Recall: 1) double complex $(C_{\bullet\bullet}, d^{hor}: C_{mn} \to C_{m-1,n}$

$d^{ver}: C_{mn} \to C_{m,n-1}$

$(d^{hor})^2 = 0, \ (d^{ver})^2 = 0$

$d^{hor} \circ d^{ver} = d^{ver} \circ d^{hor}$.

$\rightsquigarrow$ total complex $\mathrm{Tot}(C_{\bullet\bullet})$

$\mathrm{Tot}(C_{\bullet\bullet})_n := \bigoplus_{k+l=n} C_{kl}$ $\quad d: \mathrm{Tot}(C_{\bullet\bullet})_n \to \mathrm{Tot}(C_{\bullet\bullet})_{n-1}$

$d(\underset{\substack{\cap \\ C_{m,n}}}{c}) = d^{ver}(c) + (-1)^n d^{hor}(c)$

$d^2 = 0$ (homework!)

2) simplicial chain cx

$C_\bullet : \Delta^{op} \longrightarrow Ch \quad (Ch_R, \ Ch_A \text{ or } {}_A Ch)$

$\rightsquigarrow$ chain cx. of chaincx aka double complex $C^{alt}_{\bullet\bullet}$

Prop (e.g. Dugger Prop. 16.9):

$$\operatorname*{hocolim}_{\Delta^{op}} C_\bullet \underset{\text{w.e.}}{\simeq} \mathrm{Tot}(C_{\bullet\bullet}^{\mathrm{alt}})$$

alternating

<u>Example</u> of a simplicial R-module:

R comm. ring, $M \in \mathrm{Mod}_A$

A R-algebra , $N \in {}_A\mathrm{Mod}$

$$\begin{array}{ccccc}
0 & & 1 & & 2 \\
M\otimes N & \leftleftarrows\!\!\rightarrow & M\otimes A\otimes N & \text{⇇⇉⇇} & M\otimes A\otimes A\otimes N \quad \cdots \\
ma\otimes n & \longleftarrow & m\otimes a\otimes n & ma_1\otimes a_2\otimes n \longleftarrow & m\otimes a_1\otimes a_2\otimes n \\
m\otimes an & \longleftarrow & & m\otimes a_1a_2\otimes n \longleftarrow & \\
 & & & m\otimes a_1\otimes a_2 n \longleftarrow & \\
m\otimes n & \longmapsto & m\otimes 1\otimes n & & \\
 & & m\otimes a\otimes n & \longmapsto & m\otimes 1\otimes a\otimes n \\
 & & & \longmapsto & m\otimes a\otimes 1\otimes n
\end{array}$$

simplicial object

$$B_\bullet^\Delta(M,A,N):\quad B_0^\Delta(M,A,N) \leftleftarrows B_1^\Delta(M,A,N) \text{⇇⇇} \cdots$$

bar complex $\Rightarrow$ chain cx. $B_\bullet(M,A,N) :\simeq B_\bullet^\Delta(M,A,N)^{alt}$.

Observation: $B_k^\Delta(A,A,A) = A \otimes \underbrace{A \otimes \cdots \otimes A}_{k} \otimes A$ A-A-bimodule

$$M \otimes_A B_k^\Delta(A,A,A) \otimes_A N = M \otimes_A A \otimes \underbrace{A \otimes \cdots \otimes A}_{k} \otimes A \otimes_A N$$
$$= M \otimes \underbrace{A \otimes \cdots \otimes A}_{k} \otimes N$$
$$= B_k^\Delta(M,A,N).$$

$$B_\bullet(A,A,N) \in {}_A\mathrm{Ch}$$

chain map $\to$ $\downarrow \varepsilon$

$N[0]$ $\leftarrow$ chain cx concentrated in degree 0.

$$a\otimes n \in A \otimes N \longleftarrow A \otimes A \otimes N \leftarrow A \otimes A \otimes A \otimes N \leftarrow \cdots$$

$a \otimes b \otimes n \mapsto ab \otimes n - a \otimes bn$

$a\otimes n \mapsto an \in N$ (by ε)

$ab \otimes n - a \otimes bn \mapsto abn - abn = 0$

Lem: This chain map ε is a weak equivalence.

