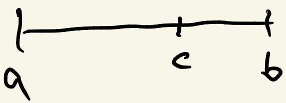
<u>Recall</u>: loc. constant multiplicate prefactorization algebra $\mathcal{F} : \mathrm{Open}(M) \longrightarrow \mathrm{Vect}$

<u>Examples</u>: i) on $M = \mathbb{R}$: $A = \mathcal{F}(\mathbb{R}) \cong \mathcal{F}((c,d))$ unital algebra

ii) on $M = [a,b]$: $A = \mathcal{F}((a,b)) \cong \mathcal{F}((c,d))$ ''

$M = \mathcal{F}([a,b)) \cong \mathcal{F}([a,c))$ right A-module

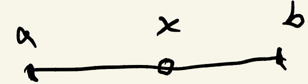
$N = \mathcal{F}((a,b]) \cong \mathcal{F}((c,b])$ left ''



Now assume that $\mathcal{F}$ in case (ii) is a <u>factorization algebra</u>, i.e. $\mathcal{F}$ is Weiss cosheaf.

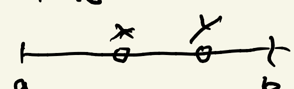
<u>Q</u>: $\mathcal{F}([a,b]) = ?$

Use Weiss cover of $[a,b]$: $\mathcal{U} = \{ U_x := [a,b] \setminus \{x\} \}_{x \in (a,b)}$

U_x : 

$\parallel$

$[a,x) \sqcup (x,b]$

$[a,x) \sqcup (x,y) \sqcup (y,b] = U_x \cap U_y$ 

$$\mathcal{F}([a,b]) \cong \check{C}([a,b],\mathcal{U})$$

$$\begin{tikzcd}
\operatorname{coeq}\Big(\bigoplus_{x} \mathcal{F}(\mathcal{U}_x) & \bigoplus_{x<y} \mathcal{F}(\mathcal{U}_x \cap \mathcal{U}_y) \Big) \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \arrow[d, phantom, "\shortparallel"] \\
\bigoplus_{x} \mathcal{F}([a,x)\amalg(x,b]) \arrow[u, phantom, "\shortparallel"] \arrow[d, phantom, "\cong"] & \bigoplus_{x<y} \mathcal{F}([a,x)\amalg(x,y)\amalg(y,b]) \arrow[d, phantom, "\cong"] \\
\bigoplus_{x} M_x \otimes N_x \arrow[d] & \bigoplus_{x<y} M_x \otimes A_{xy} \otimes N_y \arrow[l, shift right] \arrow[l, shift left] \arrow[d] \\
M \otimes N & M \otimes A \otimes N \arrow[l, shift right, "1_M \otimes \mu_N"'] \arrow[l, shift left, "\mu_M \otimes 1_N"]
\end{tikzcd}$$

multiplicative (*)

$m_x \otimes a_{xy} n_y \;\longleftarrow\; m_x \otimes a_{xy} \otimes n_y \mapsto m \otimes a \otimes n$

$\underbrace{m_x a_{xy}}_{\in M_y} \otimes \underbrace{n_y}_{\in N_y}$

(**)

vertical maps just identify the many copies of $M_x, \ldots$

note:

$$\operatorname{coeq}(M \otimes N \leftleftarrows M \otimes A \otimes N) = M \otimes_A N = M \otimes N / m \otimes an \sim ma \otimes n$$

__Claim:__ the map $(*) \xrightarrow{\Phi} (**)$ induces an isomorphism on coequalizers

Φ vertical map

Pf: surjectivity: ✓

injectivity: let $\sum_{i \in I} m^i_{x_i} \otimes n^i_{x_i}$ $\in \bigoplus_x M_x \otimes N_x$

general element in $M \otimes N$: $\sum_{i \in I} m^i \otimes n^i$

$m^i \in M$

$m^i_{x_i} \in M_{x_i}$

and assume $0 = \phi\Big[\underbrace{\sum_{i \in I} m^i_{x_i} \otimes n^i_{x_i}}_{\text{coeq}(*)}\Big] = \Big[\sum_{i \in I} m^i \otimes n^i\Big]$ $\in \text{coeq}(**)$

