Recall: A Grothendieck topology on a category $\mathcal{C}$ assigns to each $c \in \mathrm{ob}\,\mathcal{C}$ a set of sieves $J(c)$ on c called covering sieves.

A sieve on c is a subset $\underline{S} = \{d_a \xrightarrow{f_a} c \mid a \in A\} \subset \mathrm{ob}(\mathcal{C}/c)$ which is saturated w.r.t. precomposition, i.e. $(d \xrightarrow{f} c) \in \underline{S}$ and $(e \xrightarrow{g} d) \in \mathcal{C}(e,d) \Rightarrow (e \xrightarrow{f \circ g} c) \in \underline{S}$.

Ex: The Weiss topology on Mfld_n:

A Weiss covering sieve of $M \in \mathrm{Mfld}_n$ is a collection $\underline{\mathcal{U}} = \{U_a \xhookrightarrow{f_a} M\} \subset \mathrm{ob}(\mathrm{Mfld}_n / M)$ s.t.

- $(U \xhookrightarrow{f} M) \in \underline{\mathcal{U}}$, $V \xhookrightarrow{g} U \Rightarrow (V \xhookrightarrow{f \circ g} M) \in \underline{\mathcal{U}}$

- for any $S \underset{\text{finite}}{\subset} M \Rightarrow \exists (U \xhookrightarrow{f} M) \in \underline{\mathcal{U}}$ s.t. $S \subset f(U)$.

Grothendieck site = Category + Grothendieck topology

$\mathcal{C}$ cat. with a GT

$F : \mathcal{C} \to Ch$ functor

Def. F is a **homotopy cosheaf** (w.r.t. the sieves determined by GT)

e.g. Ayala-Francis Def 2.20

or F satisfies the **descent condition** if

$$F(c) \xleftarrow[\sim]{w.e.} \mathrm{hocolim}(\underline{c} \subset \mathcal{C}/c \to \mathcal{C} \xrightarrow{F} Ch) = \mathrm{hocolim}_{(d \xrightarrow{f} c) \in \underline{c}} F(d)$$

for any covering sieve $\underline{c} \subset ob(\mathcal{C}/c)$ on c

regard $\underline{c} \underset{\text{full subcategory}}{\subset} \mathcal{C}/c$ — objects: $d \xrightarrow{f} c$

morphisms:
$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "g"'] & c \\ d' \arrow[ur, "f'"'] & \end{tikzcd}$$

full: morphism in the subcat are morph. of ambient cat

Q: How does this relate to the old Čech version of descent?

based on $F(\mathcal{U}_{a_0} \cap \dots \cap \mathcal{U}_{a_n})$ if $\mathcal{C} = \mathrm{Open}(M)$

$\qquad\qquad\quad \| \qquad\qquad\qquad\qquad\quad \{\mathcal{U}_a\}$ open cover

$\mathcal{U}_{a_0} \times_{\mathcal{U}} \mathcal{U}_{a_1} \times \dots \times_{\mathcal{U}} \mathcal{U}_{a_n}$ $\qquad$ of $\mathcal{U} \in \mathrm{Open}(M)$

<u>Def:</u> $\mathcal{C}$ category with pullbacks + GT

$c \in ob(\mathcal{C})$, $\quad \underset{\sim}{c} = \{c_a \xrightarrow{f_a} c \mid a \in A\} \underset{\text{subcat}}{\subset} \mathcal{C}/c$

$F : \mathcal{C} \to Ch$ satisfies <u>Čech descent</u> if

for any $\underset{\sim}{c}$ whose saturation is a covering sieve

$$F(c) \underset{\simeq}{\xleftarrow{w.e}} \underset{\Delta^{op}}{\mathrm{hocolim}} \underbrace{\left(\bigoplus_{a_0} F(c_{a_0}) \leftrightarrows \bigoplus_{a_0,a_1} F(c_{a_0} \times_c c_{a_1}) \leftrightarrows \bigoplus_{a_0,a_1,a_2} F(c_{a_0} \times_c c_{a_1} \times_c c_{a_2}) \cdots \right)}_{\check{C}(\underset{\sim}{c}, F)}$$

<u>Thm:</u> let $\underset{\sim}{c} = \{d_a \xrightarrow{f_a} c\} \subset \mathcal{C}/c$, s.t. saturation $\hat{\underset{\sim}{c}}$
(Laura Wells Murray) is a covering sieve. Then

Prop. 3.7 $\operatorname{hocolim}_{\Delta^{op}} \check{C}(\underline{c}, F) \overset{w.e.}{\sim} \operatorname{hocolim}(\underline{\hat{C}} \subset \mathcal{C}/_c \to \mathcal{C} \xrightarrow{F} Ch)$

Cor: $F : \mathcal{C} \to Ch$ satisfies descent $\Leftrightarrow$ Čech descent.