Pf: ε is an isom. on homology $\iff$ following sequence is exact:

$$\begin{array}{llllll} \text{degree} & -1 & & 0 & & 1 \\ \widetilde{B}_\bullet(A,A,N): & N & \xleftarrow{\varepsilon} & A\otimes N & \leftarrow & A\otimes A\otimes N \leftarrow \cdots \end{array}$$

need to show homology groups vanish

idea: find a chain homotopy T between 0 and $id_{\widetilde{B}_\bullet}$

__recall:__ $F, G: C_\bullet \to D_\bullet$ chain maps

chain hmtp T from F to G: $T: C_n \to D_{n+1}$

s.t. $dT + Td = G - F$.

so need $T: \widetilde{B}_k(A,A,N) \longrightarrow \widetilde{B}_{k+1}(A,A,N)$

$k=-1$: $\quad N \longrightarrow A\otimes N$

$\qquad n \longmapsto 1\otimes n$

$k \geq 0$: $\quad A\otimes \underbrace{A\otimes \cdots \otimes A}_{k}\otimes N \longrightarrow A\otimes \underbrace{(A\otimes \cdots \otimes A)}_{k+1}\otimes N$

$a \otimes a_1 \otimes \dots \otimes a_k \otimes n \longmapsto 1 \otimes (a \otimes a_1 \dots \otimes a_k) \otimes n$

HW: $dT + Td = \mathrm{id}$

<u>Remark</u>: the chain complex $B_\bullet(A, A, N)$ is a resolution of left A-module N by free A-modules.

<u>Recall</u>: $M_\bullet, N_\bullet \in Ch_R$ $\quad R$ comm. ring

tensor product $\quad M_\bullet \otimes N_\bullet \in Ch_R$

$$(M_\bullet \otimes N_\bullet)_n = \bigoplus_{k+l=n} M_k \otimes N_l$$

$$d(\underset{M_k}{m} \otimes \underset{N_l}{n}) := dm \otimes n + (-1)^{|m|\cdot|d|} m \otimes dn$$

$$(-1)^{|m|\cdot|d|} = (-1)^k$$

More generally: A R-algebra, $M_\bullet \in Ch_A$, $N_\bullet \in {}_A Ch$

$Ch_R \ni M_\bullet \otimes_A N_\bullet$ with the same def. of degree + diff. as above

Fix $M_\bullet \in Ch_A$

$F : {}_ACh \longrightarrow Ch_R$ functor

$N_\bullet \longmapsto M_\bullet \otimes_A N_\bullet$

<u>Q</u>: Is F compatible with weak equivalences?

<u>Ex</u>: $A = R = \mathbb{Z}$ $\qquad M_\bullet = \mathbb{Z}/2[0]$

$$
\begin{array}{lccc}
\text{deg} & 0 & & 1 \\
N_\bullet : & \mathbb{Z} & \xleftarrow{\cdot 2} & \mathbb{Z} \\
\varepsilon \downarrow \ \text{weak equiv.} & \downarrow & & \\
\mathbb{Z}/2[0] & \mathbb{Z}/2 & &
\end{array}
\qquad M_\bullet \otimes \_\_ \longmapsto
$$

$$
\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, equal] & \mathbb{Z}/2 \arrow[l, "0"'] \arrow[d, equal] \\
\mathbb{Z}/2 \otimes \mathbb{Z} \arrow[d] & \mathbb{Z}/2 \otimes \mathbb{Z} \arrow[l, "id \otimes (\cdot 2)"'] \\
\mathbb{Z}/2 \otimes \mathbb{Z}/2 \arrow[d, equal] & \\
\mathbb{Z}/2 &
\end{tikzcd}
$$

this shows that tensoring is not compatible with w.e.

How can we fix this?

traditionally: $M \in \mathrm{Mod}_A$, $N \in {}_A\mathrm{Mod}$

A <u>free resolution</u> of M is an exact sequence of right A-modules

$$M \xleftarrow{\varepsilon} \underbrace{M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots}_{\hat{M}_\bullet}$$

M_i free A-mod.

i.e. $\hat{M}_\bullet \xrightarrow{\varepsilon} M$ is a weak equivalence

e.g. $\hat{M}_\bullet = B_\bullet(M,A,A)$ "cofibrant replacement of M"

<u>Def</u>: $\mathrm{Tor}_k^A(M,N) := H_k(\hat{M}_\bullet \otimes_A N) = H_k(B_\bullet(M,A,A) \otimes_A N)$

$$\| $$

$$H_k(B_\bullet(M,A,N))$$

<u>Ex</u>: $R = \mathbb{Z}$

$A = \underbrace{\mathbb{Z}[G]}_{\text{group ring}} = \{ \sum_{g \in G} k_g g \mid k_g \in \mathbb{Z} \}$

discrete group

finite sums

$\mathbb{Z}[G]$-module

$\updownarrow$

$\mathbb{Z}$-module with G-action.

$$\mathrm{Tor}_k^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = H_k(BG; \mathbb{Z})$$ group homology

$$\mathrm{Tor}_k^{\mathbb{Z}[G]}(\mathbb{Z}, N) = H_k(BG; N)$$

← "twisted" homology
aka homology with
local coefficients.