<u>Correction:</u> hocolims of top. spaces preserve weak equivalences, i.e

$C, D : I \to \mathrm{Top}$ $\qquad T : C \to D$ object-wise w.e.

$T(i) : C(i) \xrightarrow[\text{w.e.}]{\sim} D(i)$

$$\Rightarrow \quad \operatorname{hocolim}_I C \xrightarrow[\text{w.e.}]{\sim} \operatorname{hocolim}_I D$$

<u>recall:</u> $X_\bullet : \Delta^{op} \to \mathrm{Top}$ simplicial top. space

<u>geometric realization</u>

$$|X_\bullet| = \operatorname{coeq}\Big(\coprod_n X_n \times \Delta^n \leftleftarrows \coprod_{[m] \xrightarrow{f} [n]} X_n \times \Delta^m \Big)$$

morphisms in Δ

$(x, f_x t) \leftarrow (f, \underset{X_n}{x}, \underset{\Delta^m}{t})$

$(f^* x, t) \leftarrow (f, x, t)$

$f^* : X_n \to X_m$

$f_x : \Delta^m \to \Delta^n$

fat geometric realization:

$$\|X_\bullet\| = \mathrm{coeq}\left(\coprod_{[m] \rightarrowtail [n]} \quad \rightrightarrows \quad \coprod X_n \times \Delta^m \right)$$

$[m] \xrightarrow{f} [n]$ monomorphisms.

$$\downarrow$$

$$|X_\bullet|$$

pros/cons: pro for $|X_\bullet|$: it is smaller

pro for $\|X_\bullet\|$: if $X_\bullet \xrightarrow{f} Y_\bullet$ objectwise w.e.

$\Rightarrow \|X_\bullet\| \xrightarrow[\text{w.e.}]{\sim} \|Y_\bullet\|$

Thm: (Wang): $\operatorname{hocolim}_{\Delta^{op}} X_\bullet \underset{\text{w.e.}}{\sim} \|X_\bullet\|$

$X_\bullet$ simplicial top space

Ex: X top. space

$$\{\Delta^n \xrightarrow[\text{continuous}]{\sigma} X\} =: \mathrm{Sing}_n(X) \in \mathrm{Top}$$

$$\begin{aligned}
\mathrm{Sing}_\bullet(X) : \Delta^{op} &\longrightarrow \mathrm{Top} \\
[n] &\longmapsto \mathrm{Sing}_n(X)
\end{aligned}$$

$$\begin{tikzcd}
{[n]} \arrow[r, mapsto] & \mathrm{Sing}_n(X) \ni \sigma \circ f_* \\
{[m]} \arrow[u, "f"] \arrow[r, mapsto] & \mathrm{Sing}_m(X) \ni \sigma \arrow[u]
\end{tikzcd}$$

$$|\mathrm{Sing}_\bullet(X)| \xrightarrow[\text{w.e.}]{\sim} X$$

CW-complex with one n-cell for every non-degenerate n-simplex in X

$$\left(\coprod_n \mathrm{Sing}_n(X) \times \Delta^n \right) / \sim \qquad \ni \qquad (\Delta^n \xrightarrow{\sigma} X,\ t) \longmapsto \sigma(t) \in X$$

Category of Chain complexes Ch

R commutative ring

of interest to us: $R =$ field or $R = \mathbb{Z}$

Mod_R = category of modules over R

Ex: $\mathrm{Mod}_{\mathbb{Z}}$ = cat. of abelian groups = Ab

$\mathrm{Mod}_{k \leftarrow \text{field}}$ = cat. of k-vector spaces = Vect_k

$$\mathrm{Mod}_R \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$$

$$(M, N) \longmapsto M \otimes_R N =: M \otimes N$$

Mod_R is a symmetric monoidal category with $\otimes$

Q: common generalization of a ring and an algebra?

Def: A (unital) R-algebra is an R-module A together with $\mu : A \otimes A \longrightarrow A$

unit $u : R \longrightarrow A$

required to satisfy: associativity:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

unit property:

$$\begin{tikzcd} R \otimes_R A \arrow[r, "u \otimes 1"] \arrow[dr, "\cong"'] & A \otimes_R A \arrow[d, "\mu"] & A \otimes_R R \arrow[l, "1 \otimes u"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

given by R-module structure of A

Def: A R-algebra, a right A-module $M \in \mathrm{Mod}_R$ together with $M \otimes A \xrightarrow{\mu^M} M$ (satisfying associativity + unit prop.)

left A-module M: $A \otimes M \xrightarrow{\mu^M} M$ (")

Mod_A = category of right A-modules

${}_A\mathrm{Mod}$ = " left "

M right A-module

define left-multipl. by $\underset{\substack{\cap \\ A}}{a} \cdot \underset{\substack{\cap \\ M}}{m} := ma$

Problem: $a\cdot(b\cdot m) = a\cdot(mb) = (mb)a = m(ba)$

$(ab)\cdot m \neq (ba)\cdot m = m(ba)$

for non-comm. A.

