R comm. ring

A R-algebra

$M \in \mathrm{Mod}_A$, $N \in {}_A\mathrm{Mod}$

$$B_k(M,A,N) := M \otimes \underbrace{A \otimes \ldots \otimes A}_{k} \otimes N$$

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<u>Claim:</u> $B_k(M,A,A)$ is a <u>free</u> right A-module

<u>Correction:</u> we need here: A & M are free R-modules $(*)$

In particular $B_\bullet(M,A,A)$ is a free resolution of $M \in \mathrm{Mod}_A$

<u>Recall:</u> $\mathrm{Tor}_k^A(M,N) := H_k(\hat{M}_\bullet \otimes_A N)$ ($\hat{M}_\bullet$: free resolution)

$$\overset{\text{if } (*) \text{ holds}}{=} H_k(B_\bullet(M,A,A) \otimes_A N) = H_k(B_\bullet(M,A,N))$$

Def: the derived tensor product

$$M \otimes^h_A N := \hat{M}_\bullet \otimes_A N \underset{(*)}{=} B_\bullet(M, A, N)$$

(h ← "homotopy")

Rem: 1) in $Ch_{\mathbb{Z}}$ if $C_\bullet \in Ch_{\mathbb{Z}}$ has finitely gen. homology, then $C_\bullet \underset{w.e.}{\sim} \underbrace{H_\bullet(C_\bullet)}$ (chain cx. with zero differential)

(HW: prove this)

2) $C_\bullet, D_\bullet \in Ch_{\mathbb{Z}}$

Künneth formula

$$0 \to \bigoplus_{k+l=n} H_k(C_\bullet) \otimes H_l(D_\bullet) \to H_n(C_\bullet \otimes D_\bullet) \to \bigoplus_{k+l=n-1} \mathrm{Tor}(H_k(C_\bullet), H_l(D_\bullet))$$

(split)

Problem: split is not functorial, hence can't describe elements in $H_\bullet(C_\bullet \otimes D_\bullet)$ functorially in terms of $H_\bullet(C_\bullet)$ & $H_\bullet(D_\bullet)$.

Def: A R-algebra. A **trace** is a R-linear map $t : A \to R$

s.t. $t(ab) = t(ba)$.

More generally if M is an A-A-bimodule (i.e. M is a left A-module, a right A-module, and $(am)b = a(mb)$, $a, b \in A$, $m \in M$)

Ex: $M = A$ is a bimodule

$t : M \to R$ R-linear map is a **trace** if $t(am) = t(ma)$

note:

$$M \xrightarrow{\;t\;} R$$

$$M \to M/am \sim ma \to R$$

$M/am \sim ma$ ← R-module

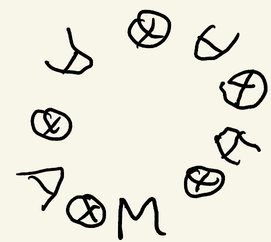
universal R-module through which each trace factors.

Hochschild Homology is the derived version of the domain of the universal trace

Def: M A-A-bimodule (free as R-module; A free R-module)

Hochschild complex

$$HHC_k(A;M) := M \otimes \underbrace{A \otimes \cdots \otimes A}_{k} =$$



differential:

$$d(m \otimes a_1 \otimes \cdots \otimes a_k)$$
$$= ma_1 \otimes a_2 \otimes \cdots \otimes a_k$$
$$- m \otimes a_1a_2 \otimes \cdots \otimes a_k$$
$$\vdots$$
$$+ (-1)^{k-1} m \otimes a_1 \otimes \cdots \otimes a_{k-1}a_k$$
$$+ (-1)^k a_k m \otimes a_1 \otimes \cdots \otimes a_{k-1}$$