$\Rightarrow \exists \underbrace{\sum_j m^j \otimes a^j \otimes n^j}_{\in M \otimes A \otimes N}$ s.t. $(1 \otimes \mu - \mu \otimes 1)\left(\sum_j m^j \otimes a^j \otimes n^j\right) = \sum_{i \in I} m^i \otimes n^i$

$(1 \otimes \mu - \mu \otimes 1)\left(\sum_j m^j \otimes a^j \otimes n^j\right) = \sum_j \left(m^j \otimes a^j n^j - m^j a^j \otimes n^j\right)$

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relations in coeq$(*)$:

$$\underset{M_x \otimes N_x}{[m_x \otimes n_x]} = \underset{M_x \otimes A_{xy} \otimes N_y}{[m_x \otimes 1_{xy} \otimes n_y]} = \underset{M_y \otimes N_y}{[m_y \otimes n_y]}$$

Want to show $\sum_{i \in I} [m^i_{x_i} \otimes n^i_{x_i}] = 0$ in coeq$(*)$

for some fixed $x \in (a,b)$ $\quad \overset{\shortparallel}{\sum_{i \in I}} [m^i_x \otimes n^i_x]$

$$0 = \left[(f-g)\left(\sum_j m^j_x \otimes a^j_{xy} \otimes n^j_y\right)\right] = \left[\sum_j \left(\underset{\substack{\cap \\ M_x \otimes N_x}}{m^j_x \otimes a^j_{xy} n^j_y} - \underbrace{m^j_x a^j_{xy}}_{\in M_y \otimes N_y} \otimes n^j_y\right)\right]$$

$$= \left[\sum_j \left(m^j_x \otimes (a^j n)_x - (m a^j)_x \otimes n^j_x\right)\right]$$

$$= \left[\sum_i m^i_x \otimes n^i_x\right]$$

<u>Upshot</u>: $\mathcal{F}([a,b]) = M \otimes_A N$.

<u>Goal</u>: we are interested in functors:

$$\mathcal{F} : \mathrm{Open}(M) \longrightarrow \mathrm{Top}$$

i) $\mathcal{U} \longmapsto \mathrm{Conf}(\mathcal{U})$

ii) $\mathcal{U} \longmapsto \mathrm{map}_c(\mathcal{U}, X)$

or $\quad \mathrm{Open}(M) \xrightarrow{\text{i) or (ii)}} \mathrm{Top} \xrightarrow{C_*} \mathrm{Ch}$

$X \longmapsto C_*(X)$

$=$ singular chain cx.

So we want to talk about factorization algebras with values in Top or Ch.

We need: a) a "tensorproduct" in Top or Ch

(i.e. we want to regard them as symm. monoidal cats)

b) to talk about the Weiss cosheaf condition we need (homotopy) colimits in these categories.

Digression on colimits in a category $\mathcal{C}$.

basic example of a colimit: pushout of a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}$$

pushout = colimit of this diagram.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

$\leftarrow$ this is a picture of a category I

$\bullet \leftrightarrow$ objects

$\rightarrow \leftrightarrow$ non-identity morphisms

a diagram in $\mathcal{C}$ of shape I is a functor

$$F : I \to \mathcal{C}$$

let $\mathrm{Fun}(I, \mathcal{C})$ be the cat. whose obj. are functors $I \xrightarrow{F} \mathcal{C}$

morphisms : nat. transformations.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[drr] & Y \arrow[dr] & \\ Z \arrow[rr] & & C \end{tikzcd}$$

such an extension of the diagram F

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

is called a <u>cocone</u> on F

<u>Def:</u> Given a diagram $F : I \to \mathcal{C}$, a cocone on F consists of an object $C \in \mathcal{C}$ and a collection of morphisms $\psi_i : F(i) \to C$ for all $i \in ob(I)$ s.t. for all morphisms $i \xrightarrow{f} j$ in I

the diagram

$$\begin{tikzcd} F(i) \arrow[rr, "F(f)"] \arrow[dr, "\psi_i"'] & & F(j) \arrow[dl, "\psi_j"] \\ & C & \end{tikzcd}$$

is commutative.