

Recall: A Grothendieck topology on a category \mathcal{C} assigns to each $c \in \text{ob } \mathcal{C}$ a set of sieves $\mathcal{J}(c)$ on c called covering sieves.

A sieve on c is a subset $\underline{c} = \{d_a \xrightarrow{f_a} c \mid a \in A\} \subset \text{ob}(\mathcal{C}/c)$ which is saturated w.r.t. precomposition, i.e. $(d \xrightarrow{f} c) \in \underline{c}$ and $(e \xrightarrow{g} d) \in \mathcal{C}(e, d) \Rightarrow (e \xrightarrow{f \circ g} c) \in \underline{c}$.

Ex: The Weiss topology on Mfld_n :

A Weiss covering sieve of $M \in \text{Mfld}_n$ is a collection $\underline{u} = \{u_a \xrightarrow{f_a} M\} \subset \text{ob}(\text{Mfld}_n/M)$ s.t. • $(u \xrightarrow{f} M) \in \underline{u}$, $v \xrightarrow{g} u \Rightarrow (v \xrightarrow{f \circ g} M) \in \underline{u}$

• for any $S \subset M$ finite $\Rightarrow \exists (u \xrightarrow{f} M) \in \underline{u}$ s.t. $S \subset f(u)$.

Grothendieck site = Category + Grothendieck topology

\mathcal{C} cat. with a GT

$F: \mathcal{C} \rightarrow \text{Ch}$ functor

Def. F is a homotopy cosheaf (w.r.t. the sieves determined by GT)

e.g. Ayala-Franz
Def 2.20

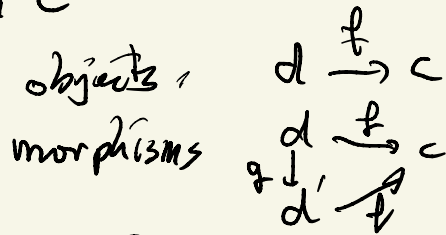
OR F satisfies the descent condition if

$$F(c) \xrightarrow[\sim]{\text{we.}} \text{hocolim}_{(d \in \mathcal{C}/c)} (\mathcal{C}/c \rightarrow \mathcal{C} \xrightarrow{F} \text{Ch}) \xrightarrow[\sim]{\text{hocolim } F(d)}$$

for any covering sieve $\mathcal{C} \in \text{ob}(\mathcal{C}/c)$ on c

regard $\mathcal{C} \subset \mathcal{C}/c$
full subcategory

full: morphism in the subcat
are morph. of ambient cat



Q: How does this relate to the old Čech version of descent?

based on $F(U_{a_0} \cap \dots \cap U_{a_n})$ if $\mathcal{C} = \text{Open}(M)$
 \parallel
 $U_{a_0} \times_U U_{a_1} \times \dots \times_U U_{a_n}$ $\{U_{a_i}\}$ open cover of $U \in \text{Open}(M)$

Def: \mathcal{C} category with pullbacks + GT
 $c \in \text{ob}(\mathcal{C})$, $\underline{c} = \{c_a \xrightarrow{f_a} c \mid a \in A\} \subset \mathcal{C}/c$
 subcat

$F: \mathcal{C} \rightarrow \text{Ch}$ satisfies Éch descent if
 for any \underline{c} whose saturation is a covering sieve

$$F(c) \xleftarrow[\cong]{\text{w.e.}} \text{holim}_{\Delta^{\text{op}}} \left(\bigoplus_{a_0} F(c_{a_0}) \right) \rightleftharpoons \bigoplus_{a_0, a_1} F(c_{a_0} \times_c c_{a_1}) \rightleftharpoons \bigoplus_{a_0, a_1, a_2} F(c_{a_0} \times_c c_{a_1} \times_c c_{a_2})$$

$\check{C}(\underline{c}, F)$

Thm: let $\underline{c} = \{c_a \xrightarrow{f_a} c\} \subset \mathcal{C}/c$, s.t. saturation $\hat{\underline{c}}$
 (Laurie Wells) Murray is a covering sieve. Then

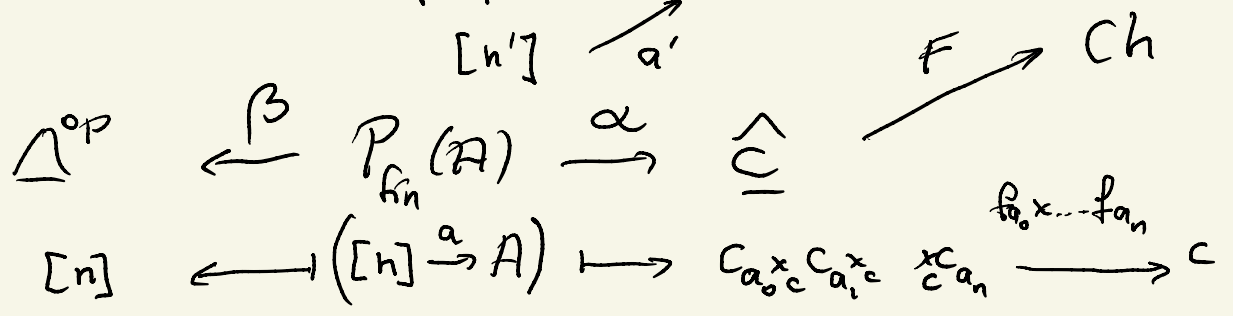
Prop. 3.7 $\text{hocolim}_{\Delta^{\text{op}}} \check{C}(C, F) \xrightarrow{\text{w.e.}} \text{hocolim} (\underbrace{\hat{C} \xrightarrow{F} C}_{\text{F}} \xrightarrow{F} Ch)$