Hochschild homology $HH_k(A;M) := H_k(HHC_\bullet(A;M))$

$$HH_*(A) := HH_*(A;A)$$

$$HH_0(A;M) = M/\mathrm{im}(d: M\otimes A \to M) = M/_{ma-am}$$

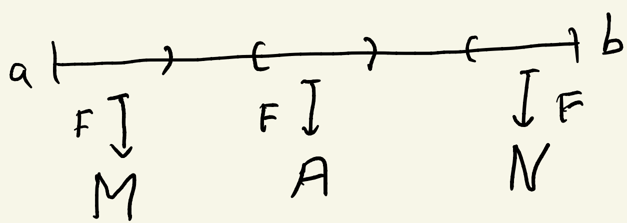
$$m\otimes a \mapsto ma - am$$

Back to fac. algs:

<u>data:</u> A R-module, $M \in Mod_A$, $N \in {}_A Mod$

(always we'll assume A, M, N are free R-modules)

Let F be a locally constant fact. alg on $[a,b]$ with values in $Ch = Ch_R$, given by

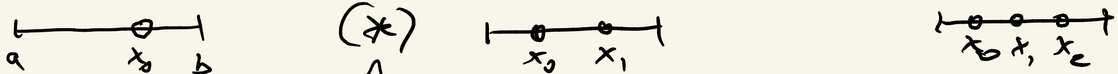


<u>Question:</u> What is $F([a,b]) \in Ch$

We need a Weiss cover of $[a,b]$:

$$\underline{\mathcal{U}} = \{ \mathcal{U}_x := [a,b] \setminus \{x\} \}_{x \in (a,b)} \quad \text{is a Weiss cover}$$

Hence



$$F([a,b]) \underset{\text{w.e.}}{\simeq} \operatorname{hocolim} \Big((*) \Big)$$

$$(*) \qquad \bigoplus_{x_0} F(\mathcal{U}_{x_0}) \leftleftarrows \bigoplus_{x_0<x_1} F(\mathcal{U}_{x_0} \cap \mathcal{U}_{x_1}) \Lleftarrow \bigoplus_{x_0<x_1<x_2} F(\mathcal{U}_{x_0} \cap \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2})$$

$$\simeq \qquad\qquad \simeq \qquad\qquad \simeq$$

$$M \otimes N \leftleftarrows M \otimes A \otimes N \Lleftarrow M \otimes A \otimes A \otimes N$$

"bloated bar complex"

Thm: $\operatorname{hocolim}(*) \xrightarrow[\sim]{\text{w.e.}} B_\bullet(M,A,N)$

We need to develop a bit more technology to prove this.

$\underline{\mathcal{U}} = \{\mathcal{U}_a\}_{a \in A}$ cover of $\mathcal{U} \underset{\text{open}}{\subset} M$

$\rightsquigarrow$ get the descent diagram;

pieces are : $F(\mathcal{U}_{a_0} \cap \dots \cap \mathcal{U}_{a_n})$ $\quad a_i \in A$

parametrized by ordered tupel $\{a_0, \dots, a_n\}$

$$\updownarrow$$

$$\Delta \ni [n] = \{0, \dots, n\} \xrightarrow{a} A$$

<u>Def</u>: build a category $\mathcal{P}_{fin}(A)$:

<u>objects</u>: $[n] \xrightarrow{a} A$

<u>morphism</u>: from $[n] \xrightarrow{a} A$ to $[m] \xrightarrow{b} A$;

order preserving map $\rightarrow f$

$$\begin{array}{ccc} [n] & \searrow^{a} & \\ f \uparrow & & A \\ [m] & \nearrow_{b} & \end{array}$$

functor $\alpha : P_{fin}(A) \longrightarrow Open(M)$

$$\begin{array}{ccc} [n] \xrightarrow{a} A & \longmapsto & U_{a_0} \cap \dots \cap U_{a_n} \\ f \uparrow & & \cap\!\mid \\ [m] \xrightarrow{b} A & \longmapsto & U_{b_0} \cap \dots \cap U_{b_m} \end{array}$$

$im(b) \subseteq im(a)$

<u>Lem:</u> $\operatorname{hocolim}\left(P_{fin}(A) \xrightarrow{\alpha} Open(M) \xrightarrow{F} Ch\right) \underset{w.e.}{\simeq} \operatorname{hocolim}_{\Delta^{op}}(*)$