}$$

$$\{0, \dots, n\} = [n] \longmapsto \left\{ \begin{array}{l} n\text{-tupels of composable} \\ \text{morphisms in } \mathcal{C} \end{array} \right\}$$

$$= \left\{ c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \to \cdots \xrightarrow{f_n} c_n \right\}$$

$$= \{ \text{functor } f : [n] \to \mathcal{C} \}$$

diagram in $\mathcal{C}$ $\longrightarrow$ $c_0 \to c_1 \to \cdots \to c_n$

shape of this diagram

$$\underset{0}{\bullet} \to \underset{1}{\bullet} \to \underset{2}{\bullet} \to \cdots \to \underset{n}{\bullet}$$

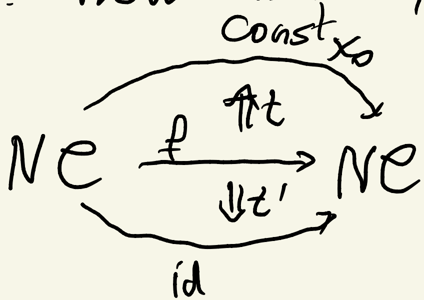
$\|$

$\{0, \dots, n\} = [n]$ $\leftarrow$ partially ordered set, thought of as category.

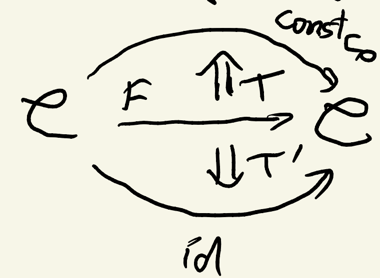
$$\begin{array}{ccc} [n] & & f \in \{ f : [n] \to \mathcal{C} \} \\ g \uparrow & \longmapsto & \downarrow g^* \\ [n'] & & f \circ g \in \{ f' : [n'] \to \mathcal{C} \} \end{array}$$

Q: how would you prove contractibility of $N(\mathcal{C})$?

$x_0 \in N\mathcal{C}$ $\quad x_0 \longleftrightarrow$ object $c_0 \in \mathcal{C}$



$\xleftarrow{\text{apply } N(\ )}$



f map, t hmtp. from f to $const_{x_0}$

t' " " id.

$\Downarrow$

$N\mathcal{C}$ is contractible.

F functor

T, T' nat. transformations.

Proof of Prop A: We need to show that

$$P_{fin}(A) \xrightarrow{\alpha} \hat{\underline{C}}$$

is hmtp. final, i.e. need to show that for $\forall\, (d \xrightarrow{f} c) \in \hat{\underline{C}}$ the slice cat. $(d \xrightarrow{f} c)/\alpha$ has contractible nerve.

$(d \xrightarrow{f} c)/\alpha$

objects: $\left([n] \xrightarrow{a} A,\ d \xrightarrow{\phi} c_{a_0} \times_c \cdots \times_c c_{a_n} \xrightarrow{f_a} c\right)$, where the composite $d \to c$ is f

think of this as a factorization of f.

morphism:

$$\begin{array}{ccc} [n] & \xrightarrow{a} & A \\ g \uparrow & \nearrow_{a'} & \\ [n'] & & \end{array} \qquad \begin{array}{ccccc} d & \xrightarrow{\phi} & c_{a_0} \times_c \cdots \times_c c_{a_n} & \xrightarrow{f_a} & c \\ & \searrow^{\phi'} & \downarrow g^* & \nearrow_{f_{a'}} & \\ & & c_{a'_0} \times_c \cdots \times_c c_{a'_{n'}} & & \end{array}$$

by definition of $\hat{\underline{C}}$ $\exists$ some $b_0 \in A$ s.t.

$$\begin{array}{ccc} d & \xrightarrow{f} & c \\ & \searrow_{\phi_0} \quad c_{b_0} \quad \nearrow_{f_{b_0}} & \end{array}$$

$([0], \phi_0) \in (d \to c)/\alpha$

$$F : (d \to c)/\hat{\underline{C}} \rightsquigarrow (d \to c)/\hat{\underline{C}}$$

$$d \xrightarrow{\phi} c_{a_0} \times_c \cdots \times_c c_{a_n} \to c \quad \longmapsto \quad d \xrightarrow{\phi_0 \times \phi} c_{b_0} \times_c c_{a_0} \times_c \cdots \times_c c_{a_n} \to c$$

$$\begin{array}{ccc} & \overset{\phi_0}{\nearrow} & c_{b_0} & \searrow & \\ & & \uparrow \tau' & & \\ d & \xrightarrow{\phi_0 \times \phi} & c_{b_0} \times_c c_{a_0} \times_c \cdots \times_c c_{a_n} & \to & c \\ & & \downarrow \tau & & \\ & \underset{\phi}{\searrow} & c_{a_0} \times_c \cdots \times_c c_{a_n} & \nearrow & \end{array}$$

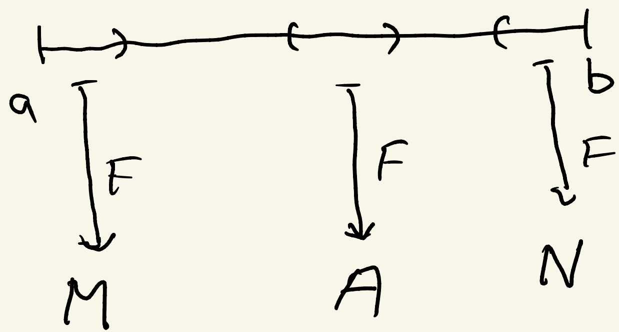
So F is a functor +

$\tau : F \Rightarrow id$, $\tau' : F \to Const([0], \phi_0)$

$\Longrightarrow N((d \to c)/\alpha)$ is contractible. $\square$

Recall: A R-algebra, $M \in \mathrm{Mod}_A$, $N \in {}_A\mathrm{Mod}$
(A, M, N are free $/R$)

Let F be a fact. algebra of $M = [a, b]$ with values in Ch



Prop: $F([a,b]) \overset{\text{w.e.}}{\sim} M \overset{h}{\underset{A}{\otimes}} N := \underset{\Delta^{op}}{\mathrm{hocolim}}\, B_\bullet(M, A, N) = \mathrm{Tot}\left(B_\bullet(M, A, N)^{alt}\right)$

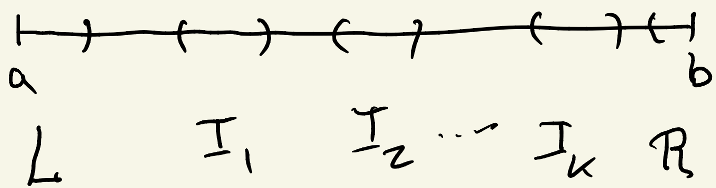
$M \overset{h}{\underset{A}{\otimes}} N$: derived tensor product

$B_\bullet(M, A, N)$: simplicial object in Ch

Proof: use a Weiss sieve on Open(M) + compute $F(M)$ using "abstract" descent property.

Let $\mathrm{Disk}_1(M) \subset \mathrm{Open}(M)$

$\mathcal{U} \in \mathrm{Disk}_1(M) \iff \mathcal{U}$ is a finite union of intervals, $a, b \in \mathcal{U}$



$\mathcal{U} = L \cup I_1 \cup \dots \cup I_k \cup R$

Note: $\mathrm{Disk}_1(M)$ is a Weiss cover.