$M \in \mathrm{Mod}_A$

$N \in {}_A\mathrm{Mod}$

$M \otimes_A N$ with relation: $m \in M$, $n \in N$, $a \in A$

$ma \otimes n = m \otimes an$

R-module

$$\mathrm{Mod}_A \times {}_A\mathrm{Mod} \xrightarrow[\otimes_A]{} \mathrm{Mod}_R$$

<u>Def:</u> A chain complex of R-modules is:

$$C_\bullet : \quad \xleftarrow{d} C_{n-1} \xleftarrow{d} C_n \xleftarrow{d} C_{n+1} \xleftarrow{d}$$

$C_i \in \mathrm{Mod}_R$

d R-module maps

$$d \circ d = 0$$

a <u>chain map</u> $f : C_\bullet \to D_\bullet$

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & C_n \arrow[l, "d"'] \arrow[d, "f_n"] & \cdots \arrow[l] \\
\cdots & D_{n-1} \arrow[l] & D_n \arrow[l, "d"] & \cdots \arrow[l, "d"']
\end{tikzcd}$$

Similarly can define chain cx. of A-modules left/right

Ch_R = cat. of chain cx. of R-modules

Ch_A = " " right A-modules

${}_A\mathrm{Ch}$ = " " Left "

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simplicial object $C_\bullet : \Delta^{op} \longrightarrow \mathrm{Ch}\ (\mathrm{Ch}_R,\ \mathrm{Ch}_A$ or ${}_A\mathrm{Ch})$

$$C_\bullet : \quad C_0 \overset{d_0}{\underset{d_1}{\leftleftarrows}} C_1 \quad C_2 \quad C_3 \quad \cdots$$

all d_i's are chain maps

$d^{alt} : C_n \longrightarrow C_{n-1}$ chain map

$$d^{alt} := \sum_{i=0}^{n} (-1)^i d_i$$

Fact: $d^{alt} \circ d^{alt} = 0$ (consequence of simplicial identities)

This is a double complex $C^{alt}_{\bullet\bullet}$

$$\left\{\begin{array}{c}\begin{tikzcd}
{} \arrow[d, "d^{ver}"'] & {} \arrow[d] & {} \arrow[d] \\
C_{0K} & C_{1K} \arrow[l, "d^{hor}"'] & C_{2K} \arrow[l] \\
{} & & \\
C_{0,K-1} \arrow[u, phantom] & C_{1,K-1} \arrow[l, "d^{hor}"'] & C_{2,K-1} \arrow[l] \\
{} & {} & {}
\end{tikzcd}\end{array}\right.$$

$$\begin{tikzcd}
{} \arrow[d, "d^{ver}"'] & {} \arrow[d] & {} \arrow[d] \\
C_{0K} \arrow[d, "d^{ver}"'] & C_{1K} \arrow[l, "d^{hor}"'] \arrow[d] & C_{2K} \arrow[l] \arrow[d] \\
C_{0,K-1} \arrow[d, "d^{ver}"'] & C_{1,K-1} \arrow[l, "d^{hor}"'] \arrow[d] & C_{2,K-1} \arrow[l] \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

d^{hor} = "horizontal differential" $\doteq d^{alt}$

d^{ver} = "vertical " " " (comes from the differential in each C_i)

<u>Def:</u> Let $C_{\bullet\bullet}$ be a <u>double complex</u>, i.e.

$d^{ver}: C_{m,n} \to C_{m,n-1}$ and $(d^{ver})^2 = 0$

$d^{hor}: C_{m,n} \to C_{m-1,n}$ $(d^{hor})^2 = 0$

$d^{ver} \circ d^{hor} = d^{hor} \circ d^{ver}$

Then the <u>totalization</u> $\mathrm{Tot}(C_{\bullet\bullet})$ is given by

a chain cx

$$\mathrm{Tot}(C_{\bullet\bullet})_k = \bigoplus_{m+n=k} C_{m,n}$$

with differential: $d = d^{ver} + (-1)^{\text{vertical degree}}\, d^{hor}$