To prove Thm, you need to relate indexing cats Δ^{op} & $\underline{\hat{C}}$.

recall: $P_{fin}(A)$ obj: $[n] \xrightarrow{a} A$, $[n] = \{0, \dots, n\}$

mor:

$$\begin{tikzcd}
{[n]} \arrow[dr, "a"] & \\
 & A \\
{[n']} \arrow[uu, "f"] \arrow[ur, "a'"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & Ch \\
\Delta^{op} & P_{fin}(A) \arrow[l, "\beta"'] \arrow[r, "\alpha"] & \underline{\hat{C}} \arrow[ur, "F"] &
\end{tikzcd}$$

$$[n] \longleftarrow\!\shortmid ([n] \xrightarrow{a} A) \longmapsto c_{a_0} \times_c c_{a_1} \times_c \cdots \times_c c_{a_n} \xrightarrow{f_{a_0} \times \cdots f_{a_n}} c$$

Prop A: $\operatorname{hocolim}_{P_{fin}(A)} \alpha^* F \xrightarrow{\text{w.e.}} \operatorname{hocolim}_{\hat{\subseteq}} F$

+

Prop B: $\operatorname{hocolim}_{\Delta^{op}} \check{C}(\subseteq, F) \xrightarrow{\text{w.e.}} \operatorname{hocolim}_{P_{fin}(A)} \alpha^* F$

$\Downarrow$

Thm.

today: proof of Prop B.

recall: (homotopy) left Kan extension: $\mathcal{C} \xrightarrow{\beta} \mathcal{D}$

$F : \mathcal{C} \to Ch$

$\rightsquigarrow (\beta_! F) : \mathcal{D} \to Ch$

$(\beta_! F)(d) = \operatorname{hocolim}(\mathcal{C}/d \to \mathcal{C} \xrightarrow{F} Ch)$

note: if $\mathcal{D} = *$ (one obj, no non-identity mor)

$\mathcal{C} \xrightarrow{T} *$ terminal functor $(T_! F)(*) = \operatorname{hocolim}_{\mathcal{C}} F$.

functoriality of left Kan extension: $(\beta\circ\gamma)_! = \beta_!\,\gamma_!$

<u>Pf. of Prop B</u>: $G = \alpha^* F : P_{fin}(A) \to Ch$

$$\operatorname*{hocolim}_{P_{fin}(A)} G = (T\circ\beta)_! G$$

$$\begin{tikzcd} \Delta^{op} \arrow[dr, "T"'] & P_{fin}(A) \arrow[l, "\beta"'] \arrow[d] \\ & * \end{tikzcd}$$

$$= T_!(\beta_! G) = \operatorname*{hocolim}_{\Delta^{op}} (\beta_! G)$$

so it suffices to show: $\beta_! G \underset{w.e.}{\simeq} \check{C}(\underset{\sim}{c}, F)$

$$(\beta_! G)([n]) \simeq \operatorname{hocolim}\left(P_{fin}(A)/_{[n]} \longrightarrow P_{fin}(A) \xrightarrow{G} Ch\right)$$

$P_{fin}(A)/_{[n]}$ objects:

$$\begin{tikzcd} {[m]} \arrow[r, "a"] & A & {[m]} \\ & & {[n]} \arrow[u, "f"] \end{tikzcd} \qquad [m]\xrightarrow{a}A \overset{\beta}{\longmapsto} [m]$$

fixed

$$\underbrace{[n] \xrightarrow{f} [m] \xrightarrow{a} A}$$

morphisms

$$\begin{tikzcd} & {[m]} \arrow[dr, "a"] & \\ {[n]} \arrow[ur, "f"] \arrow[dr, "f'"'] & & A \\ & {[m']} \arrow[uu, "g"] \arrow[ur, "a'"'] & \end{tikzcd}$$

$$P_{fin}(A)/_{[n]} = \coprod_{[n] \xrightarrow[h]{} A} P_{fin}(A)_h$$

$P_{fin}(A)_h$ ← full subcat. of $P_{fin}(A)$ consisting of objects $[n] \xrightarrow{f} [m] \xrightarrow{a} A$ s.t. $a \circ f = h$

claim: $P_{fin}(A)_h$ has a terminal object $[n] \xrightarrow{id} [n] \xrightarrow{h} A$

$$\operatorname*{hocolim}_{P_{fin}(A)/_{[n]}} G = \bigoplus_{[n] \xrightarrow{h} A} \operatorname*{hocolim}_{P_{fin}(A)_h} G = \bigoplus_{[n] \xrightarrow{h} A} G([n] \xrightarrow{h} A)$$

$$G([n] \xrightarrow{h} A) = F(\alpha([n] \xrightarrow{h} A)) = F(c_{h_0} \times_c \cdots \times_c c_{h_n})$$

$$\underbrace{}_{(\check{C}(\underline{c}, F))([n])}$$

$\Rightarrow$ Prop. B.