Grothendieck topologies

M top space $\quad \mathcal{U} \underset{\text{open}}{\subset} M$

$\{\mathcal{U}_a \hookrightarrow \mathcal{U}\}$ is a standard cover $\Leftrightarrow$ $\forall x \in \mathcal{U}\ \exists a \in A$ s.t. $x \in \mathcal{U}_a$

Weiss cover $\Leftrightarrow$ $S \subset \mathcal{U}$ finite $\exists a \in A$ s.t. $S \subset \mathcal{U}_a$

" " " disk Weiss cover $\Leftrightarrow$ " " & $\mathcal{U}_a \underset{\text{homeom.}}{\approx} \underbrace{\mathbb{R}^n \amalg \dots \amalg \mathbb{R}^n}_{\text{finite}}$ $\forall a$

Def: $S = \{\mathcal{U}_a \hookrightarrow \mathcal{U}\}$ is a sieve on $\mathcal{U}$ if for $V \hookrightarrow \mathcal{U}_a$, then $(V \hookrightarrow \mathcal{U}_a \hookrightarrow \mathcal{U}) \in S$

if $\mathcal{V} = \{\mathcal{U}_a \hookrightarrow \mathcal{U}\}$

then $\widehat{\mathcal{V}} = \{V \hookrightarrow \mathcal{U} \mid V \subset \mathcal{U}_a$ for some $\mathcal{U}_a\}$

Def: a sieve S on $\mathcal{U}$ is a standard sieve if it contains a standard cover.

S is a Weiss sieve if ..... Weiss cover

S " disk Weiss sieve ..... disk Weiss cover

Claim: A sieve $\mathcal{S}$ is a disk Weiss sieve $\Leftrightarrow$ $\mathcal{S}$ is a Weiss sieve

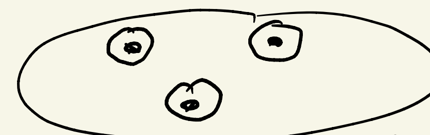
"$\Rightarrow$" tautological

$\Leftarrow$: let $\mathcal{S}$ be a Weiss sieve $\Rightarrow$ it contains a Weiss cover $\{\mathcal{U}_a \hookrightarrow \mathcal{U}\} \subset \mathcal{S}$

we need to construct a disk Weiss cover belonging to $\mathcal{S}$.

suppose $S \underset{\text{finite}}{\subset} \mathcal{U} \Rightarrow \exists\, a \in A$ s.t. $S \subset \mathcal{U}_a$

$\Rightarrow$



$\mathcal{U}_a$

union of these disks:

$V_S \approx \mathbb{R}^n \amalg \dots \amalg \mathbb{R}^n \subset \mathcal{U}_a$

$\Rightarrow (V_S \hookrightarrow \mathcal{U}) \in \mathcal{S}$

do that for any $S \subset \mathcal{U}$

to get a disk Weiss cover contained in $\mathcal{S}$ $\square$.

Def: A <u>Grothendieck topology</u> on $\mathrm{Open}(M)$ assigns to every $U \underset{\text{open}}{\subset} M$ a collection $J(U)$ of sieves on U, called <u>covering sieves</u>. Required properties:

<u>GT1</u>: (stability under pullbacks)

$J(U) \ni S = \{U_a \hookrightarrow U\}$ covering sieve on U, $V \xrightarrow{g} U$, then $g^* S$ covering sieve on V, i.e. $g^* S \in J(V)$.

$$g^* S = \{V_b \underset{f_b}{\hookrightarrow} V \mid (V_b \xrightarrow{f_b} V \xrightarrow{g} U) \in S\}$$

<u>GT2</u>: (locality of covering sieves):

S sieve on U, S' covering sieve on U

S covering sieve on U $\iff$ $\forall\, (V \xrightarrow{g} U) \in S'$, $g^* S$ is covering sieve on V

$\forall\ (g : V \to U) \in S'$

<u>GT3</u>: $\{V \hookrightarrow U\}$ is a covering sieve.
all subsets

<u>Ex</u>: standard sieves define the standard Grothendieck top on Open(M)

Weiss sieves ———— Weiss "

<u>Def</u>: category of n-manifolds: Mfld_n

<u>objects</u>: n-mfds (not necess. compact, but without ∂)

<u>morphisms</u>: embeddings $M \hookrightarrow N$

<u>note</u>: $M \in \mathrm{Mfld}_n$ over category or slice category:

$$\mathrm{Mfld}_n / M \xrightarrow{\simeq} \mathrm{Open}(M)$$

equivalence of categories.

obj: $N \overset{f}{\hookrightarrow} M \longmapsto f(N)$

morphisms:

$$\begin{tikzcd} N \arrow[d, "g"'] \arrow[dr, hook, "f"] & \\ N' \arrow[r, hook, "f'"'] & M \end{tikzcd} \longmapsto f(N) \hookrightarrow f(N')$$

Def: A Weiss cover of $M \in \mathrm{Mfld}_n$ is a collection $\{N_a \overset{f_a}{\hookrightarrow} M\}_{a \in A}$ s.t. $\underset{\text{finite}}{S \subset M}$, then $\exists\, a \in A$ s.t. $S \subset f_a(N_a)$.

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goal: generalize the definition of a GT on Open(M) to a general category $\mathcal{C}$.

Def: $c \in \mathrm{ob}\,\mathcal{C}$. A sieve on c is a collection $\mathcal{S} = \{d_a \xrightarrow{f_a} c\}_{a \in A}$ such that if $d \xrightarrow{g} d_a$, then $(d \xrightarrow{g} d_a \xrightarrow{f_a} c) \in \mathcal{S}$.

Rem: $S \subset ob(\mathcal{C}/c)$, so we can view S as a full subcategory of $\mathcal{C}/c$ (consisting of these objects + all morphisms between them)

Def: A <u>Grothendieck topology</u> on a category $\mathcal{C}$ assigns to $c \in ob\,\mathcal{C}$ a set of sieves $J(c)$ on c, called <u>covering sieves</u>.

Required properties:

<u>GT1</u>: (stability under pull-back): $d \xrightarrow{g} c$

$S \in J(c)$ (i.e. S is a covering sieve on c)

then $g^*S \in J(d)$ (g^*S " " d)

$$g^*S = \{v \xrightarrow{f} d \mid (v \xrightarrow{f} d \xrightarrow{g} c) \in S\}$$

GT2 : (locality of covering sieves)

$S = \{ d_a \xrightarrow{f_a} c \}_{a \in A}$ sieve, let S' covering sieve of $S' = \{ e \xrightarrow{g} c \}$

S is a covering sieve

$\Updownarrow$

$g^* S$ is covering sieve of e $\forall\ (e \xrightarrow{g} c) \in S'$.

GT3 : $S = \{$ all morphisms $d \xrightarrow{f} c \}$ is a covering sieve of c.

A site is a category $\mathcal{C}$ equipped with a Grothendieck topology.

e.g. Mfld_n with standard GT

" " Weiss GT.

To define descent for a functor $F : \mathrm{Open}(M) \to Ch$ we use $\underbrace{F(\mathcal{U}_{a_0} \cap \dots \cap \mathcal{U}_{a_n})}$ $\mathcal{U}_{a_i}$ part of a Weiss cover

Q: What is the analog of this with $\mathrm{Open}(M)$ replaced by a cat. $\mathcal{C}$

and ~~Weiss-cover~~ replaced by a covering sieve?