Cor: $F: C \rightarrow Ch$ satisfies descent \Leftrightarrow Čech descent.

To prove Thm, you need to relate indexing cats Δ^{op} & \hat{C} .

recall: $P_{\text{fin}}(A)$ obj: $[n] \xrightarrow{a} A$
 $\{0, \dots, n\}$

mor: $[n] \xrightarrow{a} A$
 $f \uparrow$
 $[n'] \xrightarrow{a'}$



$$\frac{\text{Prop A:}}{+} \quad \text{hocolim}_{P_{\text{fin}}(A)} d^* F \xrightarrow{\text{w.e.}} \text{hocolim}_{\hat{C}} F$$

$$\frac{\text{Prop B:}}{\Downarrow} \quad \text{hocolim}_{\Delta^{\text{op}}} \check{C}(C, F) \xrightarrow{\text{w.e.}} \text{hocolim}_{P_{\text{fin}}(A)} d^* F$$

Thm.

today: proof of Prop B.

recall: (homotopy) left Kan extension: $C \xrightarrow{\beta} D$
 $F: C \rightarrow Ch$

$$\rightsquigarrow (\beta_! F): D \rightarrow Ch$$

$$(\beta_! F)(d) = \text{hocolim} (C/d \rightarrow C \xrightarrow{F} Ch)$$

note: if $D = *$ (one obj, no non-identity mor)
 $C \xrightarrow{T} *$ terminal functor $(T_! F)(*) = \text{hocolim}_C F$.

functoriality of left Kan extension: $(\beta \circ \gamma)_!$
 $= \beta_! \gamma_!$

Pf. of Prop B: $G = d^* F : P_{fin}(A) \rightarrow Ch$

$$\text{hocolim}_{P_{fin}(A)} G = (T \circ \beta)_! G$$

$$\begin{array}{ccc} \Delta^{op} & \xleftarrow{\beta} & P_{fin}(A) \\ & \searrow T & \downarrow \alpha \\ & & * \end{array}$$

$$= T_!(\beta_! G) = \text{hocolim}_{\Delta^{op}} (\beta_! G)$$

so it suffices to show: $\beta_! G \simeq \check{C}(\underline{C}, F)$

$$(\beta_! G)([n]) = \text{hocolim}_{P_{fin}(A)/[n]} (P_{fin}(A)/[n] \rightarrow P_{fin}(A) \xrightarrow{G} Ch)$$

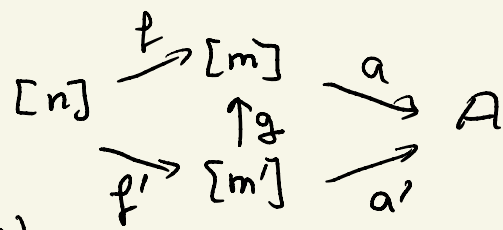
$$P_{fin}(A)/[n]$$

objects: $[m] \xrightarrow{a} A \xrightarrow{\beta} [m]$

fixed \downarrow $[n] \xrightarrow{f} [m] \xrightarrow{a} A$

$\uparrow f$
 $[n]$

morphisms



$$P_{\text{fin}}(A)/[n] = \coprod_{[n] \xrightarrow{h} A}$$

$$P_{\text{fin}}(A)_h$$

full subcat. of $P_{\text{fin}}(A)$
consisting of objects

$$[n] \xrightarrow{f} [m] \xrightarrow{a} A$$

s.t. $a \circ f = h$

claim: $P_{\text{fin}}(A)_h$

has a terminal object

$$[n] \xrightarrow{\text{id}} [n] \xrightarrow{h} A$$

$$\text{cocohim } G = \bigoplus_{P_{\text{fin}}(A)/[n] \xrightarrow{h} A}$$

$$\text{cocohim } G = \bigoplus_{P_{\text{fin}}(A)_h \xrightarrow{h} A} G([n] \xrightarrow{h} A) \\
 \parallel \\
 F(\alpha([n] \xrightarrow{h} A)) \\
 \parallel \\
 F(c_{h_0} \times c \dots \times c_{h_n})$$

\Rightarrow Prop. B,

$(\check{C}(c, F))(In)$
